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H-张量的新判定准则及其应用

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摘 要

***H*-张量在科学与工程实践等领域中有着重要的应用，但在实际中要判定*H*-张量是比较困难的。本文通过构造不同的正对角阵，结合不等式的放缩技巧，给出了*H*-张量比较实用的新判别条件。作为应用，给出了判定偶次齐次多项式正定性的新方法，并给出相应的数值算例，表明了新结论的有效性。**

关键词

***H*-张量，齐次多项式，正定性，不可约，非零元素链**

New Criterions for *H*-Tensors and Its Applications

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Abstract

***H*-tensors have important applications in science and engineering, but it is difficult to determine whether a given tensor is an *H*-tensor or not in practice. In this paper, by constructing different positive diagonal matrices and combining the technique of inequality reduction, new practical conditions for *H*-tensors are given. As applications, new methods for determining the positive definiteness of even homogeneous polynomials are presented, and the validity of new results is verified by some numerical examples.**

Keywords

***H*-Tensors, Homogeneous Multivariate Forms, Positive Definiteness, Irreducible, Nonzero Elements Chain**

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1. 引言

张量是高阶广义矩阵，广泛应用于信号和图像处理、高阶统计学、自动控制、医学成像、超图理论、弹性材料科学和工程研究与数据分析等领域中。近年来，许多专家学者对一般张量[1]-[6]或特殊结构张量的理论、性质及应用进行了广泛探讨[7]-[18]。本文继续讨论$\boldsymbol{H}$-张量的判定问题，得到了只与张量元素有关的新判别不等式，拓广了文献[11] [14] [15] [16]的结果。同时，获得了偶数阶实对称张量，即偶数阶齐次多项式正定性的新判定条件。最后，利用数值算例说明了新条件的有效性。

2. 预备知识

记$\mathbf{C}(\mathbf{R})$为复(实)数集，$N=[n]=\{1,2,\cdots,n\}$。一个m阶n维张量$\boldsymbol{A}=\left(a_{i_1i_2\cdots i_m}\right)$由$n^m$个元素构成，其中$a_{i_1i_2\cdots i_m}\in\mathbf{C}$，$i_j\in N$，$j\in[m]$ [3] [4]。若$a_{i_1i_2\cdots i_m}=a_{\pi(i_1i_2\cdots i_m)}$，$\forall\pi\in\Pi_m$，则称$\boldsymbol{A}=\left(a_{i_1i_2\cdots i_m}\right)$为对称张量[5]，其中$\Pi_m$为$m$个指标的置换群。称张量$\boldsymbol{I}=\left(\delta_{i_1i_2\cdots i_m}\right)$为单位张量[5]，若

$$\delta_{i_1i_2\cdots i_m}=\begin{cases}1, & i_1=i_2=\cdots=i_m,\\ 0, & \text{其它}.\end{cases}$$

若

$$f(\boldsymbol{x})=\sum_{i_1,\cdots,i_m\in[n]}a_{i_1i_2\cdots i_m}x_{i_1}\cdots x_{i_m}>0,\quad \boldsymbol{x}=(x_1,x_2,\cdots,x_n)^{\mathrm{T}}\in\mathbf{R}^n,\quad \boldsymbol{x}\neq 0,$$

则称m阶n次齐次多项式$f(\boldsymbol{x})$是正定的[3]。$f(\boldsymbol{x})$也可以表示为m阶n维对称张量$\boldsymbol{A}$与$\boldsymbol{x}^m$的乘积，如下

$$f(\boldsymbol{x})=\boldsymbol{A}\boldsymbol{x}^m=\sum_{i_1,\cdots,i_m\in[n]}a_{i_1i_2\cdots i_m}x_{i_1}\cdots x_{i_m}.$$

若$f(\boldsymbol{x})$是正定的，则对称张量$\boldsymbol{A}$也是正定的[3]。

定义 1 [8] 设$\boldsymbol{A}=\left(a_{i_1i_2\cdots i_m}\right)$是$m$阶$n$维张量，若存在正向量$\boldsymbol{x}=(x_1,x_2,\cdots,x_n)^{\mathrm{T}}\in\mathbf{R}^n$，满足

$$\left|a_{ii\cdots i}\right|x_i^{m-1}>\sum_{\substack{i_2,i_3,\cdots,i_m\in[n]\\ \delta_{ii_2\cdots i_m}=0}}\left|a_{ii_2\cdots i_m}\right|x_{i_2}\cdots x_{i_m},\quad \forall i\in N,$$

则称$\boldsymbol{A}$为$\boldsymbol{H}$-张量。

定义 2 [5] 设$\boldsymbol{A}=\left(a_{i_1i_2\cdots i_m}\right)$是$m$阶$n$维张量，若存在一个非空子集$\mathbf{I}\subset N$，满足

$$a_{i_1i_2\cdots i_m}=0,\quad \forall i_1\in\mathbf{I},\quad \forall i_2,\cdots,i_m\notin\mathbf{I},$$

则称$\boldsymbol{A}$是可约张量。否则，称$\boldsymbol{A}$是不可约张量。

定义 3 [9] 设$\boldsymbol{A}=\left(a_{i_1i_2\cdots i_m}\right)$是$m$阶$n$维张量，若存在指标$k_1,k_2,\cdots,k_r$，满足

$$\sum_{\substack{i_2,i_3,\cdots,i_m\in[n]\\ \delta_{k_si_2\cdots i_m}=0\\ k_{s+1}\in\{i_2,i_3,\cdots,i_m\}}}\left|a_{k_si_2\cdots i_m}\right|\neq 0,\quad s=0,1,\cdots,r,\quad \forall i,j\in N\,(i\neq j),$$

其中$k_0=i$，$k_{r+1}=j$，则称张量$\boldsymbol{A}$中有一条从指标i到指标j的非零元素链。

3. 主要结果

为讨论方便，给出如下记号：设$\boldsymbol{A}=\left(a_{i_1i_2\cdots i_m}\right)$是$m$阶$n$维张量，令

$$S^{m-1}=\left\{i_2i_3\cdots i_m : i_j\in S,\ j=2,3,\cdots,m\right\},$$

$$N^{m-1}\setminus S^{m-1}=\left\{i_2i_3\cdots i_m : i_2i_3\cdots i_m\in N^{m-1}\text{且}\, i_2i_3\cdots i_m\notin S^{m-1}\right\},$$

$$r_i\left(\boldsymbol{A}\right)=\sum_{\substack{i_2,\cdots,i_m\in[n]\\ \delta_{ii_2\cdots i_m}=0}}\left|a_{ii_2\cdots i_m}\right|=\sum_{i_2,\cdots,i_m\in[n]}\left|a_{ii_2\cdots i_m}\right|-\left|a_{ii\cdots i}\right|,$$

$$N_1=\left\{i\in N:0<\left|a_{ii\cdots i}\right|=r_i\left(\boldsymbol{A}\right)\right\},\quad N_2=\left\{i\in N:0<\left|a_{ii\cdots i}\right|<r_i\left(\boldsymbol{A}\right)\right\},$$

$$N_3=\left\{i\in N:\left|a_{ii\cdots i}\right|>r_i\left(\boldsymbol{A}\right)\right\},\quad N_0^{m-1}=N^{m-1}/\left(N_2^{m-1}\cup N_3^{m-1}\right),$$

$$\beta_i=\frac{r_i\left(\boldsymbol{A}\right)-\left|a_{ii\cdots i}\right|}{r_i\left(\boldsymbol{A}\right)},\quad M=\max_{\substack{i\in N_2\\ j\in N_3}}\left\{\beta_i,\frac{r_j\left(\boldsymbol{A}\right)}{\left|a_{jj\cdots j}\right|}\right\},$$

$$K=\max_{i\in N_3}\frac{M\sum\limits_{i_2\cdots i_m\in N_0^{m-1}}\left|a_{ii_2\cdots i_m}\right|+\sum\limits_{i_2\cdots i_m\in N_2^{m-1}}\max\limits_{j\in\{i_2,i_3,\cdots,i_m\}}\left\{\beta_j\right\}\left|a_{ii_2\cdots i_m}\right|}{r_i\left(\boldsymbol{A}\right)-\sum\limits_{\substack{i_2\cdots i_m\in N_3^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}\left|a_{ii_2\cdots i_m}\right|},$$

$$R_i\left(\boldsymbol{A}\right)=M\sum_{i_2\cdots i_m\in N_0^{m-1}}\left|a_{ii_2\cdots i_m}\right|+\sum_{i_2\cdots i_m\in N_2^{m-1}}\max_{j\in\{i_2,i_3,\cdots,i_m\}}\left\{\beta_j\right\}\left|a_{ii_2\cdots i_m}\right|+K\sum_{\substack{i_2\cdots i_m\in N_3^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}\left|a_{ii_2\cdots i_m}\right|\left(i\in N_3\right).$$

引理 1 [6] 若 $\boldsymbol{A}$ 是严格对角占优的张量，则 $\boldsymbol{A}$ 是 $\boldsymbol{H}$-张量。

引理 2 [10] 设 $\boldsymbol{A}=\left(a_{i_1i_2\cdots i_m}\right)$ 是 m 阶 n 维张量。若存在正对角阵 $\boldsymbol{X}$，满足 $\boldsymbol{AX}^{m-1}$ 是 $\boldsymbol{H}$-张量，则 $\boldsymbol{A}$ 是 $\boldsymbol{H}$-张量。

引理 3 [6] 设 $\boldsymbol{A}=\left(a_{i_1i_2\cdots i_m}\right)$ 是 m 阶 n 维张量且不可约。若

$$\left|a_{ii\cdots i}\right|\ge r_i\left(\boldsymbol{A}\right),\quad \forall i\in N,$$

且上式中至少有一个严格不等式成立，则 $\boldsymbol{A}$ 是 $\boldsymbol{H}$-张量。

引理 4 [9] 设 $\boldsymbol{A}=\left(a_{i_1i_2\cdots i_m}\right)$ 是 m 阶 n 维张量。若

1) $\left|a_{ii\cdots i}\right|\ge r_i\left(\boldsymbol{A}\right)$，$\forall i\in N$；

2) $J\left(\boldsymbol{A}\right)=\left\{i\in N:\left|a_{ii\cdots i}\right|>r_i\left(\boldsymbol{A}\right)\right\}\ne\varnothing$；

3) $\forall i\notin J\left(\boldsymbol{A}\right)$，从指标 i 到指标 j 有一条非零元素链，满足 $j\in J\left(\boldsymbol{A}\right)$；

则 $\boldsymbol{A}$ 是 $\boldsymbol{H}$-张量。

定理 1 设 $\boldsymbol{A}=\left(a_{i_1i_2\cdots i_m}\right)$ 是 m 阶 n 维张量。若 $\boldsymbol{A}$ 满足

$$\begin{aligned}\left|a_{ii\cdots i}\right|\beta_i>M&\sum_{i_2\cdots i_m\in N_0^{m-1}}\left|a_{ii_2\cdots i_m}\right|+\sum_{\substack{i_2\cdots i_m\in N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}\max_{j\in\{i_2,i_3,\cdots,i_m\}}\left\{\beta_j\right\}\left|a_{ii_2\cdots i_m}\right|\\&+\sum_{i_2\cdots i_m\in N_3^{m-1}}\max_{j\in\{i_2,i_3,\cdots,i_m\}}\left\{\frac{R_j\left(\boldsymbol{A}\right)}{\left|a_{jj\cdots j}\right|}\right\}\left|a_{ii_2\cdots i_m}\right|\end{aligned},\quad \forall i\in N_2, \tag{1}$$

且 $\left|a_{ii\cdots i}\right|\ne\sum\limits_{\substack{i_2\cdots i_m\in N_0^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}\left|a_{ii_2\cdots i_m}\right|$ $\left(\forall i\in N_1\right)$，则 $\boldsymbol{A}$ 是 $\boldsymbol{H}$-张量。

证明： 由 K 的定义知

$$K\left|a_{ii\cdots i}\right|\ge M\sum_{i_2\cdots i_m\in N_0^{m-1}}\left|a_{ii_2\cdots i_m}\right|+\sum_{\substack{i_2\cdots i_m\in N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}\max_{j\in\{i_2,i_3,\cdots,i_m\}}\left\{\beta_j\right\}\left|a_{ii_2\cdots i_m}\right|+\sum_{i_2\cdots i_m\in N_3^{m-1}}\left|a_{ii_2\cdots i_m}\right|=R_i\left(\boldsymbol{A}\right).$$

因此

$$K \geq \frac{R_i(\boldsymbol{A})}{|a_{ii\cdots i}|}，\quad \forall i \in N_3. \tag{2}$$

由式(1)得，

$$|a_{ii\cdots i}|\beta_i - M\sum_{i_2\cdots i_m \in N_0^{m-1}}|a_{ii_2\cdots i_m}| - \sum_{\substack{i_2\cdots i_m \in N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}\max_{j\in\{i_2,i_3,\cdots,i_m\}}\{\beta_j\}|a_{ii_2\cdots i_m}| - \sum_{i_2\cdots i_m \in N_3^{m-1}}\max_{j\in\{i_2,i_3,\cdots,i_m\}}\left\{\frac{R_j(\boldsymbol{A})}{|a_{jj\cdots j}|}\right\}|a_{ii_2\cdots i_m}| > 0. \tag{3}$$

而 $R_i(\boldsymbol{A}) < r_i(\boldsymbol{A})$ 且 $M \geq \frac{r_i(\boldsymbol{A})}{|a_{ii\cdots i}|}(\forall i \in N_3)$，所以

$$M > \frac{R_i(\boldsymbol{A})}{|a_{ii\cdots i}|}，\quad \forall i \in N_3. \tag{4}$$

由(3)式和(4)式得，一定存在足够小的正数 ε，使得

$$M > \frac{R_i(\boldsymbol{A})}{|a_{ii\cdots i}|} + \varepsilon，\quad \forall i \in N_3， \tag{5}$$

$$\begin{aligned}&|a_{ii\cdots i}|\beta_i - M\sum_{i_2\cdots i_m \in N_0^{m-1}}|a_{ii_2\cdots i_m}| - \sum_{\substack{i_2\cdots i_m \in N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}\max_{j\in\{i_2,i_3,\cdots,i_m\}}\{\beta_j\}|a_{ii_2\cdots i_m}|\\ &-\sum_{i_2\cdots i_m \in N_3^{m-1}}\max_{j\in\{i_2,i_3,\cdots,i_m\}}\left\{\frac{R_j(\boldsymbol{A})}{|a_{jj\cdots j}|}\right\}|a_{ii_2\cdots i_m}| > \varepsilon\sum_{i_2\cdots i_m \in N_3^{m-1}}|a_{ii_2\cdots i_m}|\end{aligned}，\quad \forall i \in N_2. \tag{6}$$

构造正对角阵 $X = diag(x_1, x_2, \cdots, x_n)$，记 $\boldsymbol{B} = \boldsymbol{A}\boldsymbol{X}^{m-1} = (b_{i_1 i_2\cdots i_m})$，其中

$$x_i = M，\quad i \in N_1；\quad x_i = \beta_i^{\frac{1}{m-1}}，\quad i \in N_2；\quad x_i = \left(\frac{R_i(\boldsymbol{A})}{|a_{ii\cdots i}|} + \varepsilon\right)^{\frac{1}{m-1}}，\quad i \in N_3.$$

而 $M \geq \frac{r_i(\boldsymbol{A}) - |a_{ii\cdots i}|}{r_i(\boldsymbol{A})}(i \in N_2)$ 且 $M > \frac{r_i(\boldsymbol{A})}{|a_{ii\cdots i}|} + \varepsilon(i \in N_3)$，故对 $\forall i \in N_1$，

$$\begin{aligned}r_i(\boldsymbol{B}) &= \sum_{\substack{i_2\cdots i_m \in N_0^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|x_{i_2}\cdots x_{i_m} + \sum_{i_2 i_3\cdots i_m \in N_2^{m-1}}|a_{ii_2\cdots i_m}|\beta_{i_2}^{\frac{1}{m-1}}\cdots\beta_{i_m}^{\frac{1}{m-1}}\\ &\quad+\sum_{i_2 i_3\cdots i_m \in N_3^{m-1}}|a_{ii_2\cdots i_m}|\left(\frac{R_{i_2}(\boldsymbol{A})}{|a_{i_2 i_2\cdots i_2}|} + \varepsilon\right)^{\frac{1}{m-1}}\cdots\left(\frac{R_{i_m}(\boldsymbol{A})}{a_{i_m i_m\cdots i_m}} + \varepsilon\right)^{\frac{1}{m-1}}\\ &\leq M\sum_{\substack{i_2\cdots i_m \in N_0^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}| + \sum_{i_2\cdots i_m \in N_2^{m-1}}\max_{j\in\{i_2,i_3,\cdots,i_m\}}\{\beta_j\}|a_{ii_2\cdots i_m}|\\ &\quad+\sum_{i_2\cdots i_m \in N_3^{m-1}}\max_{j\in\{i_2,i_3,\cdots,i_m\}}\left\{\frac{R_j(\boldsymbol{A})}{|a_{jj\cdots j}|} + \varepsilon\right\}|a_{ii_2\cdots i_m}|\\ &\leq M\sum_{\substack{i_2\cdots i_m \in N_0^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}| + M\sum_{i_2\cdots i_m \in N_2^{m-1}}|a_{ii_2\cdots i_m}| + M\sum_{i_2\cdots i_m \in N_3^{m-1}}|a_{ii_2\cdots i_m}|\\ &= Mr_j(\boldsymbol{A}) = M|a_{ii\cdots i}| = |b_{ii\cdots i}|.\end{aligned}$$

根据(6)式，对 $\forall i \in N_2$，

$$r_i(\boldsymbol{B}) = \sum_{i_2\cdots i_m \in N_0^{m-1}} |a_{ii_2\cdots i_m}| x_{i_2}\cdots x_{i_m} + \sum_{\substack{i_2 i_3\cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \beta_{i_2}^{\frac{1}{m-1}}\cdots\beta_{i_m}^{\frac{1}{m-1}}$$

$$+ \sum_{i_2 i_3\cdots i_m \in N_3^{m-1}} |a_{ii_2\cdots i_m}| \left(\frac{R_{i_2}(\boldsymbol{A})}{|a_{i_2 i_2\cdots i_2}|}+\varepsilon\right)^{\frac{1}{m-1}}\cdots\left(\frac{R_{i_m}(\boldsymbol{A})}{a_{i_m i_m\cdots i_m}}+\varepsilon\right)^{\frac{1}{m-1}}$$

$$\le M \sum_{i_2\cdots i_m \in N_0^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2 i_3\cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,i_3,\cdots,i_m\}}\{\beta_j\}|a_{ii_2\cdots i_m}|$$

$$+ \sum_{i_2\cdots i_m \in N_3^{m-1}} \max_{j\in\{i_2,i_3,\cdots,i_m\}} \frac{R_j(\boldsymbol{A})}{|a_{jj\cdots j}|}|a_{ii_2\cdots i_m}| + \varepsilon \sum_{i_2\cdots i_m \in N_3^{m-1}} |a_{ii_2\cdots i_m}|$$

$$< M \sum_{i_2\cdots i_m \in N_0^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2 i_3\cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,i_3,\cdots,i_m\}}\{\beta_j\}|a_{ii_2\cdots i_m}|$$

$$+ \sum_{i_2\cdots i_m \in N_3^{m-1}} \max_{j\in\{i_2,i_3,\cdots,i_m\}} \frac{R_j(\boldsymbol{A})}{|a_{jj\cdots j}|}|a_{ii_2\cdots i_m}| + |a_{ii\cdots i}|\beta_i - M \sum_{i_2\cdots i_m \in N_0^{m-1}} |a_{ii_2\cdots i_m}|$$

$$- \sum_{\substack{i_2\cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,i_3,\cdots,i_m\}}\{\beta_j\}|a_{ii_2\cdots i_m}| - \sum_{i_2\cdots i_m \in N_3^{m-1}} \max_{j\in\{i_2,i_3,\cdots,i_m\}}\left\{\frac{R_j(\boldsymbol{A})}{|a_{jj\cdots j}|}\right\}|a_{ii_2\cdots i_m}|$$

$$= |a_{ii\cdots i}|\beta_i = |b_{ii\cdots i}|.$$

对 $\forall i \in N_3$，由(2)式得

$$r_i(\boldsymbol{B}) = \sum_{i_2\cdots i_m \in N_0^{m-1}} |a_{ii_2\cdots i_m}| x_{i_2}\cdots x_{i_m} + \sum_{i_2 i_3\cdots i_m \in N_2^{m-1}} |a_{ii_2\cdots i_m}| \beta_{i_2}^{\frac{1}{m-1}}\cdots\beta_{i_m}^{\frac{1}{m-1}}$$

$$+ \sum_{\substack{i_2 i_3\cdots i_m \in N_3^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \left(\frac{R_{i_2}(\boldsymbol{A})}{|a_{i_2 i_2\cdots i_2}|}+\varepsilon\right)^{\frac{1}{m-1}}\cdots\left(\frac{R_{i_m}(\boldsymbol{A})}{a_{i_m i_m\cdots i_m}}+\varepsilon\right)^{\frac{1}{m-1}}$$

$$\le M \sum_{i_2\cdots i_m \in N_0^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{i_2 i_3\cdots i_m \in N_2^{m-1}} \max_{j\in\{i_2,i_3,\cdots,i_m\}}\{\beta_j\}|a_{ii_2\cdots i_m}|$$

$$+ \sum_{\substack{i_2\cdots i_m \in N_3^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,i_3,\cdots,i_m\}} \frac{R_j(\boldsymbol{A})}{|a_{jj\cdots j}|}|a_{ii_2\cdots i_m}| + \varepsilon \sum_{\substack{i_2\cdots i_m \in N_3^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|$$

$$\le M \sum_{i_2\cdots i_m \in N_0^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{i_2 i_3\cdots i_m \in N_2^{m-1}} \max_{j\in\{i_2,i_3,\cdots,i_m\}}\{\beta_j\}|a_{ii_2\cdots i_m}|$$

$$+ \sum_{\substack{i_2\cdots i_m \in N_3^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} K|a_{ii_2\cdots i_m}| + \varepsilon \sum_{\substack{i_2\cdots i_m \in N_3^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|$$

$$= R_i(\boldsymbol{A}) + \varepsilon \sum_{\substack{i_2\cdots i_m \in N_3^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| < R_i(\boldsymbol{A}) + \varepsilon \sum_{i_2\cdots i_m \in N_3^{m-1}} |a_{ii\cdots i}|$$

$$= |a_{ii\cdots i}|\left(\frac{R_i(\boldsymbol{A})}{|a_{ii\cdots i}|}+\varepsilon\right) = |b_{ii\cdots i}|.$$

综上可得 $|b_{ii\cdots i}| > r_i(\boldsymbol{B})(\forall i \in N)$。由引理 1 知 $\boldsymbol{B}$ 是 $\boldsymbol{H}$-张量，故由引理 2 知 $\boldsymbol{A}$ 是 $\boldsymbol{H}$-张量。

定理 2 设 $\boldsymbol{A}=\left(a_{i_1i_2\cdots i_m}\right)$ 是 m 阶 n 维张量且不可约。若 $\boldsymbol{A}$ 满足

$$\begin{aligned}|a_{ii\cdots i}|\beta_i \ge M\sum_{\substack{i_2\cdots i_m\in N_0^{m-1}}}|a_{ii_2\cdots i_m}|+\sum_{\substack{i_2\cdots i_m\in N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}\max_{j\in\{i_2,i_3,\cdots,i_m\}}\{\beta_j\}|a_{ii_2\cdots i_m}| \\ +\sum_{i_2\cdots i_m\in N_3^{m-1}}\max_{j\in\{i_2,i_3,\cdots,i_m\}}\left\{\frac{R_j(\boldsymbol{A})}{|a_{jj\cdots j}|}\right\}|a_{ii_2\cdots i_m}|\end{aligned}, \quad \forall i\in N_2, \tag{7}$$

且(7)式中至少有一个严格不等式成立，则 $\boldsymbol{A}$ 是 $\boldsymbol{H}$-张量。

证明： 构造正对角阵 $X=diag\left(x_1,x_2,\cdots,x_n\right)$，记 $\boldsymbol{B}=\boldsymbol{A}X^{m-1}=\left(b_{i_1i_2\cdots i_m}\right)$，其中

$$x_i=M\,,\quad i\in N_1;\quad x_i=\beta_i^{\frac{1}{m-1}},\quad i\in N_2;\quad x_i=\left(\frac{R_i(\boldsymbol{A})}{|a_{ii\cdots i}|}\right)^{\frac{1}{m-1}},\quad i\in N_3 .$$

由 M 的定义得，对 $\forall i\in N_1$，

$$\begin{aligned}r_i(\boldsymbol{B})&=\sum_{\substack{i_2\cdots i_m\in N_0^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|x_{i_2}\cdots x_{i_m}+\sum_{i_2i_3\cdots i_m\in N_2^{m-1}}|a_{ii_2\cdots i_m}|\beta_{i_2}^{\frac{1}{m-1}}\cdots\beta_{i_m}^{\frac{1}{m-1}}\\&\quad+\sum_{i_2i_3\cdots i_m\in N_3^{m-1}}|a_{ii_2\cdots i_m}|\left(\frac{R_{i_2}(\boldsymbol{A})}{|a_{i_2i_2\cdots i_2}|}\right)^{\frac{1}{m-1}}\cdots\left(\frac{R_{i_m}(\boldsymbol{A})}{a_{i_mi_m\cdots i_m}}\right)^{\frac{1}{m-1}}\\&\le M\sum_{\substack{i_2\cdots i_m\in N_0^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|+\sum_{i_2\cdots i_m\in N_2^{m-1}}\max_{j\in\{i_2,i_3,\cdots,i_m\}}\{\beta_j\}|a_{ii_2\cdots i_m}|\\&\quad+\sum_{i_2\cdots i_m\in N_3^{m-1}}\max_{j\in\{i_2,i_3,\cdots,i_m\}}\left\{\frac{R_j(\boldsymbol{A})}{|a_{jj\cdots j}|}\right\}|a_{ii_2\cdots i_m}|\\&\le M\sum_{\substack{i_2\cdots i_m\in N_0^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|+M\sum_{i_2\cdots i_m\in N_2^{m-1}}|a_{ii_2\cdots i_m}|+M\sum_{i_2\cdots i_m\in N_3^{m-1}}|a_{ii_2\cdots i_m}|\\&=Mr_j(\boldsymbol{A})=M|a_{ii\cdots i}|=|b_{ii\cdots i}|.\end{aligned}$$

根据(7)式知，对 $\forall i\in N_2$，

$$\begin{aligned}r_i(\boldsymbol{B})&=\sum_{i_2\cdots i_m\in N_0^{m-1}}|a_{ii_2\cdots i_m}|x_{i_2}\cdots x_{i_m}+\sum_{\substack{i_2i_3\cdots i_m\in N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|\beta_{i_2}^{\frac{1}{m-1}}\cdots\beta_{i_m}^{\frac{1}{m-1}}\\&\quad+\sum_{i_2i_3\cdots i_m\in N_3^{m-1}}|a_{ii_2\cdots i_m}|\left(\frac{R_{i_2}(\boldsymbol{A})}{|a_{i_2i_2\cdots i_2}|}\right)^{\frac{1}{m-1}}\cdots\left(\frac{R_{i_m}(\boldsymbol{A})}{a_{i_mi_m\cdots i_m}}\right)^{\frac{1}{m-1}}\\&\le M\sum_{i_2\cdots i_m\in N_0^{m-1}}|a_{ii_2\cdots i_m}|+\sum_{\substack{i_2i_3\cdots i_m\in N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}\max_{j\in\{i_2,i_3,\cdots,i_m\}}\{\beta_j\}|a_{ii_2\cdots i_m}|\\&\quad+\sum_{i_2\cdots i_m\in N_3^{m-1}}\max_{j\in\{i_2,i_3,\cdots,i_m\}}\frac{R_j(\boldsymbol{A})}{|a_{jj\cdots j}|}|a_{ii_2\cdots i_m}|\\&\le|a_{ii\cdots i}|\beta_i=|b_{ii\cdots i}|.\end{aligned}$$

又对 $\forall i\in N_3$，由 K 的定义知

$$
\begin{aligned}
r_i(\boldsymbol{B}) &= \sum_{i_2\cdots i_m\in N_0^{m-1}} |a_{ii_2\cdots i_m}| x_{i_2}\cdots x_{i_m} + \sum_{i_2i_3\cdots i_m\in N_2^{m-1}} |a_{ii_2\cdots i_m}| \beta_{i_2}^{\frac{1}{m-1}}\cdots\beta_{i_m}^{\frac{1}{m-1}} \\
&\quad + \sum_{\substack{i_2i_3\cdots i_m\in N_3^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \left(\frac{R_{i_2}(\boldsymbol{A})}{|a_{i_2i_2\cdots i_2}|}\right)^{\frac{1}{m-1}}\cdots\left(\frac{R_{i_m}(\boldsymbol{A})}{a_{i_mi_m\cdots i_m}}\right)^{\frac{1}{m-1}} \\
&\le M \sum_{i_2\cdots i_m\in N_0^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{i_2i_3\cdots i_m\in N_2^{m-1}} \max_{j\in\{i_2,i_3,\cdots,i_m\}}\{\beta_j\}|a_{ii_2\cdots i_m}| \\
&\quad + \sum_{\substack{i_2\cdots i_m\in N_3^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,i_3,\cdots,i_m\}} \frac{R_j(\boldsymbol{A})}{|a_{jj\cdots j}|}|a_{ii_2\cdots i_m}| \\
&\quad + \sum_{\substack{i_2\cdots i_m\in N_3^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,i_3,\cdots,i_m\}} \frac{R_j(\boldsymbol{A})}{|a_{jj\cdots j}|}|a_{ii_2\cdots i_m}| \\
&\le M \sum_{i_2\cdots i_m\in N_0^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{i_2i_3\cdots i_m\in N_2^{m-1}} \max_{j\in\{i_2,i_3,\cdots,i_m\}}\{\beta_j\}|a_{ii_2\cdots i_m}| \\
&\quad + \sum_{\substack{i_2\cdots i_m\in N_3^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} K|a_{ii_2\cdots i_m}| + \varepsilon \sum_{\substack{i_2\cdots i_m\in N_3^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \\
&= R_i(\boldsymbol{A}) = |a_{ii\cdots i}|\frac{R_i(\boldsymbol{A})}{|a_{ii\cdots i}|} = |b_{ii\cdots i}|.
\end{aligned}
$$

因此，$|b_{ii\cdots i}| \ge r_i(\boldsymbol{B})(\forall i\in N)$。因(7)式中至少有一个严格不等式成立，所以存在指标 i_0 满足 $|b_{i_0i_0\cdots i_0}| > r_{i_0}(\boldsymbol{B})$，且由 $\boldsymbol{A}$ 不可约知 $\boldsymbol{B}$ 不可约，于是由引理 3 知 $\boldsymbol{B}$ 是 $\boldsymbol{H}$-张量。从而，由引理 2 知 $\boldsymbol{A}$ 是 $\boldsymbol{H}$-张量。

记

$$
\Omega(\boldsymbol{A}) = \left\{\forall i\in N_1 : |a_{ii\cdots i}|\beta_i \ge M \sum_{i_2\cdots i_m\in N_0^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m\in N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,i_3,\cdots,i_m\}}\{\beta_j\}|a_{ii_2\cdots i_m}| \right.
\left. + \sum_{i_2\cdots i_m\in N_3^{m-1}} \max_{j\in\{i_2,i_3,\cdots,i_m\}} \left\{\frac{R_j(\boldsymbol{A})}{|a_{jj\cdots j}|}\right\}|a_{ii_2\cdots i_m}|\right\}.
$$

定理 3 设 $\boldsymbol{A} = (a_{i_1i_2\cdots i_m})$ 是 m 阶 n 维张量。若 $\boldsymbol{A}$ 满足

$$
\begin{aligned}
|a_{ii\cdots i}|\beta_i &\ge M \sum_{i_2\cdots i_m\in N_0^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m\in N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,i_3,\cdots,i_m\}}\{\beta_j\}|a_{ii_2\cdots i_m}| \\
&\quad + \sum_{i_2\cdots i_m\in N_3^{m-1}} \max_{j\in\{i_2,i_3,\cdots,i_m\}} \left\{\frac{R_j(\boldsymbol{A})}{|a_{jj\cdots j}|}\right\}|a_{ii_2\cdots i_m}|
\end{aligned}
, \quad \forall i\in N_2, \tag{8}
$$

且对 $\forall i\in N/\Omega(\boldsymbol{A})\ne\varnothing$，$\boldsymbol{A}$ 中存在从 i 到 j 的非零元素链，满足 $j\in\Omega(\boldsymbol{A})\ne\varnothing$，则 $\boldsymbol{A}$ 为 $\boldsymbol{H}$-张量。

证明： 构造正对角阵 $X = diag(x_1,x_2,\cdots,x_n)$，记 $\boldsymbol{B} = \boldsymbol{A}\boldsymbol{X}^{m-1} = (b_{i_1i_2\cdots i_m})$，其中

$$
x_i = M,\quad i\in N_1;\quad x_i = \beta_i^{\frac{1}{m-1}},\quad i\in N_2;\quad x_i = \left(\frac{R_i(\boldsymbol{A})}{|a_{ii\cdots i}|}\right)^{\frac{1}{m-1}},\quad i\in N_3.
$$

类似于定理 2 的证明，对任意的 $i\in N$，有 $|b_{ii\cdots i}| \ge r_i(\boldsymbol{B})$，且至少有一个严格不等式成立。

另一方面，若$\left|b_{ii\cdots i}\right|=r_i\left(\boldsymbol{B}\right)$，则$\forall i\in N/\Omega\left(\boldsymbol{A}\right)$。设$\boldsymbol{A}$中有从$i$到$j$的一条非零元素链，满足$j\in\Omega\left(\boldsymbol{A}\right)$，则$\boldsymbol{B}$中也有从$i$到$j$一条非零元素链，满足$\left|b_{jj\cdots j}\right|>r_j\left(\boldsymbol{B}\right)$。于是，由引理4知$\boldsymbol{B}$是$\boldsymbol{H}$-张量，再由引理2知$\boldsymbol{A}$是$\boldsymbol{H}$-张量。

例1 设$\boldsymbol{A}=\left[\boldsymbol{A}(1,:,:),\boldsymbol{A}(2,:,:),\boldsymbol{A}(3,:,:)\right]$是一个3阶3维张量，其中

$$\boldsymbol{A}(1,:,:)=\begin{pmatrix}12&1&0\\0&6&0\\1&0&16\end{pmatrix},\quad \boldsymbol{A}(2,:,:)=\begin{pmatrix}1&0&0\\0&6&0\\0&0&3\end{pmatrix},\quad \boldsymbol{A}(3,:,:)=\begin{pmatrix}1&0&0\\0&2&0\\1&0&20\end{pmatrix}.$$

则

$$\left|a_{111}\right|=12,\quad r_1\left(\boldsymbol{A}\right)=24,\quad \left|a_{222}\right|=6,\quad r_2\left(\boldsymbol{A}\right)=4,\quad \left|a_{333}\right|=20,\quad r_3\left(\boldsymbol{A}\right)=4.$$

所以$N_1=\varnothing,N_2=\{1\},N_2=\{2,3\}$。计算得

$$\beta_1=\frac{1}{2},\quad M=\frac{1}{2},\quad K=\frac{1}{6},\quad R_2\left(\boldsymbol{A}\right)=1,\quad R_3\left(\boldsymbol{A}\right)=\frac{4}{3}.$$

因为

$$\sum_{jk\in N_0^2}\left|a_{1jk}\right|+\sum_{\substack{jk\in N_2^2\\ \delta_{1jk}=0}}\max_{l\in\{j,k\}}\{\beta_l\}\left|a_{1jk}\right|+\sum_{jk\in N_3^2}\max_{l\in\{j,k\}}\frac{R_l\left(\boldsymbol{A}\right)}{\left|a_{ll}\right|}\left|a_{1jk}\right|=\frac{1}{2}\times2+0+22\times\frac{1}{6}=\frac{14}{3}<6=a_{111}\beta_1,$$

所以张量$\boldsymbol{A}$满足本文定理1的条件，故张量$\boldsymbol{A}$为$\boldsymbol{H}$-张量。但

$$\sum_{\substack{jk\in N^2/N_3^2\\ \delta_{1jk}=0}}\left|a_{1jk}\right|+\sum_{jk\in N_3^2}\max_{l\in\{j,k\}}\frac{r_l\left(\boldsymbol{A}\right)}{\left|a_{ll}\right|}\left|a_{1jk}\right|=2+\frac{1}{2}\times22=13>12=\left|a_{111}\right|,$$

且

$$\begin{aligned}&\frac{r_1\left(\boldsymbol{A}\right)}{r_1\left(\boldsymbol{A}\right)-\left|a_{111}\right|}\left[q\left(\sum_{i_2i_3\cdots i_m\in N_0^2}\left|a_{1i_2i_3}\right|+\sum_{\substack{i_2i_3\cdots i_m\in N_2^2\\ \delta_{1i_2i_3}=0}}\left|a_{1i_2i_3}\right|\right)+\sum_{i_2i_3\in N_3^2}\max_{j\in\{i_2,i_3\}}\frac{tP_j\left(\boldsymbol{A}\right)}{\left|a_{jjj}\right|}\left|a_{1i_2i_3}\right|\right]\\&=\frac{24}{24-12}\left[\frac{1}{2}(1+0+1+0)+\frac{1}{4}(16+6)\right]=\frac{26}{2}>12=\left|a_{111}\right|.\end{aligned}$$

因此$\boldsymbol{A}$不满足文献[11]中定理1的条件且$\boldsymbol{A}$不满足文献[14]中定理2的条件。

4. 应用

基于$\boldsymbol{H}$-张量的新判定条件，下面给出判定高次多元偶次齐次多项式正定性的新结论。

引理5 [6] 设$\boldsymbol{A}=\left(a_{i_1i_2\cdots i_m}\right)$是$m$阶$n$维的实对称张量，$m$是偶数，$a_{ii\cdots i}>0\left(\forall i\in N\right)$。若$\boldsymbol{A}$是$\boldsymbol{H}$-张量，则$\boldsymbol{A}$是正定的。

根据引理5，定理1，定理2和定理3，可得到以下结论。

定理4 设m阶n维张量$\boldsymbol{A}=\left(a_{i_1i_2\cdots i_m}\right)$为偶数阶实对称张量，$a_{ii\cdots i}>0\left(\forall i\in N\right)$。若$\boldsymbol{A}$满足下列条件之一：定理1的条件；或定理2的条件；或定理3的条件，则$\boldsymbol{A}$是正定的。

例2 设6次齐次多项式

$$\begin{aligned}f\left(x\right)=\boldsymbol{A}x^6=&8x_1^6+17x_2^6+164x_3^6+93x_4^6+10x_5^6+15x_6^6-6x_1x_2^5-30x_1x_3^5\\&-6x_1x_4^5-6x_2x_3^5-6x_2x_4^5-24x_3x_4^5-20x_2^3x_3^3+6x_1^5x_4,\end{aligned}$$

其中 $\boldsymbol{A}=\left(a_{i_1 i_2 i_3 i_4 i_5 i_6}\right)$ 是一个 6 阶 6 维实对称张量，且

$$
\begin{aligned}
&a_{111111}=8, a_{222222}=17, a_{333333}=164, a_{444444}=93, a_{555555}=10, a_{666666}=15,\\
&a_{122222}=a_{212222}=a_{221222}=a_{222122}=a_{222212}=a_{222221}=-1,\\
&a_{133333}=a_{313333}=a_{331333}=a_{333133}=a_{333313}=a_{333331}=-5,\\
&a_{144444}=a_{414444}=a_{441444}=a_{444144}=a_{444414}=a_{444441}=-1,\\
&a_{233333}=a_{323333}=a_{332333}=a_{333233}=a_{333323}=a_{3333332}=-1,\\
&a_{244444}=a_{424444}=a_{442444}=a_{444244}=a_{444424}=a_{444442}=-1,\\
&a_{344444}=a_{434444}=a_{444344}=a_{444344}=a_{444434}=a_{444443}=-4,\\
&a_{222333}=a_{223233}=a_{223323}=a_{223332}=a_{232233}=a_{232323}=a_{232332}=-1,\\
&a_{233223}=a_{233232}=a_{233322}=a_{333222}=a_{332322}=a_{332232}=a_{332223}=-1,\\
&a_{323322}=a_{323232}=a_{323223}=a_{322332}=a_{322323}=a_{322233}=-1,\\
&a_{411111}=a_{141111}=a_{114111}=a_{111411}=a_{111141}=a_{111141}=1.
\end{aligned}
$$

其余的 $a_{i_1 i_2 i_3 i_4 i_5 i_6}=0$。则

$$
r_1(\boldsymbol{A})=12，\ r_2(\boldsymbol{A})=17，\ r_3(\boldsymbol{A})=44，\ r_4(\boldsymbol{A})=31，\ r_5(\boldsymbol{A})=0，\ r_6(\boldsymbol{A})=0，
$$

且 $N_1=\{2\}, N_2=\{1\}, N_3=\{3,4,5,6\}$。计算得

$$
\beta_1=\frac{1}{3}，\ M=\frac{1}{3}，\ K=\frac{1}{12}，\ R_3(\boldsymbol{A})=\frac{41}{3}，\ R_4(\boldsymbol{A})=\frac{16}{3}.
$$

当 $i=1$ 时，计算得

$$
\begin{aligned}
&M\sum_{\substack{i_2i_3i_4i_5i_6\in N_0^5\\ \delta_{1i_2i_3i_4i_5i_6}=0}}\left|a_{1i_2i_3i_4i_5i_6}\right|+\sum_{i_2i_3i_4i_5i_6\in N_2^5}\max_{j\in\{i_2,i_3,i_4,i_5,i_6\}}\left\{\beta_j\right\}\left|a_{1i_2i_3i_4i_5i_6}\right|+\sum_{i_2i_3i_4i_5i_6\in N_3^5}\max_{j\in\{i_2,i_3,i_4,i_5,i_6\}}\frac{R_j(\boldsymbol{A})}{\left|a_{jjjjjj}\right|}\left|a_{1i_2i_3i_4i_5i_6}\right|\\
&=\frac{1}{3}\times 6+0+\frac{1}{12}\times 6=\frac{5}{2}<\frac{8}{3}=\left|a_{111111}\right|\beta_1,
\end{aligned}
$$

因此 $\boldsymbol{A}$ 满足本文定理 1 的条件，由定理 4 知 $\boldsymbol{A}$ 是正定的，即 $f(\boldsymbol{x})$ 是正定的。但

$$
a_{111111}=8<12=r_1(\boldsymbol{A}),
$$

$$
a_{444444}\left(a_{111111}-r_1(\boldsymbol{A})+\left|a_{144444}\right|\right)=-465<-31=r_4(\boldsymbol{A})\left|a_{144444}\right|,
$$

且

$$
\left|a_{111111}\right|=8=8=\sum_{\substack{i_2i_3i_4i_5i_6\in N^5/N_1^5\\ \delta_{1i_2i_3i_4i_5i_6}=0}}\left|a_{1i_2i_3i_4i_5i_6}\right|+\sum_{i_2i_3i_4i_5i_6\in N_1^5}\max_{j\in\{i_2,i_3,i_4,i_5,i_6\}}\frac{R_j(\boldsymbol{A})}{\left|a_{jjjjjj}\right|}\left|a_{1i_2i_3i_4i_5i_6}\right|,
$$

因此不能用[15]中的定理 3，[16]中的定理 4 和[11]中的定理 1 判断 $\boldsymbol{A}$ 的正定性。

5. 结论

本文通过构建不同的正对角矩阵，结合不等式的放缩技巧，得到了判别 ***H***-张量的新不等式，且这些不等式只涉及到张量的元素关系，因此它们是容易计算的。作为应用，给出了偶数阶实对称张量，即高次多元偶次齐次多项式正定性的判定新方法，数值例子表明了新结论的有效性。下一步，***H***-张量的高效数值迭代判定算法将是研究的重点。

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参考文献

[1] Chang, K.C., Pearson, K. and Zhang, T. (2008) Perron-Frobenius Theorem for Nonnegative Tensors. *Communications in Mathematical Sciences*, **6**, 507-520. https://doi.org/10.4310/CMS.2008.v6.n2.a12

[2] Liu, Y.J., Zhou, G.L. and Ibrahim, N.F. (2010) An Always Convergent Algorithm for the Largest Eigenvalue of an Irreducible Nonnegative Tensor. *Journal of Computational and Applied Mathematics*, **235**, 286-292. https://doi.org/10.1016/j.cam.2010.06.002

[3] Qi, L. (2005) Eigenvalues of a Real Supersymetric Tensor. *Journal of Symbolic Computation*, **40**, 1302-1324. https://doi.org/10.1016/j.jsc.2005.05.007

[4] Lim, L.H. (2005) Singular Values and Eigenvalues of Tensors: A Variational Approach. *Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing*, Puerto Vallarta, Vol. 1, 129-132.

[5] Yang, Y.N. and Yang, Q.Z. (2011) Further Results for Perron-Frobenius Theorem for Nonnegative Tensors II. *SIAM Journal on Matrix Analysis and Applications*, **32**, 1236-1250. https://doi.org/10.1137/100813671

[6] Li, C.Q., Wang, F., Zhao, J.X., *et al*. (2014) Criterions for the Positive Definiteness of Real Supersymmetric Tensors. *Journal of Computational and Applied Mathematics*, **255**, 1-14. https://doi.org/10.1016/j.cam.2013.04.022

[7] Kannana, M.R., Mondererb, N.S. and Bermana, A. (2015) Some Properties of Strong *H*-Tensors and General *H*-Tensors. *Linear Algebra and Its Applications*, **476**, 42-55. https://doi.org/10.1016/j.laa.2015.02.034

[8] Ding, W., Qi, L. and Wei, Y. (2013) *M*-Tensors and Nonsingular *M*-Tensors. *Linear Algebra and Its Applications*, **439**, 3264-3278. https://doi.org/10.1016/j.laa.2013.08.038

[9] Wang, F. and Sun, D.S. (2016) New Criteria for *H*-Tensors and an Application. *Journal of Inequalities and Applications*, **96**, 1-12. https://doi.org/10.1186/s13660-016-1041-0

[10] Wang, F., Sun, D.S., Zhao, J.X., *et al*. (2017) New Practical Criteria for *H*-Tensors and Its Application. *Linear and Multilinear Algebra*, **65**, 269-283. https://doi.org/10.1080/03081087.2016.1183558

[11] Li, Y.T., Liu, Q.L. and Qi, L. (2017) Programmable Criteria for Strong *H*-Tensors. *Numerical Algorithms*, **74**, 199-221. https://doi.org/10.1007/s11075-016-0145-4

[12] Zhang, L.P., Qi, L. and Zhou, G.L. (2014) *M*-Tensors and Some Applications. *SIAM Journal on Matrix Analysis and Applications*, **35**, 437-542. https://doi.org/10.1137/130915339

[13] Zhang, K.L. and Wang, Y.J. (2016) An *H*-Tensor Based Iterative Scheme for Identifying the Positive Definiteness of Multivariate Homogeneous Forms. *Journal of Computational and Applied Mathematics*, **305**, 1-10. https://doi.org/10.1016/j.cam.2016.03.025

[14] Bai, D.J. and Wang, F. (2021) New Methods Based *H*-Tensors for Identifying the Positive Definiteness of Multivariate Homogeneous Forms. *AIMS Mathematics*, **6**, 10281-10295. https://doi.org/10.3934/math.2021595

[15] Qi, L. and Song, Y.S. (2014) An Even Order Symmetric *B*-Tensor Is Positive Definite. *Linear Algebra and Its Applications*, **457**, 303-312. https://doi.org/10.1016/j.laa.2014.05.026

[16] Li, C.Q. and Li, Y.T. (2015) Double *B*-Tensors and Quasi-Double *B*-Tensors. *Linear Algebra and Its Applications*, **466**, 343-356. https://doi.org/10.1016/j.laa.2014.10.027

[17] Bai, D.J. and Wang, F. (2022) New Criterions-Based *H*-Tensors for Testing the Positive Definiteness of Multivariate Homogeneous Forms. *Mathematics*, **10**, Article No. 2416. https://doi.org/10.3390/math10142416

[18] Lv, C.Q. and Ma, C.F. (2021) An Iterative Scheme for Identifying the Positive Semi-Definiteness of Even-Order Real Symmetric *H*-Tensor. *Journal of Computational and Applied Mathematics*, **392**, Article ID: 113498. https://doi.org/10.1016/j.cam.2021.113498