

一类变系数二阶椭圆交面问题有效的谱元法

何 娅, 郑继会

贵州师范大学数学科学学院, 贵州 贵阳

收稿日期: 2023年3月9日; 录用日期: 2023年4月5日; 发布日期: 2023年4月12日

摘要

本文针对圆域上一类变系数二阶椭圆交面问题提出了一种有效的谱元法。首先, 根据极坐标变换公式, 极条件以及傅里叶基函数展开, 原问题被分解为一系列关于径向变量的相互独立的一维二阶问题, 并建立了弱形式和相应的离散格式。其次, 我们根据变系数的正则性, 构造了基于分片高阶多项式逼近的一种谱元方法, 再通过利用勒让德多项式的正交性质, 构造了一组适当的基函数, 使得离散变分形式中的系数矩阵在变系数为分片多项式条件下是分块对角的稀疏矩阵。最后, 我们呈现了一些数值例子, 通过数值结果验证了我们提出的算法是收敛的和高精度的。

关键词

变系数, 二阶椭圆交面问题, 降维格式, 谱元法

An Efficient Spectral Element Method for A Class of Second Order Elliptic Interface Problems with Variable Coefficients

Ya He, Jihui Zheng

School of Mathematical Sciences, Guizhou Normal University, Guiyang Guizhou

Received: Mar. 9th, 2023; accepted: Apr. 5th, 2023; published: Apr. 12th, 2023

Abstract

In this paper, we put forward an efficient spectral element method for a class of second order elliptic interface problem with variable coefficients in a circular region. Firstly, because of the polarity transformation, pole condition and fourier basis function expansion, the original problem resolve into a succession of independent one-dimensional second-order problems about radial variables. Furthermore, the weak form and relevant discrete scheme are established. Secondly,

according to the regularity of variable coefficients, we construct a spectral element method based on piecewise high-order polynomial approximation, and then by taking advantage of the orthogonal property of Legendre polynomials, we construct a set of appropriate basis functions, so that the coefficient matrix in the discrete scheme is sparse diagonal matrix in the case that the variable coefficients are piecewise polynomials. Finally, some numerical examples are presented, and the numerical results show that the proposed algorithm is convergent and high-precision.

Keywords

Variable Coefficient, Second Order Elliptic Intersection Problem, Dimension Reduction Format, Spectral Element Method

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1. 引言

在材料科学和流体动力学中,通常会涉及到一些分片光滑的变系数二阶椭圆交面问题的数值求解。常用的数值方法包括有限元方法[1] [2] [3] [4] [5]和有限差分方法[6] [7] [8]。

文献[9]应用拟合有限元方法解决椭圆界面问题,并在有限元空间的一些近似假设下,得到了能量范数估计。文献[3]利用具有特殊拟合网格的标准有限元对拟合网格有限元方法进行估计。文献[10]提出了一种无限元素方法,可以看作是一种特定的网格细化方案,用于不适合弯曲界面的椭圆界面问题。文献[11]利用 Nitsche 方法,提出了椭圆界面问题的有限元法。然而,很少有关于用谱元法去处理圆域上一类分片光滑变系数二阶椭圆交面问题的报道,因此,本文针对圆域上一类变系数二阶椭圆交面问题提出一种有效的谱元法。首先,根据极坐标变换公式,极条件以及傅里叶基函数展开,原问题被分解为一系列关于径向变量的相互独立的一维二阶资源问题,并建立了弱形式和相应的离散格式。其次,我们根据变系数的正则性,构造了基于分片高阶多项式逼近的一种谱元方法,再通过利用勒让德多项式的正交性质,构造了一组适当的基函数,使得离散变分形式中的系数矩阵在变系数为分片多项式条件下是分块对角的稀疏矩阵。最后,我们呈现了一些数值例子,通过数值结果验证了我们提出的算法是收敛的和高精度的。

本文剩余部分安排如下:在第 2 节,我们推导圆域内变系数二阶椭圆问题的降维格式。在第 3 节,我们建立相应的弱形式及其离散格式。在第 4 节,我们详细描述算法的实现过程。在第 5 节中,我们呈现了几个数值例子。

2. 降维格式

作为一个模型,我们考虑下面的分片光滑的二阶椭圆交面问题:

$$-\Delta \hat{u}(x, y) + |\hat{a}(x, y) - c| \hat{u}(x, y) = \hat{f}(x, y), (x, y) \in \Omega, \quad (1)$$

$$\hat{u}(x, y)|_{\partial\Omega} = 0, \quad (2)$$

其中 $\hat{a}(x, y) = r$ 是一个径向函数, $\Omega = \{(x, y) : x^2 + y^2 \leq R^2\}$, $c \in (0, R)$ 是一个非光滑点。下面我们将推导方程(1)~(2)基于极坐标变换 $x = r \cos \theta, y = r \sin \theta$ 的降维格式。令

$$u(r, \theta) = \hat{u}(x, y), f(r, \theta) = \hat{f}(x, y).$$

则问题(1)~(2)可以化为下面的等价形式:

$$-\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u(r, \theta)}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2 u(r, \theta)}{\partial \theta^2} + |r - c| u(r, \theta) = f(r, \theta), (r, \theta) \in D, \quad (3)$$

其中 $D = (0, R) \times (0, 2\pi)$ 。利用傅里叶基函数展开, 有

$$u(r, \theta) = \sum_{|m|=0}^{\infty} u_m(r) e^{im\theta}, f(r, \theta) = \sum_{|m|=0}^{\infty} f_m(r) e^{im\theta}. \quad (4)$$

将(4)代入(3), 利用傅里叶基函数的正交性可得到(1)~(2)等价的一系列一维二阶椭圆交面问题:

$$-\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_m}{\partial r} \right) + \frac{m^2}{r^2} u_m + |r - c| u_m = f_m, r \in (0, R), \quad (5)$$

$$u_m(R) = 0, (m = 0), \quad (6)$$

$$u_m(0) = u_m(R) = 0, (m \neq 0). \quad (7)$$

3. 弱形式和离散格式

令 $I = (0, R), \omega = r$ 。定义一个通常的带权 Sobolev 空间:

$$L_\omega^2(I) = \left\{ \sigma : \int_I \omega |\sigma|^2 dr < \infty \right\},$$

其中, 对应的内积以及范数分别如下:

$$(\sigma, \phi)_\omega = \int_I \omega \sigma \bar{\phi} dr, \|\sigma\|_\omega = \left(\int_I \omega |\sigma|^2 dr \right)^{\frac{1}{2}}.$$

进一步定义非一致带权 Sobolev 空间:

$$\begin{aligned} H_{0,\omega,m}^1(I) &= \left\{ u_m : u_m \in L_\omega^2(I), \frac{\partial u_m}{\partial r} \in L_\omega^2(I), u_m(R) = 0 \right\}, (m = 0), \\ H_{0,\omega,m}^1(I) &= \left\{ u_m : \frac{\partial^p u_m}{\partial r^p} \in L_{\omega^{2p-1}}^2(I), p = 0, 1, u_m(0) = u_m(R) = 0 \right\}, (m \neq 0), \end{aligned}$$

其中, 对应的内积以及范数分别如下:

$$\begin{aligned} (u_0, v_0)_{1,\omega,0} &= (u_0, v_0)_\omega + \left(\frac{\partial u_0}{\partial r}, \frac{\partial v_0}{\partial r} \right)_\omega, \|u_0\|_{1,\omega,0} = \sqrt{(u_0, u_0)_{1,\omega,0}}, \\ (u_m, v_m)_{1,\omega,m} &= \sum_{p=0}^1 \left(\frac{\partial^p u_m}{\partial r^p}, \frac{\partial^p v_m}{\partial r^p} \right)_{\omega^{2p-1}}, \|u_m\|_{1,\omega,m} = \sqrt{(u_m, u_m)_{1,\omega,m}}, (m \neq 0). \end{aligned}$$

则(5)~(7)的一种弱形式为: 找到 $u_m \in H_{0,\omega,m}^1(I)$, 使得

$$\mathcal{A}_m(u_m, v_m) = F(v_m), \forall v_m \in H_{0,\omega,m}^1(I), \quad (8)$$

其中

$$\begin{aligned} \mathcal{A}_m(u_m, v_m) &= \int_I r u'_m v'_m + \frac{m^2}{r} u_m v_m + r |r - c| u_m v_m dr, \\ F_m(v_m) &= \int_I r f_m v_m dr. \end{aligned}$$

P_N 是次数不超过 N 次的多项式, 定义逼近空间:

$$X_{Nm} = \left\{ \sigma_{mN} \in C(I) : \sigma_{mN}|_{[0,c]} \in P_N, \sigma_{mN}|_{[c,R]} \in P_N \right\} \cap H_{0,\omega,m}^1(I).$$

则弱形式(8)相应的离散格式为: 找到 $u_{mN} \in X_{Nm}$, 使得

$$\mathcal{A}_m(u_{mN}, v_{mN}) = F_m(v_{mN}), v_{mN} \in X_{Nm}. \quad (9)$$

4. 算法的有效实现

我们将在这一节详细描述算法的实现过程。首先, 我们构造谱元逼近空间中的一组基函数。令

$$\varphi_i(t) = \frac{1}{\sqrt{4i+6}}(L_i(t) - L_{i+2}(t)), (i = 0, 1, \dots, N-2),$$

其中 $t \in \hat{I} = [-1, 1]$, $L_i(t)$ 表示 i 次勒让德多项式。我们分别定义如下的内部基函数:

$$\begin{aligned} \phi_{1,i}(x) &= \begin{cases} \varphi_i(t_1(x)), & x \in I_1, \\ 0, & x \in I_2. \end{cases}, i = 0, 1, \dots, N-2, \\ \phi_{2,i}(x) &= \begin{cases} 0, & x \in I_1, \\ \varphi_i(t_2(x)), & x \in I_2, \end{cases}, i = 0, 1, \dots, N-2, \end{aligned}$$

其中 $I_1 = (0, c)$, $I_2 = (c, R)$, $t_1(x) : I_1 \rightarrow (-1, 1)$, $t_2(x) : I_2 \rightarrow (-1, 1)$ 。则 X_{Nm} 的所有内部基函数为:

$$\mathcal{X}_{Nm} = \bigcup_{j=1,2} \text{span}\{\phi_{j,0}, \phi_{j,1}, \dots, \phi_{j,N-2}\}$$

定义 c 处的交面基函数:

$$\begin{aligned} \psi_1(x) &= \begin{cases} \psi_{11} = \frac{1}{2}(1-t_1(x)), & x \in I_1, \\ \psi_{12} = 0, & x \in I_2. \end{cases} \\ \psi_2(x) &= \begin{cases} \psi_{21} = \frac{1}{2}(1+t_1(x)), & x \in I_1, \\ \psi_{22} = \frac{1}{2}(1-t_2(x)), & x \in I_2. \end{cases} \end{aligned}$$

则逼近空间 X_{Nm} 可表示为:

$$X_{N0} = \mathcal{X}_{N0} \oplus \text{span}\{\psi_1(x), \psi_2(x)\}; X_{Nm} = \mathcal{X}_{Nm} \oplus \text{span}\{\psi_2(x)\}.$$

1) 当 $m=0$ 时, 对离散格式(9)进行积分区间标准化:

$$\begin{aligned} &\frac{2}{c} \int_{-1}^1 r_1 u'_{0N} v'_{0N} dt_1 + \frac{c}{2} \int_{-1}^1 r_1(c-r_1) u_{0N} v_{0N} dt_1 \\ &+ \frac{2}{R-c} \int_{-1}^1 r_2 u'_{0N} v'_{0N} dt_2 + \frac{R-c}{2} \int_{-1}^1 r_2(r_2-c) u_{0N} v_{0N} dt_2 \\ &= \frac{c}{2} \int_{-1}^1 r_1 f_0 v_{0N} dt_1 + \frac{R-c}{2} \int_{-1}^1 r_2 f_0 v_{0N} dt_2, \end{aligned}$$

其中 $r_1(t_1) = \frac{1+t_1}{2}c$, $r_2(t_2) = \frac{1+t_2}{2}(R-c)+c$ 。将近似解 u_{0N} 展开为:

$$u_{0N} = \sum_{k=1}^2 \sum_{i=0}^{N-2} u_{0i}^k \phi_{k,i} + \sum_{k=1}^2 u_{0k} \psi_k.$$

则可得到离散格式(9)等价的矩阵形式为:

$$\begin{pmatrix} A_1^0 & 0 & B_1^0 & B_2^0 \\ 0 & A_2^0 & 0 & B_3^0 \\ B_1^{0T} & 0 & c_1 & c_2 \\ B_2^{0T} & B_3^{0T} & c_3 & c_4 \end{pmatrix} U_0 = \begin{pmatrix} F_1^0 \\ F_2^0 \\ f_1 \\ f_2 \end{pmatrix},$$

其中

$$\begin{aligned} A_1^0 &= (a_{ij}^{01}), A_2^0 = (a_{ij}^{02}), B_1^0 = (b_{10}^0, b_{11}^0, \dots, b_{1,N-2}^0)^T, B_2^0 = (b_{20}^0, b_{21}^0, \dots, b_{2,N-2}^0)^T, \\ B_3^0 &= (b_{30}^0, b_{31}^0, \dots, b_{3,N-2}^0)^T, F_1^0 = (f_{10}^0, f_{11}^0, \dots, f_{1,N-2}^0)^T, F_2^0 = (f_{20}^0, f_{21}^0, \dots, f_{2,N-2}^0)^T, \\ U_0 &= (u_{00}^1, u_{01}^1, \dots, u_{0,N-2}^1, u_{00}^2, u_{01}^2, \dots, u_{0,N-2}^2, u_{01}, u_{02})^T, \end{aligned}$$

其中

$$\begin{aligned} a_{ij}^{01} &= \frac{2}{c} \int_{-1}^1 r_1 \phi'_{1,i} \phi'_{1,j} dt_1 + \frac{c}{2} \int_{-1}^1 r_1 (c - r_1) \phi_{1,i} \phi_{1,j} dt_1, \\ a_{ij}^{02} &= \frac{2}{R - c} \int_{-1}^1 r_2 \phi'_{2,i} \phi'_{2,j} dt_2 + \frac{R - c}{2} \int_{-1}^1 r_2 (r_2 - c) \phi_{2,i} \phi_{2,j} dt_2, \\ b_{1i}^0 &= \frac{2}{c} \int_{-1}^1 r_1 \psi'_{11} \phi'_{1,i} dt_1 + \frac{c}{2} \int_{-1}^1 r_1 (c - r_1) \psi_{11} \phi_{1,i} dt_1, \\ b_{2i}^0 &= \frac{2}{c} \int_{-1}^1 r_1 \psi'_{21} \phi'_{1,i} dt_1 + \frac{c}{2} \int_{-1}^1 r_1 (c - r_1) \psi_{21} \phi_{1,i} dt_1, \\ b_{3i}^0 &= \frac{2}{R - c} \int_{-1}^1 r_2 \psi'_{22} \phi'_{2,i} dt_2 + \frac{R - c}{2} \int_{-1}^1 r_2 (r_2 - c) \psi_{22} \phi_{2,i} dt_2, \\ c_1 &= \frac{2}{c} \int_{-1}^1 r_1 \psi'_{11} \psi'_{11} dt_1 + \frac{c}{2} \int_{-1}^1 r_1 (c - r_1) \psi_{11} \psi_{11} dt_1, \\ c_2 &= \frac{2}{c} \int_{-1}^1 r_1 \psi'_{11} \psi'_{21} dt_1 + \frac{c}{2} \int_{-1}^1 r_1 (c - r_1) \psi_{11} \psi_{21} dt_1, \\ c_3 &= \frac{2}{c} \int_{-1}^1 r_1 \psi'_{21} \psi'_{11} dt_1 + \frac{c}{2} \int_{-1}^1 r_1 (c - r_1) \psi_{21} \psi_{11} dt_1, \\ c_4 &= \frac{2}{c} \int_{-1}^1 r_1 \psi'_{21} \psi'_{21} dt_1 + \frac{c}{2} \int_{-1}^1 r_1 (c - r_1) \psi_{21} \psi_{21} dt_1 \\ &\quad + \frac{2}{R - c} \int_{-1}^1 r_2 \psi'_{22} \psi'_{22} dt_2 + \frac{R - c}{2} \int_{-1}^1 r_2 (r_2 - c) \psi_{22} \psi_{22} dt_2, \\ f_{1i}^0 &= \frac{c}{2} \int_{-1}^1 r_1 f_0 \phi_{1,i} dt_1, f_{2i}^0 = \frac{R - c}{2} \int_{-1}^1 r_2 f_0 \phi_{2,i} dt_2, \\ f_1 &= \frac{c}{2} \int_{-1}^1 r_1 f_0 \psi_{11} dt_1, f_2 = \frac{c}{2} \int_{-1}^1 r_1 f_0 \psi_{21} dt_1 + \frac{R - c}{2} \int_{-1}^1 r_2 f_0 \psi_{22} dt_2. \end{aligned}$$

2) 当 $m \neq 0$ 时, 对(9)进行积分区间标准化:

$$\begin{aligned} &\frac{2}{c} \int_{-1}^1 r_1 u'_{mN} v'_{mN} dt_1 + \frac{c}{2} \int_{-1}^1 \frac{m^2}{r_1} u_{mN} v_{mN} dt_1 \\ &\quad + \frac{c}{2} \int_{-1}^1 r_1 (c - r_1) u_{mN} v_{mN} dt_1 + \frac{2}{R - c} \int_{-1}^1 r_2 u'_{mN} v'_{mN} dt_2 \\ &\quad + \frac{R - c}{2} \int_{-1}^1 \frac{m^2}{r_2} u_{mN} v_{mN} dt_2 + \frac{R - c}{2} \int_{-1}^1 r_2 (r_2 - c) u_{mN} v_{mN} dt_2 \\ &= \frac{c}{2} \int_{-1}^1 r_1 f_m v_{mN} dt_1 + \frac{R - c}{2} \int_{-1}^1 r_2 f_m v_{mN} dt_2. \end{aligned}$$

将近似解 u_{mN} 展开为:

$$u_{mN} = \sum_{k=1}^2 \sum_{i=0}^{N-2} u_{mi}^k \phi_{k,i} + u_m \psi_2,$$

则可得到离散格式(9)等价的矩阵形式为:

$$\begin{pmatrix} A_1^m & 0 & B_1^m \\ 0 & A_2^m & B_2^m \\ B_1^{mT} & B_2^{mT} & c_5 \end{pmatrix} U_m = \begin{pmatrix} F_1^m \\ F_2^m \\ f_3 \end{pmatrix},$$

其中

$$\begin{aligned} A_1^m &= (a_{ij}^{m1}), A_2^m = (a_{ij}^{m2}), B_1^m = (b_{10}^m, b_{11}^m, \dots, b_{1,N-2}^m)^T, \\ B_2^m &= (b_{20}^m, b_{21}^m, \dots, b_{2,N-2}^m)^T, F_1^m = (f_{10}^m, f_{11}^m, \dots, f_{1,N-2}^m)^T, \\ F_2^m &= (f_{20}^m, f_{21}^m, \dots, f_{2,N-2}^m)^T, U_m = (u_{m1}^1, u_{m1}^2, \dots, u_{m,N-2}^1, u_{m0}^2, u_{m1}^2, \dots, u_{m,N-2}^2, u_m)^T, \end{aligned}$$

其中

$$\begin{aligned} a_{ij}^{m1} &= a_{ij}^{01} + \frac{c}{2} \int_{-1}^1 \frac{m^2}{r_1} \phi_{1,i} \phi_{1,j} dt_1, a_{ij}^{m2} = a_{ij}^{02} + \frac{R-c}{2} \int_{-1}^1 \frac{m^2}{r_2} \phi_{2,i} \phi_{2,j} dt_2, \\ b_{1i}^m &= b_{2i}^0 + \frac{c}{2} \int_{-1}^1 \frac{m^2}{r_1} \psi_{21} \phi_{1,i} dt_1, b_{2i}^m = b_{3i}^0 + \frac{R-c}{2} \int_{-1}^1 \frac{m^2}{r_2} \psi_{22} \phi_{2,i} dt_2, \\ c_5 &= c_4 + \frac{c}{2} \int_{-1}^1 \frac{m^2}{r_1} \psi_{21} \psi_{21} dt_1 + \frac{R-c}{2} \int_{-1}^1 \frac{m^2}{r_2} \psi_{22} \psi_{22} dt_2, f_{1i}^m = \frac{c}{2} \int_{-1}^1 r_1 f_m \phi_{1,i} dt_1, \\ f_{2i}^m &= \frac{R-c}{2} \int_{-1}^1 r_2 f_m \phi_{2,i} dt_2, f_3 = \frac{c}{2} \int_{-1}^1 r_1 f_m \psi_{21} dt_1 + \frac{R-c}{2} \int_{-1}^1 r_2 f_m \psi_{22} dt_2. \end{aligned}$$

5. 数值实验

为了验证该算法的有效性, 我们将呈现一些数值实验。我们在 MATLAB2017b 平台上编程计算。令数值解与近似解之间的误差如下:

$$e(\hat{u}(x, y), \hat{u}_N(x, y)) = \|\hat{u}(x, y) - \hat{u}_{NM}(x, y)\|_{L^\infty(\Omega)} = \|u(r, \theta) - u_{NM}(r, \theta)\|_{L^\infty(\Omega)},$$

其中

$$u_{NM}(r, \theta) = \sum_{|m|=0}^M u_{mN}(r) e^{im\theta}. \quad (10)$$

例 1 我们取函数 $u = (x^2 + y^2 - 1)e^{(x+y)}$, $R = 1, c = 0.5$, 当取不同的 M 和 N 时, 我们在表 1 中分别列出了数值解与近似解之间的误差结果。

Table 1. Error outcome between numerical solution and approximate solution for diverse N and M

表 1. 当取不同的 N 和 M 时, 数值解与近似解之间的误差结果

N	$M = 4$	$M = 8$	$M = 12$	$M = 16$
20	3.1136e-04	1.8268e-08	1.8769e-07	6.7240e-04
25	3.8403e-04	1.9285e-08	1.9873e-14	1.0750e-08
30	4.1052e-04	1.9436e-08	1.9554e-13	8.7985e-15
35	4.1779e-04	1.8569e-08	2.0101e-13	1.1102e-15

为了更加直观的表明算法的收敛性和高精度, 在图 1 中依次给出了数值解与近似解的图像, 在图 2 中依次给出了数值解与 $N = 30, M = 10$ 和 $N = 40, M = 20$ 时的近似解之间的误差图像。

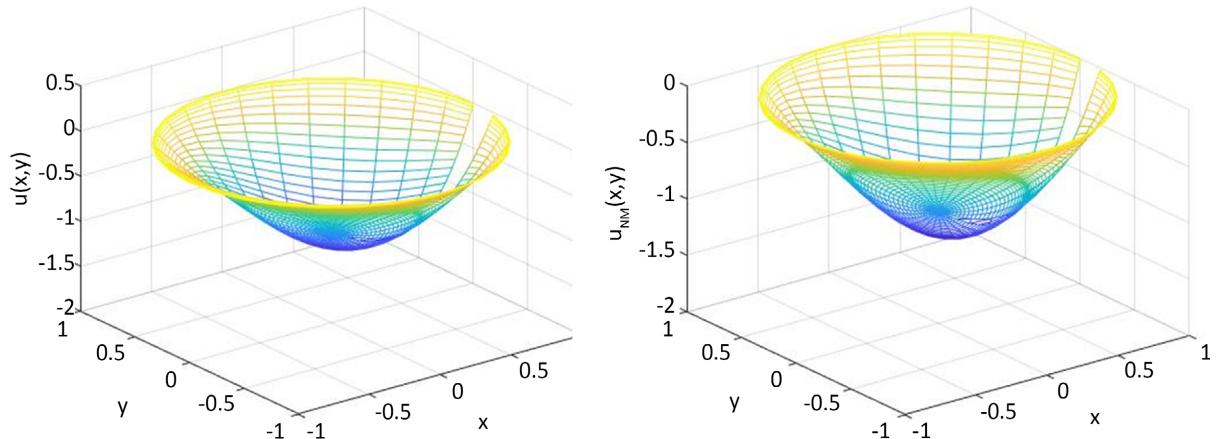


Figure 1. Graphics of numerical solution (left) and approximate solutions (right) for $N = 30$ and $M = 12$
图 1. 数值解(左)与 $N = 30$ 和 $M = 12$ 时的近似解(右)的绘图

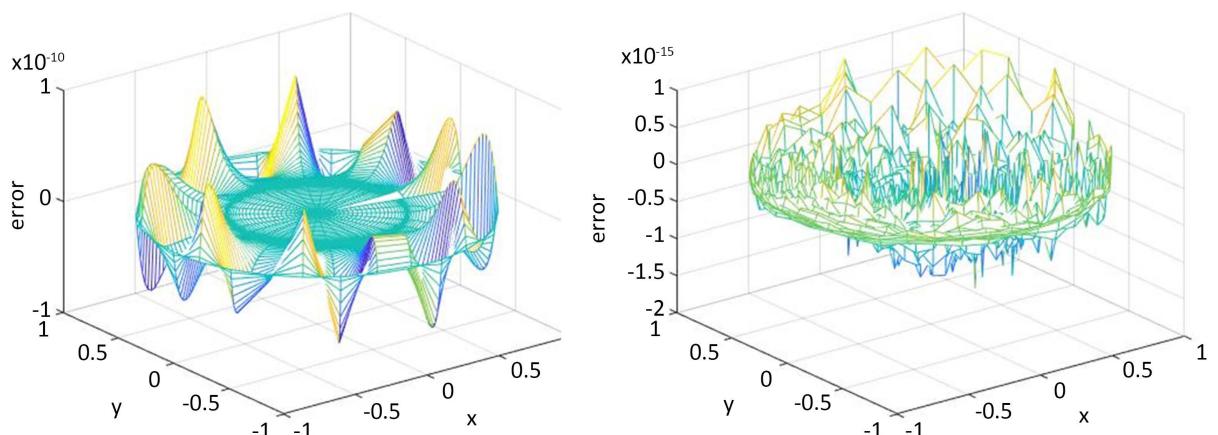


Figure 2. Error Graphics between the numerical solution and the approximate solution when $N = 30, M = 10$ (left) and $N = 40, M = 20$ (right)

图 2. 数值解与 $N = 30, M = 10$ (左)和 $N = 40, M = 20$ (右)时的近似解之间的误差绘图

从表 1 的数据可以知道, 当 $N \geq 30, M \geq 12$ 时, 近似解 $u_{mN}(x, y)$ 达到了大约 10^{-13} 的精度。最后, 从图 1 和图 2 也能得到我们提出的算法是收敛的和高精度。

参考文献

- [1] Bordas, S.P., Burman, E., Larson, M.G., et al. (2018) Geometrically Unfitted Finite Element Methods and Applications: Proceedings of the UCL Workshop 2016. Springer, Berlin. <https://doi.org/10.1007/978-3-319-71431-8>
- [2] Bramble, J.H. and King, J.T. (1996) A Finite Element Method for Interface Problems in Domains with Smooth Boundaries and Interfaces. *Advances in Computational Mathematics*, **6**, 109-138. <https://doi.org/10.1007/BF02127700>
- [3] Chen, Z. and Zou, J. (1998) Finite Element Methods and Their Convergence for Elliptic and Parabolic Interface Problems. *Numerische Mathematik*, **79**, 175-202. <https://doi.org/10.1007/s002110050336>
- [4] Li, Z. (1998) The Immersed Interface Method Using a Finite Element Formulation. *Applied Numerical Mathematics*, **27**, 253-267. [https://doi.org/10.1016/S0168-9274\(98\)00015-4](https://doi.org/10.1016/S0168-9274(98)00015-4)
- [5] Chou, S.H., Kwak, D.Y. and Wee, K.T. (2010) Optimal Convergence Analysis of an Immersed Interface Finite Element Method. *Advances in Computational Mathematics*, **33**, 149-168. <https://doi.org/10.1007/s10444-009-9122-y>

-
- [6] Angelova, I.T. and Vulkov, L.G. (2007) High-Order Finite Difference Schemes for Elliptic Problems with Intersecting Interfaces. *Applied Mathematics and Computation*, **187**, 824-843. <https://doi.org/10.1016/j.amc.2006.08.165>
 - [7] Smith, G.D., Smith, G.D. and Smith, G.D.S. (1985) Numerical Solution of Partial Differential Equations: Finite Difference Methods. Oxford University Press, Oxford.
 - [8] Jarvis, D.A. and Noye, B.J. (2003) Finite Difference Solution to the Poisson Equation at an Intersection of Interfaces. *ANZIAM Journal*, **45**, C632-C645. <https://doi.org/10.21914/anziamj.v45i0.913>
 - [9] Babuška, I. (1970) The Finite Element Method for Elliptic Equations with Discontinuous Coefficients. *Computing*, **5**, 207-213. <https://doi.org/10.1007/BF02248021>
 - [10] Han, H. (1982) The Numerical Solutions of Interface Problems by Infinite Element Method. *Numerische Mathematik*, **39**, 39-50. <https://doi.org/10.1007/BF01399310>
 - [11] Hansbo, A. and Hansbo, P. (2002) An Unfitted Finite Element Method, Based on Nitsche's Method, for Elliptic Interface Problems. *Computer Methods in Applied Mechanics and Engineering*, **191**, 5537-5552. [https://doi.org/10.1016/S0045-7825\(02\)00524-8](https://doi.org/10.1016/S0045-7825(02)00524-8)