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带一个自相容源的(3 + 1)-维KP-I方程的广义Dromion结构

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摘　要

本文利用Hirota双线性方法构造了具有一个自相容源的(3 + 1)-维KP-I方程(KPIESCS)的指数局部化解。我们得到了该方程广义dromion型解和多dromion解。

关键词

指数局部化解，带自相容源的(3 + 1)-维KP-I方程，双线性方法

Generalized Dromion Structures of the (3 + 1)-Dimensional Kadomtsev-Petviashvili I Equation with a Self-Consistent Source

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Abstract

Exponentially localized solutions to the (3 + 1)-dimensional Kadomtsev-Petviashvili I equation with a self-consistent source (KPIESCS) are constructed by the Hirota bilinear method. The generalized dromion type solutions and multi-dromion solutions are obtained.

Keywords

Exponentially Localized Solutions, (3 + 1)-Dimensional Kadomtsev-Petviashvili I Equation with a

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Self-Consistent Source, Hirota Bilinear Method

1. 引言

自从 Zabusky 和 Kruskal 在非线性波的数值研究中首次提出“孤子”一词后，孤子理论引起了人们的广泛关注[1]。孤子理论作为非线性科学的一个分支，在数学和物理中有着广泛的应用[2] [3]。此外，具有自相容源的方程在物理学中具有重要的实际意义，因为它们可以反映不同波之间的相互作用[4] [5] [6] [7]。目前，求解这些方程的方法主要有两种。一种是基于 Lax 对的分析方法，如达布变换方法[8] [9]，贝克隆变换方法[10] [11]和逆散射方法[12] [13] [14]。另一种广泛使用的方法是 Hirota 双线性方法，它是一种代数方法而不是解析方法[15] [16]。

对于高维可积模型，最重要的特性之一是发现了称为 dromion 的指数局部化结构[17]。通常，dromion 由两个或多个非平行线 ghost 孤子驱动[18] [19]。在具有奇偶时间对称势的(2 + 1)维 KdV 方程、(3 + 1)维条件可积系统和(2 + 1)维非线性薛定谔方程中，得到了 dromion 结构[20] [21] [22]。此外，在 Mel'nikov 方程的情况下，得到了更一般的具有空间变化振幅的 dromion 型解以及包含的多 dromion 解[23]。

在本文中，我们将重点讨论如下形式的带一个自相容源的(3 + 1)维 Kadomtsev-Petviashvili 方程(KPIESCS)广义形式的 dromion 结构

$$\begin{cases} \left(u_t + 6uu_x + u_{xxx} + 8\kappa\left|\phi\right|_x^2\right)_x - u_{yy} + u_{zz} = 0, \\ i\phi_y = 2\phi_{xx} + 2u\phi, \\ i\phi_z = \phi_{xx} + u\phi. \end{cases} \tag{1}$$

其中，u 是长波振幅，ϕ 是复的短波包，κ 满足条件 $\kappa^2 = 1$。文献[24]讨论了该方程一般有理解的显式表示。

2. 方程(1)的局部解

通过因变量转换 $u = 2(\ln f)_{xx}, \phi = g/f$，方程(1)被转换成双线性方程

$$\begin{cases} \left(D_x D_t + D_x^4 - D_y^2 + D_z^2\right) f \cdot f + 8\kappa g \cdot \bar{g} = 0, \\ \left(iD_y - 2D_x^2\right) g \cdot f = 0, \\ \left(iD_z - D_x^2\right) g \cdot f = 0. \end{cases} \tag{2}$$

以 ε 幂级数的形式展开 f, g，如下所示

$$g = \varepsilon g^{(1)} + \varepsilon^3 g^{(3)} + \cdots, \quad f = 1 + \varepsilon^2 f^{(2)} + \varepsilon^4 f^{(4)} + \cdots, \tag{3}$$

将(3)代入(2)可以得到一组线性方程

$$\begin{cases} O(\varepsilon): ig_z^{(1)} = g_{xx}^{(1)}, \quad ig_y^{(1)} = 2g_{xx}^{(1)}, \\ O(\varepsilon^2): f_{xxxx}^{(2)} + f_{xt}^{(2)} - f_{yy}^{(2)} + f_{zz}^{(2)} = 4\kappa\left|g^{(1)}\right|^2, \\ \qquad\qquad \vdots \end{cases} \tag{4}$$

为了得到孤立子解，令

$$g^{(1)}=\sum_{j=1}^{N}\exp\left(\psi_j\right),\psi_j=l_jx+m_jy+n_jz+\omega_jt+\psi_j^{(0)},in_j=l_j^2,m_j=2n_j,\tag{5}$$

其中 $l_j,m_j,\omega_j,\psi_j^{(0)}$ 是复的待定系数。当 $N=1$ 时我们得到

$$f^{(2)}=\exp\left(\psi_1+\psi_1^*+2A\right),\exp\left(2A\right)=\frac{\kappa}{4l_{1R}^4+l_{1R}\omega_{1R}-3n_{1R}^2},\tag{6}$$

这里 $l_1=l_{1R}+il_{1I},m_1=m_{1R}+im_{1I},n_1=n_{1R}+in_{1I},\omega_1=\omega_{1R}+i\omega_{1I}$， $g^{(2j-1)}=f^{(2j)}=0\left(j\ge 2\right)$。此时我们得到孤立子解

$$\begin{cases}u=2l_{1R}^2\operatorname{sech}^2\left(\psi_{1R}+A\right),\\ \phi=\sqrt{\dfrac{4l_{1R}^4+l_{1R}\omega_{1R}-12l_{1R}^2l_{1I}^2}{4\kappa}}\operatorname{sech}\left(\psi_{1R}+A\right)e^{i\psi_{1I}},\end{cases}\tag{7}$$

其中 $\psi_{1R}=l_{1R}x+m_{1R}y+2l_{1R}l_{1I}z+\omega_{1R}t+\psi_{1R}^{(0)},\psi_{1I}=l_{1I}x+m_{1I}y-\left(l_{1R}^2-l_{1I}^2\right)z+\omega_{1I}t+\psi_{1I}^{(0)}$。

特别地，为了得到类似文献[19]中的(1,1)-dromion 解，我们取

$$f=1+e^{\psi_1+\psi_1^*}+e^{\psi_2+\psi_2^*}+Ke^{\psi_1+\psi_1^*+\psi_2+\psi_2^*},g=\rho e^{\psi_1+\psi_2},\tag{8}$$

其中

$$\psi_1=px+my+nz,\quad \psi_2=qt,\tag{9}$$

并且 $K>0$ 是一个待定实数， p,m,n,q 是待定复系数。将(8)式代入(2)式中的第一个方程可以得到当 $3n_R^2=4p_R^4$ 时满足

$$\kappa|\rho|^2=p_Rq_R\left(K-1\right),\tag{10}$$

然后再将(8)和(10)代入(2)式中的第二个方程得到 $m=2n,n=-ip^2,p_R^2=3p_I^2$。

因此，我们得到如下形式的指数衰减解

$$\begin{cases}u=\dfrac{8p_R^2\left(1+e^{\psi_2+\psi_2^*}\right)\left(e^{\psi_1+\psi_1^*}+Ke^{\psi_1+\psi_1^*+\psi_2+\psi_2^*}\right)}{\left(1+e^{\psi_1+\psi_1^*}+e^{\psi_2+\psi_2^*}+Ke^{\psi_1+\psi_1^*+\psi_2+\psi_2^*}\right)^2},\\ \phi=\dfrac{\rho e^{\psi_1+\psi_2}}{1+e^{\psi_1+\psi_1^*}+e^{\psi_2+\psi_2^*}+Ke^{\psi_1+\psi_1^*+\psi_2+\psi_2^*}},\end{cases}\tag{11}$$

此外，可以看到(4)式中的第一个方程只和自变量 x,y,z 有关，因此可以引入一些关于自变量 t 的任意函数来寻找更一般形式的解。例如，选择

$$g^{(1)}=\sum_{j=1}^{N}\exp\left(\psi_j\right),\psi_j=l_jx+m_jy+n_jz+a_j\left(t\right)+\psi_j^{(0)},in_j=l_j^2,m_j=2n_j,\tag{12}$$

或者

$$g^{(1)}=\sum_{j=1}^{N}a_j\left(t\right)\exp\left(\psi_j\right),\psi_j=l_jx+m_jy+n_jz+\psi_j^{(0)},in_j=l_j^2,m_j=2n_j.\tag{13}$$

为了说明此种情况，不妨假设

$$g^{(1)}=a\left(t\right)e^{\psi_1},\psi_1=l_1x+m_1y+n_1z,\tag{14}$$

这里 $a(t)$ 是一个复函数，复系数 l_1, n_1, m_1 满足色散关系 $l_1^2 = in_1, m_1 = 2n_1$。将(14)代入(4)式第二个方程，我们得到

$$f^{(2)} = b(t)\mathrm{e}^{\psi_1+\psi_1^*}, \tag{15}$$

其中 $b(t)$ 是一个满足如下关系的实函数

$$8l_{1R}^4 b(t) + l_{1R} b'(t) - 6n_{1R}^2 b(t) = 2\kappa |a(t)|^2. \tag{16}$$

于是我们得到方程(1)如下形式的解

$$\begin{cases} u = 2l_{1R}^2 \operatorname{sech}^2\left[\psi_{1R} + \dfrac{1}{2}\ln b(t)\right], \\ \phi = \dfrac{a(t)}{2\sqrt{b(t)}} \mathrm{e}^{i\psi_{1I}} \operatorname{sech}\left[\psi_{1I} + \dfrac{1}{2}\ln b(t)\right], \end{cases} \tag{17}$$

其中 ψ_{1R}, ψ_{1I} 分别是 ψ_1 的实部和虚部。显然，此时 u 是一个曲线孤子，自相容源项 ϕ 拥有丰富的结构。例如，如果选择

$$\frac{a(t)}{\sqrt{b(t)}} = 2\operatorname{sech}(c_0 + c_1 t), \tag{18}$$

我们可以从(16)式中求得

$$b(t) = \exp\left(\frac{8\kappa\tanh(c_0 + c_1 t) + (6n_{1R}^2 - 8l_{1R}^4)t}{l_{1R}}\right), \tag{19}$$

于是可以得到

$$\phi = \mathrm{e}^{i\psi_{1I}} \operatorname{sech}(c_0 + c_1 t) \operatorname{sech}\left[l_{1R}(x + 4l_{1I}y + 2l_{1I}z) + \frac{1}{2l_{1R}}\left(8\kappa\tanh(c_0 + c_1 t) + (6n_{1R}^2 - 8l_{1R}^4)t\right)\right]. \tag{20}$$

这就是一个 dromion 型解。

如果选择

$$\frac{a(t)}{\sqrt{b(t)}} = \frac{2}{(t + t_0)^2 + 1}, \tag{21}$$

我们就得到一个代数衰减的局部解

$$\phi = \frac{\mathrm{e}^{i\psi_{1I}}}{(t + t_0)^2 + 1} \operatorname{sech}\left\{l_{1R}(x + 4l_{1I}y + 2l_{1I}z) + \frac{1}{2l_{1R}}\left[(6n_{1R}^2 - 8l_{1R}^4)t + 8\kappa\int\frac{\mathrm{d}t}{\left[(t + t_0)^2 + 1\right]^2}\right]\right\}, \tag{22}$$

这个解沿着 t 方向衰减的速度要比解(20)慢很多。

通过扩展上述步骤，我们可以构造广义形式的局部解。实际上，可以得到如下形式的多 dromion 解

$$\phi_N = \left[\frac{\sum_{j=1}^{M} a_j(t)}{2\sqrt{b(t)}}\right] \mathrm{e}^{i\psi_{1I}} \operatorname{sech}\left[\psi_{1R} + \frac{1}{2}\ln b(t)\right]. \tag{23}$$

这里 $a_j(t), j=1,2,\cdots,N$ 是关于自变量 t 的任意函数且满足

$$8l_{1R}^4 b(t)+l_{1R}b'(t)-6n_{1R}^2 b(t)=2\kappa\sum_{p=1}^{M}\sum_{q=1}^{M}a_p(t)a_q^*(t), \tag{24}$$

为了说明这种解，我们取 $M=2$ 为例。令

$$\frac{a_1(t)}{\sqrt{b(t)}}=2\operatorname{sech}(t+\eta_1), \frac{a_2(t)}{\sqrt{b(t)}}=2\operatorname{sech}(t+\eta_2), \tag{25}$$

求解式(24)得到

$$b(t)=\exp\left[\frac{\left(6n_{1R}^2-8l_{1R}^4\right)t+8\kappa\left(\tanh(t+\eta_1)+\tanh(t+\eta_2)+2\int\operatorname{sech}(t+\eta_1)\operatorname{sech}(t+\eta_2)\mathrm{d}t\right)}{l_{1R}}\right], \tag{26}$$

此时就可以到一个高阶 dromion 解

$$\begin{aligned}\phi_2 = \mathrm{e}^{i\psi_{1I}}\left[\operatorname{sech}(t+\eta_1)+\operatorname{sech}(t+\eta_2)\right]\operatorname{sech}\Bigg[l_{1R}\left(x+4l_{1I}y+2l_{1I}z\right)+\frac{1}{2l_{1R}}\Big(\left(6n_{1R}^2-8l_{1R}^4\right)t \\ +8\kappa\left(\tanh(t+\eta_1)+\tanh(t+\eta_2)+2\int\operatorname{sech}(t+\eta_1)\operatorname{sech}(t+\eta_2)\mathrm{d}t\right)\Big)\Bigg].\end{aligned} \tag{27}$$

3. 总结

本文中，我们基于 Hirota 双线性方法找到了(3 + 1)维 KPIESCS 的指数局部化解。我们特别构造了局域 dromion 解和诱导 dromion 解。

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