Advances in Applied Mathematics 应用数学进展, 2020, 9(7), 1092-1102
Published Online July 2020 in Hans. http://www.hanspub.org/journal/aam
https://doi.org/10.12677/aam.2020.97129

Hans汉斯

Representation and Extension of Hom-Jordan Triple System

Lele Guo

School of Mathematics, Liaoning Normal University, Dalian Liaoning
Email: 1543636265@qq.com

Received: Jul. 5th, 2020; accepted: Jul. 21st, 2020; published: Jul. 28th, 2020

Abstract

This paper mainly studies the representation and extension of Hom-Jordan triple system. Firstly, we give the definition of representation of Hom-Jordan triple system, the method of judging the representation and an example of representation. At the same time, the condition that the dual of Hom-Jordan triple system is the representation is given. Secondly, two methods of extension of Hom-Jordan triple system are found. One is to construct an extension by a special linear function, and the other is the T^*-extension of Hom-Jordan triple system.

Keywords

Hom-Jordan Triple System, Representation, T^*-Extension

Hom-约当三系的表示与扩张

郭乐乐

辽宁师范大学数学学院，辽宁 大连
Email: 1543636265@qq.com

收稿日期：2020年7月5日；录用日期：2020年7月21日；发布日期：2020年7月28日

摘　要

本文主要介绍了Hom-约当三系的表示与扩张。首先，给出了Hom-约当三系表示的定义、判断表示的方法以及表示的例子，同时给出Hom-约当三系的表示的对偶是表示所满足的条件。其次，找到Hom-约当三系扩张的两种方法，其一是利用一个特殊的线性函数而构造出的扩张，其二是Hom-约当三系的T^*-扩张。

文章引用: 郭乐乐. Hom-约当三系的表示与扩张[J]. 应用数学进展, 2020, 9(7): 1092-1102.
DOI: 10.12677/aam.2020.97129

关键词

Hom-约当三系，表示，T^*-扩张

1. 引言

约当代数和约当三系与对称流形有密切联系，实的紧的约当三系与对称 R-空间之间存在一一对应关系[1]。此外，约当三系与结合代数也有密切联系。在结合代数$(A,\circ)$上定义运算$a\circ(b\circ c)+b\circ(a\circ c)+(a\circ b)\circ c$可以得到约当三系[2]。约当三系作为独立的代数体系已经有很多相关的结果，例如约当三系的分类[3]、约当三系的表示[4]、约当三系的上同调[5]等。Hom-约当三系是[6]中引入的，是约当三系的推广。本文将考虑 Hom-约当三系的表示和扩张。

2. Hom-约当三系的表示

本文所指的线性空间都是数域 K 上的有限维线性空间。

定义 1.1 [5]设 J 是线性空间，$\alpha_1,\alpha_2\in End(J)$，$\{\cdot,\cdot,\cdot\}:J\times J\times J\to J$ 是三线性映射，如果对于任意的 $x,y,z,u,v\in J$ 满足下列条件

$$\{x,y,z\}=\{z,y,x\}, \tag{1.1}$$

$$\begin{aligned}&\{\alpha_1(x),\alpha_2(y),\{z,u,v\}\}-\{\alpha_1(z),\alpha_2(u),\{x,y,v\}\}\\&=\{\{x,y,z\},\alpha_1(u),\alpha_2(v)\}-\{\alpha_1(z),\{y,x,u\},\alpha_2(v)\},\end{aligned} \tag{1.2}$$

则称$(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2)$为 Hom-约当三系。

当$\alpha_1=id,\alpha_2=id$ 时，$(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2)$为约当三系。

定义 1.2 设$(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2)$为 Hom-约当三系，$V$ 是线性空间，$A_1,A_2\in End(V)$，$\theta_{12},\theta_{13}:J\times J\to End(V)$ 为双线性映射，如果对于任意的 $x,y,z,u\in J$ 有

$$\theta_{13}(x,y)=\theta_{13}(y,x), \tag{1.3}$$

$$\begin{aligned}&\theta_{12}(\alpha_2(x),\alpha_1(u))\theta_{12}(z,y)-\theta_{13}(\alpha_1(z),\alpha_2(x))\theta_{13}(y,u)\\&-\theta_{12}(\{z,u,x\},\alpha_2(y))A_1+\theta_{12}(\alpha_1(z),\alpha_2(u))\theta_{12}(x,y)=0,\end{aligned} \tag{1.4}$$

$$\begin{aligned}&\theta_{12}(\alpha_2(y),\alpha_1(u))\theta_{13}(x,z)-\theta_{13}(\alpha_1(z),\alpha_2(y))\theta_{12}(u,x)\\&-\theta_{13}(\alpha_1(x),\{z,u,y\})A_2+\theta_{12}(\alpha_1(z),\alpha_2(u))\theta_{13}(x,y)=0,\end{aligned} \tag{1.5}$$

$$\begin{aligned}&\theta_{12}(\alpha_2(z),\alpha_1(u))\theta_{12}(x,y)-\theta_{12}(\alpha_2(z),\{y,x,u\})A_1\\&-\theta_{12}(\alpha_1(x),\alpha_2(y))\theta_{12}(z,u)+\theta_{12}(\{x,y,z\},\alpha_2(u))A_1=0,\end{aligned} \tag{1.6}$$

$$\begin{aligned}&\theta_{13}(\{x,y,z\},\alpha_2(u))A_1-\theta_{13}(\alpha_1(z),\alpha_2(u))\theta_{12}(y,x)\\&-\theta_{12}(\alpha_1(x),\alpha_2(y))\theta_{13}(z,u)+\theta_{13}(\alpha_1(z),\{x,y,u\})A_2=0,\end{aligned} \tag{1.7}$$

$$\begin{aligned}&\theta_{12}\left(\{x,y,z\},\alpha_1(u)\right)A_2-\theta_{12}\left(\alpha_1(z),\{y,x,u\}\right)A_2\\&-\theta_{12}\left(\alpha_1(x),\alpha_2(y)\right)\theta_{12}(z,u)+\theta_{12}\left(\alpha_1(z),\alpha_2(u)\right)\theta_{12}(x,y)=0,\end{aligned}\tag{1.8}$$

则称 $(V,\theta_{12},\theta_{13},A_1,A_2)$ 是 $(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2)$ 的表示。

例 1.1 设 $(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2)$ 为 Hom-约当三系，$L:J\times J\to End(J)$，$M:J\times J\to End(J)$ 为双线性映射，其中 $L(x,y)(z)=\{x,y,z\}$，$M(x,y)(z)=\{x,z,y\}$ $(\forall x,y,z\in J)$，则 $(J,L,M,\alpha_1,\alpha_2)$ 为 $(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2)$ 的表示，称为伴随表示。

证明：直接验证可知对于 $(J,L,M,\alpha_1,\alpha_2)$ 有(1.3)~(1.8)成立。

定理 1.1 设 $(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2)$ 为Hom-约当三系，V是线性空间，$A_1,A_2\in End(V)$，$\theta_{12},\theta_{13}:J\times J\to End(V)$ 为双线性映射，在 $J\oplus V$ 上定义

$$\{x+a,y+b,z+c\}=\{x,y,z\}+\theta_{12}(z,y)a+\theta_{13}(x,z)b+\theta_{12}(x,y)c,$$

$$(\alpha_1+A_1)(x+a)=\alpha_1(x)+A_1(a),$$

$$(\alpha_2+A_2)(x+a)=\alpha_2(x)+A_2(a),$$

其中 $\forall x,y,z\in J,\forall a,b,c\in V$，则 $(J\oplus V,\{\cdot,\cdot,\cdot\},\alpha_1+A_1,\alpha_2+A_2)$ 为 Hom-约当三系当且仅当 $(V,\theta_{12},\theta_{13},A_1,A_2)$ 是 $(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2)$ 的表示。

证明：$(J\oplus V,\{\cdot,\cdot,\cdot\},\alpha_1+A_1,\alpha_2+A_2)$ 为 Hom-约当三系当且仅当下列等式成立

$$\{x+a,y+b,z+c\}=\{z+c,y+b,x+a\},\tag{1.9}$$

$$\begin{aligned}&\{\{x+a,y+b,z+c\},(\alpha_1+A_1)(u+d),(\alpha_2+A_2)(v+e)\}\\&-\{(\alpha_1+A_1)(z+c),\{y+b,x+a,u+d\},(\alpha_2+A_2)(v+e)\}\\&-\{(\alpha_1+A_1)(x+a),(\alpha_2+A_2)(y+b),\{z+c,u+d,v+e\}\}\\&+\{(\alpha_1+A_1)(z+c),(\alpha_2+A_2)(u+d),\{x+a,y+b,v+e\}\}=0,\end{aligned}\tag{1.10}$$

其中 $\forall x+a,y+b,z+c,u+d,v+e\in J\oplus V$。

由于

$$\begin{aligned}&\{x+a,y+b,z+c\}-\{z+c,y+b,x+a\}\\&=\{x,y,z\}-\{z,y,x\}+\theta_{12}(z,y)a-\theta_{12}(z,y)a+\theta_{13}(x,z)b\\&\quad-\theta_{13}(z,x)b+\theta_{12}(x,y)c-\theta_{12}(x,y)c,\end{aligned}$$

因此(1.9)成立当且仅当 θ_{13} 对称，即(1.3)式成立。

(1.10)式左边直接计算得

$$\begin{aligned}&\{\{x+a,y+b,z+c\},(\alpha_1+A_1)(u+d),(\alpha_2+A_2)(v+e)\}\\&-\{(\alpha_1+A_1)(z+c),\{y+b,x+a,u+d\},(\alpha_2+A_2)(v+e)\}\\&-\{(\alpha_1+A_1)(x+a),(\alpha_2+A_2)(y+b),\{z+c,u+d,v+e\}\}\\&+\{(\alpha_1+A_1)(z+c),(\alpha_2+A_2)(u+d),\{x+a,y+b,v+e\}\}\end{aligned}$$

$$
\begin{aligned}
&=\{\{x,y,z\},\alpha_1(u),\alpha_2(v)\}-\{\alpha_1(z),\{y,x,u\},\alpha_2(v)\}\\
&\quad-\{\alpha_1(x),\alpha_2(y),\{z,u,v\}\}+\{\alpha_1(z),\alpha_2(u),\{x,y,v\}\}\\
&\quad+\big(\theta_{12}(\alpha_2(v),\alpha_1(u))\theta_{12}(z,y)-\theta_{13}(\alpha_1(z),\alpha_2(v))\theta_{13}(y,u)\\
&\quad-\theta_{12}(\{z,u,v\},\alpha_2(y))A_1+\theta_{12}(\alpha_1(z),\alpha_2(u))\theta_{12}(v,y)\big)(a)\\
&\quad+\big(\theta_{12}(\alpha_2(v),\alpha_1(u))\theta_{13}(x,z)-\theta_{13}(\alpha_1(z),\alpha_2(v))\theta_{12}(u,x)\\
&\quad-\theta_{13}(\alpha_1(x),\{z,u,v\})A_2+\theta_{12}(\alpha_1(z),\alpha_2(u))\theta_{13}(x,v)\big)(b)\\
&\quad+\big(\theta_{12}(\alpha_2(v),\alpha_1(u))\theta_{12}(x,y)-\theta_{12}(\alpha_2(v),\{y,x,u\})A_1\\
&\quad-\theta_{12}(\alpha_1(x),\alpha_2(y))\theta_{12}(v,u)+\theta_{12}(\{x,y,v\},\alpha_2(u))A_1\big)(c)\\
&\quad+\big(\theta_{13}(\{x,y,z\},\alpha_2(v))A_1-\theta_{13}(\alpha_1(z),\alpha_2(v))\theta_{12}(y,x)\\
&\quad-\theta_{12}(\alpha_1(x),\alpha_2(y))\theta_{13}(z,v)+\theta_{13}(\alpha_1(z),\{x,y,v\})A_2\big)(d)\\
&\quad+\big(\theta_{12}(\{x,y,z\},\alpha_1(u))A_2-\theta_{12}(\alpha_1(z),\{y,x,u\})A_2\\
&\quad-\theta_{12}(\alpha_1(x),\alpha_2(y))\theta_{12}(z,u)+\theta_{12}(\alpha_1(z),\alpha_2(u))\theta_{12}(x,y)\big)(e)
\end{aligned}
$$

由 $(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2)$ 为 Hom-约当三系，则等式(1.2)成立。因此，(1.10)式成立当且仅当(1.4)~(1.8)成立。因此，结论成立。

设 V,W 为线性空间，$\varphi:V\to W$ 为线性映射，定义 $\varphi^*:W^*\to V^*$，其中

$$\langle\varphi^*(f),v\rangle=\langle f,\varphi(v)\rangle(\forall f\in W^*,v\in V),$$

则 φ^* 为线性映射，称为 φ 的对偶映射。

设 $(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2)$ 为 Hom-约当三系，V 为线性空间，$\theta:J\times J\to End(V)$ 为线性映射，定义 $\theta^*:J\times J\to End(V^*)$，其中

$$\langle\theta^*(x,y)f,v\rangle=\langle f,\theta(x,y)v\rangle(\forall x,y\in J,v\in V,f\in V^*),$$

则 θ^* 为线性映射。

定理 1.2 设 $(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2)$ 为 Hom-约当三系，$(V,\theta_{12},\theta_{13},A_1,A_2)$ 是 $(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2)$ 的表示，则 $(V^*,\theta_{12}^*,\theta_{13}^*,A_1^*,A_2^*)$ 也是 $(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2)$ 的表示当且仅当满足下列条件

$$\begin{aligned}&\theta_{12}(z,y)\theta_{12}(\alpha_2(x),\alpha_1(u))-\theta_{13}(y,u)\theta_{13}(\alpha_1(z),\alpha_2(x))\\&-A_1\theta_{12}(\{z,u,x\},\alpha_2(y))+\theta_{12}(x,y)\theta_{12}(\alpha_1(z),\alpha_2(u))=0,\end{aligned}\tag{1.11}$$

$$\begin{aligned}&\theta_{13}(x,z)\theta_{12}(\alpha_2(y),\alpha_1(u))-\theta_{12}(u,x)\theta_{13}(\alpha_1(z),\alpha_2(y))\\&-A_2\theta_{13}(\alpha_1(x),\{z,u,y\})+\theta_{13}(x,y)\theta_{12}(\alpha_1(z),\alpha_2(u))=0,\end{aligned}\tag{1.12}$$

$$\begin{aligned}&\theta_{12}(x,y)\theta_{12}(\alpha_2(z),\alpha_1(u))-A_1\theta_{12}(\alpha_2(z),\{y,x,u\})\\&-\theta_{12}(z,u)\theta_{12}(\alpha_1(x),\alpha_2(y))+A_1\theta_{12}(\{x,y,z\},\alpha_2(u))=0,\end{aligned}\tag{1.13}$$

$$\begin{aligned}&A_1\theta_{13}(\{x,y,z\},\alpha_2(u))-\theta_{12}(y,x)\theta_{13}(\alpha_1(z),\alpha_2(u))\\&-\theta_{13}(z,u)\theta_{12}(\alpha_1(x),\alpha_2(y))+A_2\theta_{13}(\alpha_1(z),\{x,y,u\})=0,\end{aligned}\tag{1.14}$$

$$A_2\theta_{12}(\{x,y,z\},\alpha_1(u))-A_2\theta_{12}(\alpha_1(z),\{y,x,u\})$$
$$-\theta_{12}(z,u)\theta_{12}(\alpha_1(x),\alpha_2(y))+\theta_{12}(x,y)\theta_{12}(\alpha_1(z),\alpha_2(u))=0, \tag{1.15}$$

其中 $\forall x,y,z,u\in J$ 。

证明：$(V^*,\theta_{12}^*,\theta_{13}^*,A_1^*,A_2^*)$ 为 $(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2)$ 的表示当且仅当 $(V^*,\theta_{12}^*,\theta_{13}^*,A_1^*,A_2^*)$ 满足等式(1.3)~(1.8)。$\forall x,y,z,u\in J, f\in V^*, v\in V$ ，直接计算得

$$\langle[\theta_{13}^*(x,y)-\theta_{13}^*(y,x)]f,v\rangle=\langle f,[\theta_{13}(x,y)-\theta_{13}(y,x)]v\rangle,$$

$$\langle[\theta_{12}^*(\alpha_2(x),\alpha_1(u))\theta_{12}^*(z,y)-\theta_{13}^*(\alpha_1(z),\alpha_2(x))\theta_{13}^*(y,u)$$
$$-\theta_{12}^*(\{z,u,x\},\alpha_2(y))A_1^*+\theta_{12}^*(\alpha_1(z),\alpha_2(u))\theta_{12}^*(x,y)]f,v\rangle$$
$$=\langle f,[\theta_{12}(z,y)\theta_{12}(\alpha_2(x),\alpha_1(u))-\theta_{13}(y,u)\theta_{13}(\alpha_1(z),\alpha_2(x))$$
$$-A_1\theta_{12}(\{z,u,x\},\alpha_2(y))+\theta_{12}(x,y)\theta_{12}(\alpha_1(z),\alpha_2(u))]v\rangle,$$

$$\langle[\theta_{12}^*(\alpha_2(y),\alpha_1(u))\theta_{13}^*(x,z)-\theta_{13}^*(\alpha_1(z),\alpha_2(y))\theta_{12}^*(u,x)$$
$$-(\alpha_1(x),\{z,u,y\})A_2^*+\theta_{12}^*(\alpha_1(z),\alpha_2(u))\theta_{13}^*(x,y)]f,v\rangle$$
$$=\langle f,[\theta_{13}(x,z)\theta_{12}(\alpha_2(y),\alpha_1(u))-\theta_{12}(u,x)\theta_{13}(\alpha_1(z),\alpha_2(y))$$
$$-A_2(\alpha_1(x),\{z,u,y\})+\theta_{13}(x,y)\theta_{12}(\alpha_1(z),\alpha_2(u))]v\rangle,$$

$$\langle[\theta_{12}^*(\alpha_2(z),\alpha_1(u))\theta_{12}^*(x,y)-\theta_{12}^*(\alpha_2(z),\{y,x,u\})A_1^*$$
$$-\theta_{12}^*(\alpha_1(x),\alpha_2(y))\theta_{12}^*(z,u)+\theta_{12}^*(\{x,y,z\},\alpha_2(u))A_1^*]f,v\rangle$$
$$=\langle f,[\theta_{12}(x,y)\theta_{12}(\alpha_2(z),\alpha_1(u))-A_1\theta_{12}(\alpha_2(z),\{y,x,u\})$$
$$-\theta_{12}(z,u)\theta_{12}(\alpha_1(x),\alpha_2(y))+A_1\theta_{12}(\{x,y,z\},\alpha_2(u))]v\rangle,$$

$$\langle[\theta_{13}^*(\{x,y,z\},\alpha_2(u))A_1^*-\theta_{13}^*(\alpha_1(z),\alpha_2(u))\theta_{12}^*(y,x)$$
$$-\theta_{12}^*(\alpha_1(x),\alpha_2(y))\theta_{13}^*(z,u)+\theta_{13}^*(\alpha_1(z),\{x,y,u\})A_2^*]f,v\rangle$$
$$=\langle f,[A_1\theta_{13}(\{x,y,z\},\alpha_2(u))-\theta_{12}(y,x)\theta_{13}(\alpha_1(z),\alpha_2(u))$$
$$-\theta_{13}(z,u)\theta_{12}(\alpha_1(x),\alpha_2(y))+A_2\theta_{13}(\alpha_1(z),\{x,y,u\})]v\rangle,$$

$$\langle[\theta_{12}^*(\{x,y,z\},\alpha_1(u))A_2^*-\theta_{12}^*(\alpha_1(z),\{y,x,u\})A_2^*$$
$$-\theta_{12}^*(\alpha_1(x),\alpha_2(y))\theta_{12}^*(z,u)+\theta_{12}^*(\alpha_1(z),\alpha_2(u))\theta_{12}^*(x,y)]f,v\rangle$$
$$=\langle f,[A_2\theta_{12}(\{x,y,z\},\alpha_1(u))-A_2\theta_{12}(\alpha_1(z),\{y,x,u\})$$
$$-\theta_{12}(z,u)\theta_{12}(\alpha_1(x),\alpha_2(y))+\theta_{12}(x,y)\theta_{12}(\alpha_1(z),\alpha_2(u))]v\rangle,$$

由 $(V,\theta_{12},\theta_{13},A_1,A_2)$ 是 $(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2)$ 的表示可知 $\theta_{13}(x,y)=\theta_{13}(y,x)$ ，则 $(V^*,\theta_{12}^*,\theta_{13}^*,A_1^*,A_2^*)$ 上等式(1.3)~(1.8)成立等价于等式(1.11)~(1.15)成立。因此结论成立。

推论 1.1 设 $(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2)$ 为 Hom-约当三系，则 $(J^*,L^*,M^*,\alpha_1^*,\alpha_2^*)$ 为 $(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2)$ 的表示当且仅当 J 上的运算 $\{\cdot,\cdot,\cdot\}$ 满足下列条件

$$\begin{aligned}&\{z,y,\{\alpha_2(x),\alpha_1(u),v\}\}-\{y,\{\alpha_1(z),v,\alpha_2(x)\},u\}\\&-\alpha_1(\{\{z,u,x\},\alpha_2(y),v\})+\{x,y,\{\alpha_1(z),\alpha_2(u),v\}\}=0,\end{aligned}\tag{1.16}$$

$$\begin{aligned}&\{x,\{\alpha_2(y),\alpha_1(u),v\},z\}-\{u,x,\{\alpha_1(z),v,\alpha_2(y)\}\}\\&-\alpha_2(\{\alpha_1(x),v,\{z,u,y\}\})+\{x,\{\alpha_1(z),\alpha_2(u),v\},y\}=0,\end{aligned}\tag{1.17}$$

$$\begin{aligned}&\{x,y,\{\alpha_2(z),\alpha_1(u),v\}\}-\alpha_1(\{\alpha_2(z),\{y,x,u\},v\})\\&-\{z,u,\{\alpha_1(x),\alpha_2(y),v\}\}+\alpha_1(\{x,y,z\},\alpha_2(u),v)=0,\end{aligned}\tag{1.18}$$

$$\begin{aligned}&\alpha_1(\{x,y,z\},v,\alpha_2(u))-\{y,x,\{\alpha_1(z),v,\alpha_2(u)\}\}\\&-\{z,\{\alpha_1(x),\alpha_2(y),v\},u\}+\alpha_2(\{\alpha_1(z),v,\{x,y,u\}\})=0,\end{aligned}\tag{1.19}$$

$$\begin{aligned}&\alpha_2(\{x,y,z\},\alpha_1(u),v)-\alpha_2(\{\alpha_1(z),\{y,x,u\},v\})\\&-\{z,u,\{\alpha_1(x),\alpha_2(y),v\}\}+\{x,y,\{\alpha_1(z),\alpha_2(u),v\}\}=0,\end{aligned}\tag{1.20}$$

其中 $\forall x,y,z,u,v\in J$ 。

证明：利用定理 1.2，取 $\theta_{12}=L,\theta_{13}=M$ ，通过计算可直接得出。

设 $(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2)$ 为 Hom-约当三系，V,W 为线性空间，$\theta':J\times J\to End(V)$，$\theta'':J\times J\to End(W)$ 为双线性映射，$A'\in End(V)$，$A''\in End(W)$，定义 $\theta'\otimes\theta'':J\times J\to End(V\otimes W)$，$A'\otimes A''\in End(V\otimes W)$，其中

$$\theta'\otimes\theta''(x,y)(v\otimes w)=\theta'(x,y)v\otimes w+v\otimes\theta''(x,y)w,$$

$$A'\otimes A''(v\otimes w)=A'(v)\otimes A''(w),$$

其中 $\forall x,y\in J,v\in V,w\in W$ 。

定理 1.3 设 $(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2)$ 为 Hom-约当三系，$(V,\theta'_{12},\theta'_{13},A'_1,A'_2),(W,\theta''_{12},\theta''_{13},A''_1,A''_2)$ 均为 $(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2)$ 的表示，则 $(V\otimes W,\theta'_{12}\otimes\theta''_{12},\theta'_{13}\otimes\theta''_{13},A'_1\otimes A''_1,A'_2\otimes A''_2)$ 是 $(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2)$ 的表示当且仅当 $(V,\theta'_{12},\theta'_{13},A'_1,A'_2),(W,\theta''_{12},\theta''_{13},A''_1,A''_2)$ 满足下列条件

$$\begin{aligned}&\theta'_{12}(\{z,u,x\},\alpha_2(y))A'_1(v)\otimes w-\theta'_{12}(\{z,u,x\},\alpha_2(y))A'_1(v)\otimes A''_1(w)\\&+v\otimes\theta''_{12}(\{z,u,x\},\alpha_2(y))A''_1(w)-A'_1(v)\otimes\theta''_{12}(\{z,u,x\},\alpha_2(y))A''_1(w)\\&+\theta'_{12}(\alpha_2(x),\alpha_1(u))(v)\otimes\theta''_{12}(z,y)(w)+\theta'_{12}(z,y)(v)\otimes\theta''_{12}(\alpha_2(x),\alpha_1(u))(w)\\&-\theta'_{13}(y,u)(v)\otimes\theta''_{13}(\alpha_1(z),\alpha_2(x))(w)-\theta'_{13}(\alpha_1(z),\alpha_2(x))(v)\otimes\theta''_{13}(y,u)(w)\\&+\theta'_{12}(x,y)(v)\otimes\theta''_{12}(\alpha_1(z),\alpha_2(u))(w)+\theta'_{12}(\alpha_1(z),\alpha_2(u))(v)\otimes\theta''_{13}(x,y)(w)=0,\end{aligned}\tag{1.21}$$

$$\begin{aligned}&\theta'_{13}(\alpha_1(x),\{z,u,y\})A'_2(v)\otimes w-\theta'_{13}(\alpha_1(x),\{z,u,y\})A'_2(v)\otimes A''_2(w)\\&+v\otimes\theta''_{13}(\alpha_1(x),\{z,u,y\})A''_2(w)-A'_2(v)\otimes\theta''_{13}(\alpha_1(x),\{z,u,y\})A''_2(w)\\&+v\otimes\theta''_{13}(\alpha_1(x),\{z,u,y\})A''_2(w)-A'_2(v)\otimes\theta''_{13}(\alpha_1(x),\{z,u,y\})A''_2(w)\\&+\theta'_{13}(x,z)(v)\otimes\theta''_{12}(\alpha_2(y),\alpha_1(u))(w)+\theta'_{12}(\alpha_2(y),\alpha_1(u))(v)\otimes\theta''_{13}(x,z)(w)\\&+\theta'_{13}(x,y)(v)\otimes\theta''_{12}(\alpha_1(z),\alpha_2(u))(w)+\theta'_{12}(\alpha_1(z),\alpha_2(u))(v)\otimes\theta''_{13}(x,y)(w)=0,\end{aligned}\tag{1.22}$$

$$\begin{aligned}
&\theta_{12}'\left(\alpha_2\left(z\right),\alpha_1\left(u\right)\right)\theta_{12}'\left(x,y\right)\left(v\right)\otimes w+\theta_{12}'\left(x,y\right)\left(v\right)\otimes\theta_{12}''\left(\alpha_2\left(z\right),\alpha_1\left(u\right)\right)\left(w\right)\\
&+\theta_{12}'\left(\alpha_2\left(z\right),\alpha_1\left(u\right)\right)\left(v\right)\otimes\theta_{12}''\left(x,y\right)\left(w\right)+v\otimes\theta_{12}''\left(\alpha_2\left(z\right),\alpha_1\left(u\right)\right)\theta_{12}''\left(x,y\right)\left(w\right)\\
&-\theta_{12}'\left(\alpha_2\left(z\right),\left\{y,x,u\right\}A_1'\left(v\right)\otimes A_1''\left(w\right)-A_1'\left(v\right)\otimes\theta_{12}''\left(\alpha_2\left(z\right),\left\{y,x,u\right\}\right)A_1''\left(w\right)\\
&-\theta_{12}'\left(\alpha_1\left(x\right),\alpha_2\left(y\right)\right)\theta_{12}'\left(z,u\right)\left(v\right)\otimes w-\theta_{12}'\left(z,u\right)\left(v\right)\otimes\theta_{12}''\left(\alpha_1\left(x\right),\alpha_2\left(y\right)\right)w\\
&-\theta_{12}'\left(\alpha_1\left(x\right),\alpha_2\left(y\right)\right)\left(v\right)\otimes\theta_{12}''\left(z,u\right)\left(w\right)-v\otimes\theta_{12}''\left(\alpha_1\left(x\right),\alpha_2\left(y\right)\right)\theta_{12}''\left(z,u\right)w\\
&+\theta_{12}'\left(\left\{x,y,z\right\},\alpha_2\left(u\right)\right)A_1'\left(v\right)\otimes A_1''\left(w\right)+A_1'\left(v\right)\otimes\theta_{12}''\left(\left\{x,y,z\right\},\alpha_2\left(u\right)\right)A_1''\left(w\right)=0,
\end{aligned}\tag{1.23}$$

$$\begin{aligned}
&\theta_{13}'\left(\left\{x,y,z\right\},\alpha_2\left(u\right)\right)A_1'\left(v\right)\otimes A_1''\left(w\right)+A_1'\left(v\right)\otimes\theta_{13}''\left(\left\{x,y,z\right\},\alpha_2\left(u\right)\right)A_1''\left(w\right)\\
&-\theta_{13}'\left(\alpha_1\left(z\right),\alpha_2\left(u\right)\right)\theta_{12}'\left(y,x\right)\left(v\right)\otimes\left(w\right)-\theta_{12}'\left(y,x\right)\left(v\right)\otimes\theta_{13}''\left(\alpha_1\left(z\right),\alpha_2\left(u\right)\right)\left(w\right)\\
&-\theta_{13}'\left(\alpha_1\left(z\right),\alpha_2\left(u\right)\right)\left(v\right)\otimes\theta_{12}''\left(y,x\right)\left(w\right)-v\otimes\theta_{13}''\left(\alpha_1\left(z\right),\alpha_2\left(u\right)\right)\theta_{12}''\left(y,x\right)\left(w\right)\\
&-\theta_{12}'\left(\alpha_1\left(x\right),\alpha_2\left(y\right)\right)\theta_{13}'\left(z,u\right)\left(v\right)\otimes w-\theta_{13}'\left(z,u\right)\left(v\right)\otimes\theta_{12}''\left(\alpha_1\left(x\right),\alpha_2\left(y\right)\right)\left(w\right)\\
&-\theta_{12}'\left(\alpha_1\left(x\right),\alpha_2\left(y\right)\right)\left(v\right)\otimes\theta_{13}''\left(z,u\right)\left(w\right)-v\otimes\theta_{12}''\left(\alpha_1\left(x\right),\alpha_2\left(y\right)\right)\theta_{13}''\left(z,u\right)\left(w\right)\\
&+\theta_{13}'\left(\alpha_1\left(z\right),\left\{x,y,u\right\}\right)A_1'\left(v\right)\otimes A_2''\left(w\right)+A_2'(v)\otimes\theta_{13}''\left(\alpha_1\left(z\right),\left\{x,y,u\right\}\right)A_2''\left(w\right)=0,
\end{aligned}\tag{1.24}$$

$$\begin{aligned}
&\theta_{12}'\left(\left\{x,y,z\right\},\alpha_1\left(u\right)\right)A_2'\left(v\right)\otimes A_2''\left(w\right)+A_2'\left(v\right)\otimes\theta_{12}''\left(\left\{x,y,z\right\},\alpha_1\left(u\right)\right)A_2''\left(w\right)\\
&-\theta_{12}'\left(\alpha_1\left(z\right),\left\{y,x,u\right\}\right)A_2'\left(v\right)\otimes A_2''\left(w\right)-A_2'\left(v\right)\otimes\theta_{12}''\left(\alpha_1\left(z\right),\left\{y,x,u\right\}\right)A_2''\left(w\right)\\
&-\theta_{12}'\left(\alpha_1\left(x\right),\alpha_2\left(y\right)\right)\theta_{12}'\left(z,u\right)\left(v\right)\otimes w-\theta_{12}'(z,u)(v)\otimes\theta_{12}''\left(\alpha_1\left(x\right),\alpha_2\left(y\right)\right)\left(w\right)\\
&-\theta_{12}'\left(\alpha_1\left(x\right),\alpha_2\left(y\right)\right)\left(v\right)\otimes\theta_{12}''\left(z,u\right)w-v\otimes\theta_{12}''\left(\alpha_1\left(x\right),\alpha_2\left(y\right)\right)\theta_{12}''\left(z,u\right)\left(w\right)\\
&+\theta_{12}'\left(\alpha_1\left(z\right),\alpha_2\left(u\right)\right)\theta_{12}'\left(x,y\right)\left(v\right)\otimes w+\theta_{12}'\left(x,y\right)\left(v\right)\otimes\theta_{12}''\left(\alpha_1\left(z\right),\alpha_2\left(u\right)\right)\left(w\right)\\
&+\theta_{12}'\left(\alpha_1\left(z\right),\alpha_2\left(u\right)\right)\left(v\right)\otimes\theta_{12}''\left(x,y\right)\left(w\right)+v\otimes\theta_{12}''\left(\alpha_1\left(z\right),\alpha_2\left(u\right)\right)\theta_{12}''\left(x,y\right)\left(w\right)=0,
\end{aligned}\tag{1.25}$$

其中 $\forall x,y,z,u\in J,v\in V,w\in W$ 。

证明：由 $\theta_{12}'\otimes\theta_{12}'',\theta_{13}'\otimes\theta_{13}'',A_1'\otimes A_1'',A_2'\otimes A_2''$ 的定义知为线性映射。$\left(V\otimes W,\theta_{12}'\otimes\theta_{12}'',\theta_{13}'\otimes\theta_{13}'',A_1'\otimes A_1'',A_2'\otimes A_2''\right)$ 是 $\left(J,\left\{\cdot,\cdot,\cdot\right\},\alpha_1,\alpha_2\right)$ 的表示当且仅当对于 $\left(V\otimes W,\theta_{12}'\otimes\theta_{12}'',\theta_{13}'\otimes\theta_{13}'',A_1'\otimes A_1'',A_2'\otimes A_2''\right)$ 有(1.3)~(1.8)成立。由于

$$\begin{aligned}
&\theta_{13}'\otimes\theta_{13}''\left(x,y\right)\left(v\otimes w\right)-\theta_{13}'\otimes\theta_{13}''\left(y,x\right)\left(v\otimes w\right)\\
&=\theta_{13}'\left(x,y\right)v\otimes w-\theta_{13}'\left(y,x\right)v\otimes w+v\otimes\theta_{13}''\left(x,y\right)w-v\otimes\theta_{13}''\left(y,x\right)w,
\end{aligned}$$

则 $\theta_{13}'\otimes\theta_{13}''$ 满足等式(1.3)当且仅当 $\theta_{13}',\theta_{13}''$ 满足等式(1.3)。在 $V\otimes W$ 上等式(1.4)左边

$$\begin{aligned}
&\theta_{12}'\otimes\theta_{12}''\left(\alpha_2\left(x\right),\alpha_1\left(u\right)\right)\theta_{12}'\otimes\theta_{12}''\left(z,y\right)\left(v\otimes w\right)-\theta_{13}'\otimes\theta_{13}''\left(\alpha_1\left(z\right),\alpha_2\left(x\right)\right)\theta_{13}'\otimes\theta_{13}''\left(y,u\right)\left(v\otimes w\right)\\
&-\theta_{12}'\otimes\theta_{12}''\left(\left\{z,u,x\right\},\alpha_2\left(y\right)\right)A_1'\otimes A_1''\left(v\otimes w\right)+\theta_{12}'\otimes\theta_{12}''\left(\alpha_1\left(z\right),\alpha_2\left(u\right)\right)\theta_{12}'\otimes\theta_{12}''\left(x,y\right)\left(v\otimes w\right)\\
&=\theta_{12}'\left(\left\{z,u,x\right\},\alpha_2\left(y\right)\right)A_1'\left(v\right)\otimes w-\theta_{12}'\left(\left\{z,u,x\right\},\alpha_2\left(y\right)\right)A_1'\left(v\right)\otimes A_1''\left(w\right)\\
&\quad+v\otimes\theta_{12}''\left(\left\{z,u,x\right\},\alpha_2\left(y\right)\right)A_1''\left(w\right)-A_1'\left(v\right)\otimes\theta_{12}''\left(\left\{z,u,x\right\},\alpha_2\left(y\right)\right)A_1''\left(w\right)\\
&\quad+\theta_{12}'\left(\alpha_2\left(x\right),\alpha_1\left(u\right)\right)\left(v\right)\otimes\theta_{12}''\left(z,y\right)\left(w\right)+\theta_{12}'\left(z,y\right)\left(v\right)\otimes\theta_{12}''\left(\alpha_2\left(x\right),\alpha_1\left(u\right)\right)\left(w\right)\\
&\quad-\theta_{13}'\left(y,u\right)\left(v\right)\otimes\theta_{13}''\left(\alpha_1\left(z\right),\alpha_2\left(x\right)\right)\left(w\right)-\theta_{13}'\left(\alpha_1\left(z\right),\alpha_2\left(x\right)\right)\left(v\right)\otimes\theta_{13}''\left(y,u\right)\left(w\right)\\
&\quad+\theta_{12}'\left(x,y\right)\left(v\right)\otimes\theta_{12}''\left(\alpha_1\left(z\right),\alpha_2\left(u\right)\right)\left(w\right)+\theta_{12}'\left(\alpha_1\left(z\right),\alpha_2\left(u\right)\right)\left(v\right)\otimes\theta_{12}''\left(x,y\right)\left(w\right),
\end{aligned}$$

因此 $\left(V\otimes W,\theta_{12}'\otimes\theta_{12}'',\theta_{13}'\otimes\theta_{13}'',A_1'\otimes A_1'',A_2'\otimes A_2''\right)$ 满足等式(1.4)当且仅当等式(1.21)式成立。同理，$\left(V\otimes W,\theta_{12}'\otimes\theta_{12}'',\theta_{13}'\otimes\theta_{13}'',A_1'\otimes A_1'',A_2'\otimes A_2''\right)$ 满足等式(1.5)当且仅当等式(1.22)成立，满足等式(1.6)当且仅当

等式(1.23)成立，满足等式(1.7)当且仅当等式(1.24)成立，满足等式(1.8)当且仅当等式(1.25)成立，因此，结论成立。

3. Hom-约当三系的扩张

定理 2.1 设$\left(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2\right)$为 Hom-约当三系，$\varphi: J\times J\times J\to F$ 是三线性函数，在 $J\oplus V=\left\{x_1+\lambda_1 c \middle| x_1\in J,\lambda_1\in F\right\}$ 上定义

$$\left\{x_1+\lambda_1 c,x_2+\lambda_2 c,x_3+\lambda_3 c\right\}=\left\{x_1,x_2,x_3\right\}+\varphi\left(x_1,x_2,x_3\right)c,$$

$$\left(\alpha_1+id\right)\left(x_1+\lambda_1 c\right)=\alpha_1\left(x_1\right)+\lambda_1 c,$$

$$\left(\alpha_2+id\right)\left(x_1+\lambda_1 c\right)=\alpha_2\left(x_1\right)+\lambda_1 c,$$

则$\left(J\oplus V,\{\cdot,\cdot,\cdot\},\alpha_1+id,\alpha_2+id\right)$为 Hom-约当三系的充分必要条件是$\varphi$满足

$$\varphi\left(x_1,x_2,x_3\right)=\varphi\left(x_3,x_2,x_1\right), \tag{2.1}$$

$$\begin{aligned}&\varphi\left(\alpha_1\left(x_1\right),\alpha_2\left(x_2\right),\left\{x_3,x_4,x_5\right\}\right)-\varphi\left(\left\{x_1,x_2,x_3\right\},\alpha_1\left(x_4\right),\alpha_2\left(x_5\right)\right)\\&+\varphi\left(\alpha_1\left(x_3\right),\left\{x_2,x_1,x_4\right\},\alpha_2\left(x_5\right)\right)-\varphi\left(\alpha_1\left(x_3\right),\alpha_2\left(x_4\right),\left\{x_1,x_2,x_5\right\}\right)=0,\end{aligned} \tag{2.2}$$

其中$\forall x_1,x_2,x_3,x_4,x_5\in J,\lambda_1,\lambda_2,\lambda_3\in F$ 。

证明：$\forall x_1,x_2,x_3,x_4,x_5\in J,\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5\in F$ ，则$\left(J\oplus V,\{\cdot,\cdot,\cdot\},\alpha_1+id,\alpha_2+id\right)$为 Hom-约当三系当且仅当下列等式成立

$$\left\{x_1+\lambda_1 c,x_2+\lambda_2 c,x_3+\lambda_3 c\right\}=\left\{x_3+\lambda_3 c,x_2+\lambda_2 c,x_1+\lambda_1 c\right\}, \tag{2.3}$$

$$\begin{aligned}&\left\{\left(\alpha_1+id\right)\left(x_1+\lambda_1 c\right),\left(\alpha_2+id\right)\left(x_2+\lambda_2 c\right),\left\{x_3+\lambda_3 c,x_4+\lambda_4 c,x_5+\lambda_5 c\right\}\right\}\\&-\left\{\left(\alpha_1+id\right)\left(x_3+\lambda_3 c\right),\left(\alpha_2+id\right)\left(x_4+\lambda_4 c\right),\left\{x_1+\lambda_1 c,x_2+\lambda_2 c,x_5+\lambda_5 c\right\}\right\}\\&-\left\{\left\{x_1+\lambda_1 c,x_2+\lambda_2 c,x_3+\lambda_3 c\right\},\left(\alpha_1+id\right)\left(x_4+\lambda_4 c\right),\left(\alpha_2+id\right)\left(x_5+\lambda_5 c\right)\right\}\\&+\left\{\left(\alpha_1+id\right)\left(x_3+\lambda_3 c\right),\left\{x_2+\lambda_2 c,x_1+\lambda_1 c,x_4+\lambda_4 c\right\},\left(\alpha_2+id\right)\left(x_5+\lambda_5 c\right)\right\}=0,\end{aligned} \tag{2.4}$$

由$\left(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2\right)$为 Hom-约当三系知$\left\{x_1,x_2,x_3\right\}=\left\{x_3,x_2,x_1\right\}$，则等式(2.3)成立当且仅当$\varphi$满足等式(2.1)。由$\left(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2\right)$为 Hom-约当三系知等式(1.2)成立，则等式(2.4)成立等价于等式(2.2)成立。因此，结论成立。

定理 2.2 设$\left(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2\right)$为 Hom-约当三系，且满足(1.11)~(1.15)，$\omega: J\times J\times J\to J^*$ 为三线性映射，在 $J\oplus J^*$ 上定义

$$\left\{x+a^*,y+b^*,z+c^*\right\}=\left\{x,y,z\right\}+\omega\left(x,y,z\right)+L^*\left(x,y\right)c^*+L^*\left(z,y\right)a^*+M^*\left(x,z\right)b^*,$$

$$\left(\alpha_1+\alpha_1^*\right)\left(x+a^*\right)=\alpha_1\left(x\right)+\alpha_1^*\left(a^*\right),$$

$$\left(\alpha_2+\alpha_2^*\right)\left(x+a^*\right)=\alpha_2\left(x\right)+\alpha_2^*\left(a^*\right),$$

其中$\forall x,y,z\in J,a^*,b^*,c^*\in J^*$，则$\left(J\oplus J^*,\{\cdot,\cdot,\cdot\},\alpha_1+\alpha_1^*,\alpha_2+\alpha_2^*\right)$为 Hom-约当三系当且仅当$\omega$满足

$$\omega\left(x,y,z\right)=\omega\left(z,y,x\right), \tag{2.5}$$

$$\begin{aligned}&\omega\left(\alpha_1(x),\alpha_2(y),\{z,u,v\}\right)-\omega\left(\alpha_1(z),\alpha_2(u),\{x,y,v\}\right)\\&+\omega\left(\alpha_1(z),\{y,x,u\},\alpha_2(v)\right)-\omega\left(\{x,y,z\},\alpha_1(u),\alpha_2(v)\right)\\&+L^*\left(\alpha_1(x),\alpha_2(y)\right)\omega(z,u,v)-L^*\left(\alpha_1(z),\alpha_2(u)\right)\omega(x,y,v)\\&-L^*\left(\alpha_2(v),\alpha_1(u)\right)\omega(x,y,z)+M^*\left(\alpha_1(z),\alpha_2(v)\right)\omega(y,x,u)=0,\end{aligned}\tag{2.6}$$

其中 $\forall x,y,z,u,v\in J$，此时称 $J\oplus J^*$ 为 J 的 T^*-扩张。

证明：显然，在 $J\oplus J^*$ 上定义的新的运算对三个变量都是线性的。$\forall x,y,z,u,v\in J$，$a^*,b^*,c^*,d^*,e^*\in J^*$，$\left(J\oplus J^*,\{\cdot,\cdot,\cdot\},\alpha_1+\alpha_1^*,\alpha_2+\alpha_2^*\right)$ 为 Hom-约当三系当且仅当在 $J\oplus J^*$ 上等式(1.1)、(1.2)成立。

由于

$$\begin{aligned}&\left\{x+a^*,y+b^*,z+c^*\right\}-\left\{z+c^*,y+b^*,x+a^*\right\}\\&=\{x,y,z\}-\{z,y,x\}+\omega(x,y,z)-\omega(z,y,x)+L^*(x,y)c^*\\&\quad-L^*(x,y)c^*+L^*(z,y)a^*-L^*(z,y)a^*+M^*(x,z)b^*-M^*(z,x)b^*,\end{aligned}$$

由于 $M^*(x,z)=M^*(z,x)$，因此在 $J\oplus J^*$ 上等式(1.1)成立等价于等式(2.5)成立。

在 $J\oplus J^*$ 上等式(1.2)左边直接计算得

$$\begin{aligned}&\left\{\left(\alpha_1+\alpha_1^*\right)\left(x+a^*\right),\left(\alpha_2+\alpha_2^*\right)\left(y+b^*\right),\left\{z+c^*,u+d^*,v+e^*\right\}\right\}\\&-\left\{\left(\alpha_1+\alpha_1^*\right)\left(z+c^*\right),\left(\alpha_2+\alpha_2^*\right)\left(u+d^*\right),\left\{x+a^*,y+b^*,v+e^*\right\}\right\}\\&-\left\{\left\{x+a^*,y+b^*,z+c^*\right\},\left(\alpha_1+\alpha_1^*\right)\left(u+d^*\right),\left(\alpha_2+\alpha_2^*\right)\left(v+e^*\right)\right\}\\&+\left\{\left(\alpha_1+\alpha_1^*\right)\left(z+c^*\right),\left\{y+b^*,x+a^*,u+d^*\right\},\left(\alpha_2+\alpha_2^*\right)\left(v+e^*\right)\right\}\\&=\left\{\alpha_1(x),\alpha_2(y),\{z,u,v\}\right\}-\left\{\alpha_1(z),\alpha_2(u),\{x,y,v\}\right\}-\left\{\{x,y,z\},\alpha_1(u),\alpha_2(v)\right\}\\&+\left\{\alpha_1(z),\{y,x,u\},\alpha_2(v)\right\}+\omega\left(\alpha_1(x),\alpha_2(y),\{z,u,v\}\right)-\omega\left(\alpha_1(z),\alpha_2(u),\{x,y,v\}\right)\\&-\omega\left(\{x,y,z\},\alpha_1(u),\alpha_2(v)\right)+\omega\left(\alpha_1(z),\{y,x,u\},\alpha_2(v)\right)+L^*\left(\alpha_1(x),\alpha_2(y)\right)\omega(z,u,v)\\&-L^*\left(\alpha_1(z),\alpha_2(u)\right)\omega(x,y,v)-L^*\left(\alpha_2(v),\alpha_1(u)\right)\omega(x,y,z)+M^*\left(\alpha_1(z),\alpha_2(v)\right)\omega(y,x,u)\\&+\Big(L^*\left(\{z,u,v\},\alpha_2(y)\right)\alpha_1^*-L^*\left(\alpha_1(z),\alpha_2(u)\right)L^*(v,y)\\&-L^*\left(\alpha_2(v),\alpha_1(u)\right)L^*(z,y)+M^*\left(\alpha_1(z),\alpha_2(v)\right)M^*(y,u)\Big)\left(a^*\right)\\&+\Big(M^*\left(\alpha_1(x),\{z,u,v\}\right)\alpha_2^*-L^*\left(\alpha_1(z),\alpha_2(u)\right)M^*(x,v)\\&-L^*\left(\alpha_2(v),\alpha_1(u)\right)M^*(x,z)+M^*\left(\alpha_1(z),\alpha_2(v)\right)L^*(u,x)\Big)\left(b^*\right)\\&+\Big(L^*\left(\alpha_1(x),\alpha_2(y)\right)L^*(v,u)-L^*(\{x,y,v\},\alpha_2(u))\alpha_1^*\\&-L^*\left(\alpha_2(v),\alpha_1(u)\right)L^*(x,y)+L^*\left(\alpha_2(v),\{y,x,u\}\right)\alpha_1^*\Big)\left(c^*\right)\\&+\Big(L^*\left(\alpha_1(x),\alpha_2(y)\right)M^*(z,v)-M^*\left(\alpha_1(z),\{x,y,v\}\right)\alpha_2^*\\&-M^*\left(\{x,y,z\},\alpha_2(v)\right)\alpha_1^*+M^*\left(\alpha_1(z),\alpha_2(v)\right)L^*(y,x)\Big)\left(d^*\right)\\&+\Big(L^*\left(\alpha_1(x),\alpha_2(y)\right)L^*(z,u)-L^*\left(\alpha_1(z),\alpha_2(u)\right)L^*(x,y)\\&-L^*\left(\{x,y,z\},\alpha_1(u)\right)\alpha_2^*+L^*\left(\alpha_1(z),\{y,x,u\}\right)\alpha_2^*\Big)\left(e^*\right)\end{aligned}$$

由 $\left(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2\right)$ 为 Hom-约当三系，等式(1.2)成立。又由 $\left(J^*,L^*,M^*,\alpha_1^*,\alpha_2^*\right)$ 为 $\left(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2\right)$ 的表

示知对于$\left(J^*,L^*,M^*,\alpha_1^*,\alpha_2^*\right)$有(1.4)~(1.8)成立，则在$J\oplus J^*$上等式(1.2)成立等价于等式(2.6)成立。因此结论成立。

定义 2.1 设$\left(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2\right)$为 Hom-约当三系，$f$为$J$上的双线性函数，若满足

$$f\left(\{x,y,z\},u\right)=f\left(z,\{x,y,u\}\right)\left(\forall x,y,z,u\in J\right),$$

$$f\left(\{x,y,z\},u\right)=f\left(y,\{x,u,z\}\right)\left(\forall x,y,z,u\in J\right),$$

则称f为J上的不变函数。

定义 2.2 设$\left(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2\right)$为 Hom-约当三系，若$J$上存在非退化的、不变的、对称的双线性函数$f$，则称$\left(J,f\right)$为度量 Hom-约当三系。

定理 2.3 设$\left(J,\{\cdot,\cdot,\cdot\},\alpha_1,\alpha_2\right)$为 Hom-约当三系且满足

$$\{x,y,z\}=\{x,z,y\}\left(\forall x,y,z\in J\right), \tag{2.7}$$

三线性映射$\omega:J\times J\times J\rightarrow J^*$满足(2.5)、(2.6)并且

$$\omega(z,u,x)y=\omega(x,y,u)z=\omega(x,y,z)u\left(\forall x,y,z,u\in J\right), \tag{2.8}$$

在$J\oplus J^*$上定义

$$q\left(x+a^*,y+b^*\right)=a^*(y)+b^*(x)\left(\forall x,y\in J,a^*,b^*\in J^*\right),$$

则$\left(J\oplus J^*,q\right)$是度量 Hom-约当三系。

证明：由ω满足等式(2.5)、(2.6)知$\left(J\oplus J^*,\{\cdot,\cdot,\cdot\},\alpha_1+\alpha_1^*,\alpha_2+\alpha_2^*\right)$为 Hom-约当三系。

对于$y+b^*$，若对于$\forall x+a^*\in J\oplus J^*$满足$q\left(x+a^*,y+b^*\right)=a^*(y)+b^*(x)=0$，取$a^*=0$，则$q\left(x,y+b^*\right)=b^*(x)=0$，因此$b^*=0$。取$x=0$，则$q\left(a^*,y+b^*\right)=a^*(y)=0$，因此$y=0$，从而$y+b^*=0$，$q$是非退化的。

因为$q\left(x+a^*,y+b^*\right)=a^*(y)+b^*(x)=q\left(y+b^*,x+a^*\right)$，所以$q$是对称的。

$\forall x,y,z,u\in J,a^*,b^*,c^*,d^*\in J^*$，直接计算得

$$\begin{aligned}&q\left(\left\{x+a^*,y+b^*,z+c^*\right\},u+d^*\right)\\&=q\left(\{x,y,z\}+\omega(x,y,z)+L^*(x,y)c^*+L^*(z,y)a^*+M^*(x,z)b^*,u+d^*\right)\\&=\omega(x,y,z)u+L^*(x,y)c^*(u)+L^*(z,y)a^*(u)+M^*(x,z)b^*(u)+d^*\left(\{x,y,z\}\right),\end{aligned}$$

$$\begin{aligned}&q\left(z+c^*,\left\{x+a^*,y+b^*,u+d^*\right\}\right)\\&=q\left(z+c^*,\{x,y,u\}+\omega(x,y,u)+L^*(x,y)d^*+L^*(u,y)a^*+M^*(x,u)b^*\right)\\&=\omega(x,y,u)z+L^*(x,y)d^*(z)+L^*(u,y)a^*(z)+M^*(x,u)b^*(z)+c^*\left(\{x,y,u\}\right),\end{aligned}$$

由(2.7)知

$$M^*(x,u)b^*(z)=\left\langle b^*,\{x,z,u\}\right\rangle=M^*(x,z)b^*(u),$$

此外，

$$L^*(x,y)a^*(u)=L^*(u,y)a^*(z),L^*(x,y)c^*(u)=c^*\left(\{x,y,u\}\right),L^*(x,y)d^*(z)=d^*\left(\{x,y,z\}\right),$$

因此，

$$q\left(\left\{x+a^*,y+b^*,z+c^*\right\},u+d^*\right)=q\left(z+c^*,\left\{x+a^*,y+b^*,u+d^*\right\}\right).$$

$$\begin{aligned}&q\left(\left\{x+a^*,y+b^*,z+c^*\right\},u+d^*\right)\\&=q\left(\{x,y,z\}+\omega(x,y,z)+L^*(x,y)c^*+L^*(z,y)a^*+M^*(x,z)b^*,u+d^*\right)\\&=\omega(x,y,z)u+L^*(x,y)c^*(u)+L^*(z,y)a^*(u)+M^*(x,z)b^*(u)+d^*\left(\{x,y,z\}\right),\end{aligned}$$

$$\begin{aligned}&q\left(y+b^*,\left\{z+c^*,\ u+d^*,x+a^*\right\}\right)\\&=q\left(y+b^*,\{z,u,x\}+\omega(z,u,x)+L^*(z,u)a^*+L^*(x,u)c^*+M^*(z,x)d^*\right)\\&=\omega(z,u,x)(y)+L^*(z,u)a^*(y)+L^*(x,u)c^*(y)+M^*(z,x)d^*(y)+b^*\left(\{z,u,x\}\right),\end{aligned}$$

由(2.7)知

$$L^*(z,u)a^*(y)=\left\langle a^*,\{z,u,y\}\right\rangle=L^*(z,y)a^*(u),$$

$$L^*(x,u)c^*(y)=\left\langle c^*,\{x,u,y\}\right\rangle=L^*(x,y)c^*(u),$$

此外，

$$b^*\left(\{z,u,x\}\right)=M^*(x,z)b^*(u),M^*(z,x)d^*(y)=d^*\left(\{x,y,z\}\right),$$

因此，

$$q\left(\left\{x+a^*,y+b^*,z+c^*\right\},u+d^*\right)=q\left(y+b^*,\left\{z+c^*,u+d^*,x+a^*\right\}\right),$$

则 q 是不变的，结论成立。

4. 结论

本文给出了 Hom-约当三系的表示的定义及扩张的两种方法，作为约当三系的推广，还可以对 Hom-约当三系进入深入研究，比如还可以研究它的上同调，并应用到其他领域。

参考文献

[1] Chu, C.-H. (2008) Jordan Triples and Riemannian Symmetric Spaces. *Advances in Mathematics*, **219**, 2029-2057. https://doi.org/10.1016/j.aim.2008.08.001

[2] Jacobson, N. (1949) Lie and Jordan Triple Systems. *American Journal of Mathematics*, **71**, 149-170. https://doi.org/10.2307/2372102

[3] Neher, E. (2007) On the Classification of Lie and Jordan Triple Systems. *Communications in Algebra*, **13**, 2615-2667. https://doi.org/10.1080/00927878508823293

[4] Loos, O. (1973) Representations of Jordan Triples. *Transactions of the American Mathematical Society*, **185**, 199-211. https://doi.org/10.1090/S0002-9947-1973-0327857-5

[5] Chu, C.-H. and Russo, B. (2016) Cohomology of Jordan Triples via Lie Algebras. *Topics in Functional Analysis and Algebra*. arXiv:1512.03347 [math.OA]

[6] Yau, D. (2012) On N-Ary Hom-Nambu and Hom-Nambu-Lie Algebras. *Journal of Geometry and Physics*, **62**, 506-522. https://doi.org/10.1016/j.geomphys.2011.11.006