Advances in Applied Mathematics 应用数学进展, 2021, 10(1), 24-36
Published Online January 2021 in Hans. http://www.hanspub.org/journal/aam
https://doi.org/10.12677/aam.2021.101003

关于等熵可压缩Navier-Stokes-Poisson方程组的一些先验估计

周 杰

中央民族大学理学院，北京
Email: xhsdxzj@163.com

收稿日期：2020年12月11日；录用日期：2020年12月26日；发布日期：2021年1月13日

摘 要

关于研究等熵可压缩的**Navier-Stokes-Poisson**方程对**Cauchy**问题的弱解研究。我们需要有一些关于等熵可压缩**Navier-Stokes-Poisson**方程组的一些先验估计。本文我们主要研究带有**Poisson**项的基本能量估计、**B. Desjardin**的估计方法。

关键词

Navier-Stokes-Poisson方程，存在性，弱解

Local Weak Solution of the Barotropic Compressible Navier-Stokes-Poisson Equations

Jie Zhou

College of Science, Minzu University of China, Beijing
Email: xhsdxzj@163.com

Received: Dec. 11^{th}, 2020; accepted: Dec. 26^{th}, 2020; published: Jan. 13^{th}, 2021

文章引用: 周杰. 关于等熵可压缩Navier-Stokes-Poisson方程组的一些先验估计[J]. 应用数学进展, 2021, 10(1): 24-36. DOI: 10.12677/aam.2021.101003

Abstract

On the study of weak solutions of barotropic compressible Navier-Stokes-Poisson equation to Cauchy problem. We need some prior estimates for barotropic compressible Navier-Stokes-Poisson equations. We mainly use energy estimation, B. Desjardin's estimation method.

Keywords

Navier-Stokes-Poisson Equations, Existence, Weak Solution

1. 介绍

本文研究等熵可压缩Navier-Stokes-Poisson方程中一些先验估计。等熵可压缩流体可以用Navier-Stokes-Poisson方程来表示

$$\begin{cases} \partial_t\rho + div(\rho u) = 0. \\ \partial_t(\rho u) + div(\rho u \otimes u) + \nabla p = \mu\Delta u + (\lambda+\mu)\nabla div u + \rho\nabla\Phi. \\ \lambda\Delta\Phi = \rho. \end{cases} \tag{1.1}$$

这里

$$p = a\rho^\gamma (a > 0, \gamma > 1), \tag{1.2}$$

这里ρ, u, Φ 分别代表密度，速度，重力势能，粘性系数μ, λ满足物理要求

$$\mu > 0, \mu + \frac{3}{2}\lambda \geq 0 \tag{1.3}$$

对于Cauchy 问题，我们试图在$\mathbb{R}^3$ 寻找一个解$(\rho(x,t), u(x,t))$满足(1.1)

$$u(x,t) \to 0, \rho(x,t) \to 0, as|x| \to \infty$$

且初始条件

$$\rho|_{t=0} = \rho_0, u|_{t=0} = u_0, x \in \mathbb{R}^3. \tag{1.4}$$

满足

$$\begin{cases} 0 \le \rho_0 \in L^\infty(\Omega). \\ u_0 \in H^1(\Omega)^3. \end{cases} \tag{1.5}$$

对于等熵常粘性系数可压缩Navier-Stokes方程的数学理论已经被许多先驱工作者建立。Vaigant [1]指出局部经典解可能产生有限时间奇点。Lions [2]证明了绝热指数$\gamma > \frac{5}{3}$的三维等熵可压缩Navier-Stokes方程允许初始真空的弱解的整体存在性且Feireisl-Novotny-Petzeltova [3] 将这个条件推广到$\gamma > \frac{3}{2}$。Jiang-Zhang [4]的论文证明了在条件$\gamma > 1$下，球对称流初值较大时的全局弱解。Hoff [5] 的论文证明了多维可压缩Navier-stokes 方程的整体解。Hoff 论证中一个很好的特点是他的准则里包含了表现出某种特殊奇异性的解，见Hoff [6]。Huang [7]还验证了可压缩球对称Navier-Stokes方程弱解的存在唯一性。Cho-Kim [8]得到了密度可能包含真空的局部强经典解的存在性。Huang-Li-Xin [9]证明了三维等熵可压缩Navier-Stokes 方程具有大振荡和真空的经典解的全局适定性。对于可压缩Navier-Stokes-Poisson方程：Kobayashi-Suzuki [10]研究了可压缩Navier-Stokes-Poisson弱解的存在性。关于Navier-Stokes-Poisson系统的整体经典解，特别是对大时间行为的分析，我们在Zhang-Tan [11]的论文中,针对Cauchy 问题在压力密度函数上有不同的假设。Huang-Yan [12]文章的启发，我们研究是否存在与泊松项相似的结果。因此需要的准备工作是研究两个重要估计。

我们的主要结果如下：

定理 1.1 *假设$\gamma > 1$，存在$T_0 \in (0, +\infty)$和一个弱解(ρ, u, Φ) 满足Navier-Stokes-Poisson方程(1.1)，使得所有的$T < T_0$成立。*

$$\begin{cases} \rho \in L^\infty((0,T) \times \Omega) \cap ([0,T]; L^q(\Omega)), forallq \in [1, \infty). \\ \rho \dot{u}, \sigma(t)^{\frac{1}{2}} \nabla \dot{u} \in L^2((0,T) \times \Omega)^{3\times 3}. \\ \nabla u \in L^\infty(0,T; L^2(\Omega))^{3\times 3}. \end{cases} \tag{1.6}$$

这里的

$$\dot{u} = u_t + u \cdot \nabla u, \sigma(t) = min\{1, t\} \tag{1.7}$$

2. 预备知识

有效粘性通量F与旋度ω，被定义为如下：

$$F = (2\mu + \lambda) divu - p, \omega = \nabla \times u. \tag{2.1}$$

引理 2.1 *存在一个正常数C且仅依赖于μ 和λ 对于任何$r \in [2,6]$满足以下结果。*

$$||F||_{L^6} \le C||\nabla F||_{L^2} \tag{2.2}$$

$$||\nabla F||_{L^r}+||\nabla\omega||_{L^r}\le C||\rho\dot{u}||_{L^r} \tag{2.3}$$

$$||F||_{L^r}+||\omega||_{L^r}\le C||\rho\dot{u}||_{L^2}^{(3r-6)/(2r)}(||\nabla u||_{L^2}+||p||_{L^2})^{(6-r)/(2r)} \tag{2.4}$$

$$||\nabla u||_{L^r}\le C(||F||_{L^r}+||\omega||_{L^r}+||p||_{L^r}) \tag{2.5}$$

$$||\nabla u||_{L^r}\le C||\nabla u||_{L^2}^{(6-r)/(2r)}(||\rho\dot{u}||_{L^2}+||p||_{L^6})^{(3r-6)/(2r)} \tag{2.6}$$

3. 两个重要的先验估计

在本节中，我们假设(ρ,u,Φ)是初始问题在$[0,T]\times\Omega$上关于$(1.1)-(1.5)$的解，我们用C代表正的连续常数且只依赖于初始值和时间T。

为了证明定理1.1我们将简要描述其主要想法。
令

$$\Psi(t)=1+||\rho(\cdot,t)||_{L^\infty(\Omega)}+||p(\cdot,t)||_{L^2(\Omega)}^2+||\nabla u(\cdot,t)||_{L^2(\Omega)}^2+\sigma(t)\int_\Omega\rho(\cdot,t)\dot{u}(\cdot,t)^2dx$$

$$=1+||\rho(\cdot,t)||_{L^\infty(\Omega)}+\beta(t)+\gamma(t) \tag{3.1}$$

其中设

$$\beta(t)=||p(\cdot,t)||_{L^2(\Omega)}^2+||\nabla u(\cdot,t)||_{L^2(\Omega)}^2$$

$$\gamma(t)=\sigma(t)\int_\Omega\rho(\cdot,t)\dot{u}(\cdot,t)^2dx \tag{3.2}$$

第一步，我们有基本能量估计带$Poisson$项

引理 3.1 基本能量估计：其中对任何$0\le t\le T$成立.

$$\int_\Omega(\rho\frac{|u|^2}{2}+\frac{a\rho^\gamma}{\gamma-1}+\frac{\lambda}{2}|\nabla\Phi|^2)+\int_0^t\int_\Omega(\mu|\nabla u|^2+(\mu+\lambda)|divu|^2)dxd\tau$$

$$\le\int_\Omega(\rho_0\frac{|u_0|^2}{2}+\frac{a\rho_0^\gamma}{\gamma-1}+\frac{\lambda}{2}|\nabla\Phi_0|^2) \tag{3.3}$$

证明 3.1 在动量方程两端乘以检验函数u且在$(0,t)\times\Omega$上积分,我们得到.

$$\int_0^t\int_\Omega \rho u_t u dx d\tau+\int_0^t\int_\Omega \rho u\cdot\nabla u\cdot u dx d\tau+\int_0^t\int_\Omega \nabla p u dx d\tau$$

$$-\int_0^t\int_\Omega \mu\Delta u\cdot u dx d\tau-\int_0^t\int_\Omega (\mu+\lambda)\nabla divu\cdot u dx d\tau-\int_0^t\int_\Omega \rho\nabla\Phi\cdot u dx d\tau=0$$

对每一项进行处理，利用分部积分法:

$\int_\Omega \rho u_t u dx=\int_\Omega \rho(\frac{|u|^2}{2})_t dx=\frac{d}{dt}\int_\Omega \rho\frac{|u|^2}{2}dx-\int_\Omega \rho_t\frac{|u|^2}{2}dx$

接着:

$\int_\Omega \rho(u\cdot\nabla)u\cdot u dx=\int_\Omega \rho(u\cdot\nabla)\frac{|u|^2}{2}dx=\int_\Omega div(\frac{|u|^2}{2}u)dx-\int_\Omega div(\rho u)\frac{|u|^2}{2}dx$

$\int_\Omega u\nabla p dx=\int_\Omega a\gamma\rho u\cdot\nabla\frac{\rho^{\gamma-1}}{\gamma-1}dx=\frac{\gamma}{\gamma-1}\int_\Omega a\rho_t\rho^{\gamma-1}dx=\frac{d}{dt}\int_\Omega a\frac{\rho^\gamma}{\gamma-1}dx$

再次使用分部积分法则:

$\int_\Omega \Delta u\cdot u dx=-\int_\Omega |\nabla u|^2 dx$

$\int_\Omega(\mu+\lambda)\nabla divu\cdot u dx=-\int_\Omega(\mu+\lambda)|divu|^2 dx$

利用分部积分与质量方程得到:

$\int_\Omega \rho\nabla\Phi\cdot u dx=-\int_\Omega \Phi\cdot div(\rho u)=\int_\Omega \Phi\cdot\rho_t dx=\int_\Omega \Phi\lambda\Delta\Phi_t dx=-\frac{d}{dt}\int_\Omega \lambda\frac{|\nabla\Phi|^2}{2}dx$

所有式子相加得到:

$$\int_\Omega(\rho\frac{|u|^2}{2}+\frac{a\rho^\gamma}{\gamma-1}+\frac{\lambda}{2}|\nabla\Phi|^2)dx+\int_0^t\int_\Omega(\mu|\nabla u|^2+(\mu+\lambda)|divu|^2)dx d\tau$$

$$\leq\int_\Omega(\rho_0\frac{|u_0|^2}{2}+\frac{a\rho_0^\gamma}{\gamma-1}+\frac{\lambda}{2}|\nabla\Phi_0|^2)dx=E_0$$

第二步：我们回忆在Desjardins [13]中对$\beta(t)$ 的估计，利用其想法。

引理 3.2 在论文 [13]中，这里存在一个单调递增的光滑函数$\Psi_0(x)$使得：

$$\beta(t)+\int_0^t\int_\Omega \rho|\dot{u}|^2 \le C+C\int_0^t \Psi_0(||\rho||_{L^\infty})\delta(s)(1+\beta(s))^3ds \tag{3.4}$$

这里：正如(3.2)定义

$$\beta(t)=||p(\cdot,t)||^2_{L^2(\Omega)}+||\nabla u(\cdot,t)||^2_{L^2(\Omega)}$$

证明 3.2 在动量方程两端乘以检验函数u_t且在$(0,t)\times\Omega$上积分，我们得到。

$$\int_0^t\int_\Omega \rho|\partial_t u|^2dxds+\frac{1}{2}\int_\Omega(\mu|\nabla u(t,x)|^2+(\mu+\lambda)|divu(t,x)|^2)dx$$

$$+\int_0^t\int_\Omega \nabla p(\rho)\cdot\partial_t udxds\le C\int_\Omega|\nabla u_0|^2dx+\int_0^t\int_\Omega|\sqrt{\rho}\partial_t u|_{L^2(\Omega)}\cdot(|\sqrt{\rho}(u\cdot\nabla)u|_{L^2(\Omega)}+|\sqrt{\rho}\nabla\Phi|_{L^2(\Omega)})ds$$

现在，利用连续性方程得到

$\int_\Omega \partial_t u\cdot\nabla p(\rho)dx=-\frac{d}{dt}\int_\Omega p(\rho)divudx+\int_\Omega \partial_t p(\rho)divudx,$

$=-\frac{d}{dt}\int_\Omega p(\rho)divudx+\frac{1}{\lambda+2\mu}\frac{d}{dt}\int_\Omega \kappa(\rho)dx+\frac{1}{\lambda+2\mu}\int_\Omega p(\rho)u\cdot\nabla Fdx$

$+\frac{1}{(\lambda+2\mu)^2}\int_\Omega p(\rho)^2(\rho p'(\rho)-p(\rho))dx-\frac{1}{(\lambda+2\mu)^2}\int_\Omega(\rho p'(\rho)-p(\rho))F^2dx$

这里

$$\kappa(s)=p(s)^2-\frac{s}{2}\int_0^s\frac{p(z)}{z^2}dz$$

结果，可以得到：

$$\int_0^t\int_\Omega \rho|\partial_t u|^2dxds+\frac{1}{2}\int_\Omega(\mu|\nabla u(t,x)|^2+(\mu+\lambda)|divu(t,x)|^2)dx$$

$$+\frac{1}{(\lambda+2\mu)^2}\int_\Omega p(\rho)^2(\rho p'(\rho)-p(\rho))dx+\frac{1}{2\mu+\lambda}\int_\Omega \kappa(\rho)dx$$

$$\le\int_\Omega p(\rho(t,x))divu(t,x)dx-\int_\Omega p(\rho_0)divu_0dx+C$$

$$+C\int_0^t\int_\Omega(|\rho p'(\rho)-p(\rho)|F^2+|p(\rho)u||\nabla F|+|\sqrt{\rho}u\cdot\nabla u|^2+|\sqrt{\rho}\cdot\nabla\Phi|^2)dxds$$

利用$Gagliardo-Nirenberg$的不等式：

$$||f||^2_{L^4(f)}\le C||f||^{\frac{1}{2}}_{L^2(\Omega)}||\nabla f||^{\frac{3}{2}}_{L^2(\Omega)}.$$

我们观察到当$\varepsilon > 0$取得足够小有：

$\int_\Omega p(\rho(t,x))|divu(t,x)|dx$

$$\leq \frac{1}{2}(\lambda+\mu-\varepsilon)\int_\Omega (divu(t,x))^2dx + \frac{1}{2}\frac{1}{(\lambda+\mu-\varepsilon)}\int_\Omega p(\rho(t,x))^2dx$$

这里$p = p_{\alpha,\gamma}$并且$\gamma > 1$,有$2\gamma - \frac{3}{2} > \frac{1}{2}$，我们有$\kappa(t) \geq \delta p(t)^2$对于一些$\delta > \frac{1}{2}$。一般得，我们可以推出：存在一个$\upsilon \in C^0(R^+) \cap C^\infty(0,\infty)$其中$\upsilon(t) \geq c_1 t^{2\gamma} c_1$ 为正数，t 足够大，有：

$$\int_0^t\int_\Omega (\rho|\partial_t u|^2 + p(\rho)^2(\rho p'(\rho) - p(\rho)))dxds + \int_\Omega |\nabla u(t,x)|^2dx + \int_\Omega \upsilon(\rho(t,x))dx$$

$$\leq C + C\int_0^t\int_\Omega (|p(\rho)u\nabla F| + |\rho p'(\rho) - p(\rho)|F^2 + |\sqrt{\rho}(u\cdot\nabla)u|^2 + |\sqrt{\rho}\nabla\Phi|^2)dxds$$

$$\leq C + C\int_0^t (|\frac{p(\rho(s,.))}{\sqrt{\rho(s,.)}}|_{L^\infty(\Omega)}|\sqrt{\rho}u(s,.)|_{L^2(\Omega)}|\nabla F(s,.)|_{L^2(\Omega)} + |p'(\rho) - p(\rho)|_{L^\infty(\Omega)}|F(s,.)|^2_{L^2(\Omega)}$$

$$+|\sqrt{\rho}u(s,.)|^2_{L^4(\Omega)}|\nabla u(s,.)|^2_{L^4(\Omega)} + |\rho(s,.)|_{L^\infty(\Omega)}|\nabla\Phi|^2_{L^2(\Omega)})ds$$

我们想要得到Pu，F有界，假设ρ在$L^\infty(\Omega)$先验有界，利用上式的不等式有$(t > 0)$。这里$Q = \nabla\triangle^{-1}div$，$P = I - Q$。

$$\int_0^t\int_\Omega \rho|\partial_t u|^2dxds + \int_\Omega |\nabla u(t,x)|^2dx + \int_\Omega |p(\rho(t,x))|^2dx \leq C + C\int_0^t |g(\rho(s,.))|_{L^\infty(\Omega)}$$

$$\times|\nabla F(s,.)|_{L^2(\Omega)} + |h(\rho(s,.))|_{L^\infty(\Omega)}|\nabla u(s,.)|^2_{L^2(\Omega)} + |\sqrt{\rho}u(s,.)|^2_{L^4(\Omega)}(|\nabla Pu(s,.)|^2_{L^4(\Omega)} + |RF(s,.)|^2_{L^4(\Omega)}$$

$$+|Rp(\rho(s,.))|^2_{L^4(\Omega)}) + |\rho(s,.)|_{L^\infty(\Omega)}|\nabla\Phi|^2_{L^2(\Omega)})ds + |i(\rho(s,.))|_{L^\infty(\Omega)}ds \quad (3.5)$$

注：$g(s) = p(s)/\sqrt{s}, h(s) = |sp'(s) - p(s)|$和$i(s) = h(s)p(s)^2, R_i\Delta^{-1/2}\partial_i$. 接下来，我们再使用一次动量方程

$$\mu\Delta Pu = P(\rho\partial_t u) + P(\rho u\cdot\nabla u) - P(\rho\nabla\Phi)$$

$$\nabla F = Q(\rho\partial_t u) + Q(\rho u\cdot\nabla u) - Q(\rho\nabla\Phi)$$

因此，有

$$|\nabla F|^2_{L^2(\Omega)} + |\Delta Pu|^2_{L^2(\Omega)} \leq C|\rho(s,.)|_{L^\infty(\Omega)}(|\sqrt{\rho}\partial_t u(s,.)|^2_{L^2(\Omega)} + |\sqrt{\rho}u\cdot\nabla u(s,.)|^2_{L^2(\Omega)}$$

$$+|\rho(s,.)|_{L^\infty(\Omega)}|\nabla\Phi|^2_{L^2(\Omega)}$$

利用$Gagliardo-Nirenberg$的不等式：

$$||f||^2_{L^4(f)} \le C||f||^{\frac{1}{2}}_{L^2(\Omega)}||\nabla f||^{\frac{3}{2}}_{L^2(\Omega)}.$$

我们可以推演出来：

$$|\nabla Pu(s,.)|^2_{L^4(\Omega)} + |RF(s,.)|^2_{L^4(\Omega)} \le C(|\nabla u(s,.)|_{L^2(\Omega)} + |p(\rho(s,.))|_{L^2(\Omega)})^{\frac{1}{2}}$$
$$\times(|\Delta Pu(s,.)|_{L^2(\Omega)} + |\nabla F(s,.)|_{L^2(\Omega)})^{\frac{3}{2}}$$

因此，我们得到：

$$|g(\rho(s,.))|_{L^\infty(\Omega)}|\nabla F(s,.)|_{L^2(\Omega)} + |\sqrt{\rho}u(s,.)|^2_{L^4(\Omega)}(|\nabla Pu(s,.)|^2_{L^4(\Omega)} + |RF(s,.)|^2_{L^4(\Omega)})$$
$$\le \varepsilon(|\Delta Pu(s,.)|^2_{L^2(\Omega)} + |\nabla F(s,.)|^2_{L^2(\Omega)}) + \frac{C}{\varepsilon}|\sqrt{\rho}u(s,.)|^4_{L^4(\Omega)}$$
$$\times(|\nabla u(s,.)|^2_{L^2(\Omega)} + |p(\rho(s,.))|^2_{L^2(\Omega)}) + \frac{C}{\varepsilon}\Psi(|\rho(s,.)|_{L^\infty(\Omega)}) \tag{3.6}$$

观察：

$$|\sqrt{\rho}u(s,.)|^4_{L^4(\Omega)} \le C|\sqrt{\rho}u(s,.)|_{L^2(\Omega)}|\sqrt{\rho}u(s,.)|^3_{L^6(\Omega)}$$
$$\le C|\rho(s,.)|^{\frac{3}{2}}_{L^\infty(\Omega)}(1+|\nabla u(s,.)|_{L^2(\Omega)})^3 \tag{3.7}$$

最后，我们得到：

$$\int_0^t \rho|\partial_t u|^2 dxds + \int_\Omega |\nabla u(t,x)|^2 dx + \int_\Omega |p(\rho(t,x))|^2 dx$$
$$\le C+C\int_0^t \Psi(|\rho(s,.)|_{L^\infty(\Omega)})((1+|\nabla u(s,.)|^2_{L^2(\Omega)})^3(1+|\nabla u(s,.)|^2_{L^2(\Omega)}+|p(\rho(s,.))|^2_{L^2(\Omega)})+|\nabla\Phi|^2_{L^2(\Omega)})ds$$

让我们定义：β和δ。

$$\beta(t) = |\nabla u(t,.)|^2_{L^2(\Omega)} + |p(\rho(t,.))|^2_{L^2(\Omega)}$$
$$\delta(s) = 1 + |\nabla u(s,.)|^2_{L^2(\Omega)} + |\nabla\Phi(s,.)|^2_{L^2(\Omega)})$$

得到：

$$\int_0^t (\rho|\partial_t u|^2_{L^2(\Omega)} + |\nabla F|^2_{L^2(\Omega)} + |\Delta Pu|^2_{L^2(\Omega)})ds + \beta(t)$$

$$\leq C + C\int_0^t \Psi_0|\rho(s,.)|_{L^\infty(\Omega)})\delta(s)(1+\beta(s))^3 ds$$

这里：$\Psi_0(|\rho(s,.)|_{L^\infty(\Omega)}) = max\{(\gamma-1)|\rho|^{\gamma}_{L^\infty}, |\rho|_{L^\infty}, |\rho|^{2\gamma-1}_{L^\infty}, |\rho(s,.)|^{\frac{3}{2}}_{L^\infty}\}$

Ψ 代表非负递增的函数.事实上，在本文中，它和常数C一样的使用方法，以简化概念。在动量方程两端乘以检验函数$\sigma^m\dot{u}$且在Ω上积分得。

$$\int_\Omega \sigma^m \rho|\dot{u}|^2 dx = \int_\Omega -\sigma^m \dot{u}\nabla p + \mu\cdot\nabla u\cdot\sigma^m\cdot\dot{u} + (\mu+\lambda)\nabla divu\cdot\sigma^m\dot{u} + \sigma^m\dot{u}\cdot\rho\nabla\Phi dx \tag{3.8}$$

利用分部积分以及$\dot{u} = u_t + u\cdot\nabla u$

$-\int_\Omega \sigma^m \dot{u}\nabla p = \int_\Omega \sigma^m div\dot{u}p = \int_\Omega \sigma^m divu_t p + \sigma^m div(u\cdot\nabla u)pdx$

$= (\int_\Omega \sigma^m divu\cdot pdx)_t - \int_\Omega m\sigma^{m-1}\sigma' divu\cdot pdx - \int_\Omega \sigma^m divu\cdot p_t dx$

$+\int_\Omega \sigma^m div(u\cdot\nabla u)pdx$

利用等式$p_t + div(pu) + (\gamma-1)pdivu = 0$

$-\int_\Omega \sigma^m divu\cdot p_t dx = \int_\Omega \sigma^m divu(div(pu) + (\gamma-1)pdivu)dx$

$= \int_\Omega \sigma^m \partial_i u^i\cdot(\partial_j(pu^j)) + \sigma^m(\gamma-1)p(divu)^2 dx$

$= -\int_\Omega \sigma^m \partial_j\partial_i u^i pu^j + \sigma^m(\gamma-1)p(divu)^2 dx$

$\int_\Omega \sigma^m div(u\cdot\nabla u)pdx = \int_\Omega \sigma^m\partial_j(u_i\cdot\partial_i u^j)pdx = \int_\Omega \sigma^m\partial_j u_i\cdot\partial_i u^j p + u_i\partial_i\partial_j u^j Pdx$

以及

$-\int_\Omega \sigma^m\dot{u}\nabla p = (\int_\Omega \sigma^m divu\cdot pdx)_t - \int_\Omega m\sigma^{m-1}\sigma' divu\cdot pdx + \sigma^m(\gamma-1)p(divu)^2 dx$

$+\int_\Omega \sigma^m\partial_j u_i\cdot\partial_i u^j p$

利用$Holder$不等式

$-\int_\Omega \sigma^m \dot{u}\nabla p \le (\int_\Omega \sigma^m divu\cdot pdx)_t + m\sigma^{m-1}||p||_{L^2}||\nabla u||_{L^2} + \sigma^m||p||_{L^2}||\nabla u||_{L^2}$

$\le (\int_\Omega \sigma^m divu\cdot pdx)_t + C||\nabla u||_{L^2}^2 + C||p|^2||_{L^\infty}$

接着：

$\int_\Omega \mu\sigma^m \Delta u\cdot \dot{u}dx = \int_\Omega \mu\sigma^m\Delta u\cdot(u_t + u\cdot\nabla u)dx$

$= \int_\Omega \mu\sigma^m\Delta u\cdot u_t dx + \int_\Omega \mu\sigma^m\Delta u\cdot u\cdot\nabla u dx$

令$\int_\Omega \mu\sigma^m\Delta u\cdot u\cdot\nabla udx = I$

$= -\int_\Omega \mu\sigma^m\frac{|\nabla u|_t^2}{2} + I$

$= -\int_\Omega (\mu\sigma^m\frac{|\nabla u|^2}{2}dx)_t + \int_\Omega \mu m\sigma^{m-1}\sigma'\frac{|\nabla u|^2}{2}dx + I$

$I = \int_\Omega \mu\sigma^m\partial_{ii}u^j(u^k\cdot\partial_k u^j)dx$

利用分部积分可以得到

$I = -\int_\Omega \mu\sigma^m\partial_i u^j\partial_i u^k\cdot\partial_k u^j dx - \int_\Omega \mu\sigma^m\partial_i u^j u^k\cdot\partial_{ik}u^j dx$

令$M = \int_\Omega \mu\sigma^m\partial_i u^j u^k\cdot\partial_{ik}u^j dx$

$I = -\int_\Omega \mu\sigma^m\partial_i u^j\partial_i u^k\cdot\partial_k u^j dx + M + \int_\Omega \mu\sigma^m\partial_i u^j\partial_k u^k\cdot\partial_i u^j dx$

$I \le C\cdot\sigma^m\int_\Omega |\nabla u|^3 dx$

$\int_\Omega \mu\sigma^m\Delta u\cdot\dot{u}dx \le -\frac{\mu}{2}\sigma^m(||\nabla u||_{L^2}^2)_t + Cm\sigma^{m-1}\sigma'||\nabla u||_{L^2}^2 + C\int_\Omega \sigma^m|\nabla u|^3 dx$

利用$Holder$不等式

$$\int_\Omega \sigma^m\dot{u}\cdot\rho\nabla\Phi dx \le ||\rho||_{L^\infty}^{\frac{1}{2}}||\sqrt{\rho}\dot{u}||_{L^2}||\nabla\Phi||_{L^2} \le \varepsilon||\sqrt{\rho}\dot{u}||_{L^2}^2 + \frac{C}{\varepsilon}||\rho||_{L^\infty}||\nabla\Phi||_{L^2}^2$$

最后得到：

$\Rightarrow \int_\Omega \sigma^m \rho|\dot{u}|^2 dx + [\sigma^m(\frac{\mu}{2}||\nabla u||_{L^2}^2 + \frac{\mu+\lambda}{2}||divu||_{L^2}^2 - \int_\Omega divup]_t$

$\leq (C + Cm\sigma^{m-1}\sigma^{'})||\nabla u||_{L^2}^2 + Cm\sigma^{m-1}\sigma^{'} C_0 + C\int_\Omega |\nabla u|^3 dx + C||p||_{L^\infty}^2 + \frac{C}{\varepsilon}||\rho||_{L^\infty}||\nabla \Phi||_{L^2}^2$

$\int_\Omega divup \leq ||p||_{L^\infty}^2 + \frac{\lambda+\mu}{4}||divu||_{L^2}^2$

令$L = [\sigma^m(\frac{\mu}{2}||\nabla u||_{L^2}^2 + \frac{\mu+\lambda}{2}||divu||_{L^2}^2 - \int_\Omega divup]_t$

$L \geq \frac{\sigma^m \mu}{2}||\nabla u||_{L^2}^2 + \frac{\mu+\lambda}{2}||divu||_{L^2}^2 - \frac{\lambda+\mu}{4}||divu||_{L^2}^2 - ||p||_{L^\infty}^2$

对于$\int_\Omega |\nabla u|^3 dx$这一项

由*(2.6)*

$$||\nabla u||_{L^r} \leq C||\nabla u||_{L^2}^{(6-r)/(2r)}(||\rho \dot{u}||_{L^2} + ||p||_{L^6})^{(3r-6)/(2r)}.$$

可以推导：

$||\nabla u||_{L^3}^3 \leq ||\nabla u||_{L^2}^{\frac{3}{2}}(||\rho \dot{u}||_{L^2}^{\frac{3}{2}} + ||p||_{L^6}^{\frac{3}{2}})$

$\leq \varepsilon||\rho \dot{u}||_{L^2}^2 + \frac{C}{\varepsilon}||\nabla u||_{L^2}^6 + \varepsilon||\nabla u||_{L^2}^2 + C||p||_{L^6}^6$

令：

$L = \sigma^m(\frac{\mu}{2}||\nabla u||_{L^2}^2 + \frac{\mu+\lambda}{2}||divu||_{L^2}^2 - \int_\Omega divup)_t$

$L \geq \frac{\sigma^m \mu}{2}||\nabla u||_{L^2}^2 + \frac{\mu+\lambda}{2}||divu||_{L^2}^2 - \frac{\lambda+\mu}{4}||divu||_{L^2}^2 - ||p||_{L^\infty}^2$

总结以上推论，我们得到结果：

$$\int_0^t (\rho|\partial_t u|_{L^2(\Omega)}^2 + |\sqrt{\rho} u \cdot \nabla u(s,.)|_{L^2(\Omega)}^2 + |\nabla F|_{L^2(\Omega)}^2 + |\Delta P u|_{L^2(\Omega)}^2) ds + \beta(t)$$

$$\leq C + C\int_0^t \Psi_0(||\rho(s,.)||_{L^\infty})\delta(s)(1+\beta(s))^3 ds$$

参考文献

[1] Vaigant, V.A. (1993) An Example of the Nonexistence "in the Large" with Respect to Time of a Solution to the Navier-Stokes Equations of a Compressible Viscous Barotropic Fluid. *Dinamika Sploshn. Sredy*, **107**, 39-48.

[2] Lions, P.L. (1998) Mathematical Topics in Fluid Mechanics. Vol. 2. Compressible Models. Oxford University Press, New York.

[3] Feireisl, E., Novotny, A. and Petzeltova, H. (2001) On the Existence of Globally Defined Weak Solution to the Navier-Stokes Equations. *Journal of Mathematical Fluid Mechanics*, **3**, 358-392. https://doi.org/10.1007/PL00000976

[4] Jiang, S. and Zhang, P. (2001) On Spherically Symmetric Solutions of the Compressible Isentropic Navier-Stokes Equations. *Communications in Mathematical Physics*, **215**, 559-581. https://doi.org/10.1007/PL00005543

[5] Hoff, D. (1995) Global Solutions of the Navier-Stokes Equations for Multidimensional Compressible Flow with Discontinuous Initial Data. *Journal of Differential Equations*, **120**, 215-254. https://doi.org/10.1006/jdeq.1995.1111

[6] Hoff, D. (2002) Dynamics of Singularity Surfaces for Compressible, Viscous Flows in Two Space Dimensions. *Communications on Pure and Applied Mathematics*, **55**, 1365-1407. https://doi.org/10.1002/cpa.10046

[7] Huang, X.D. (2017) Existence and Uniqueness of Weak Solutions of the Compressible Spherical Symmetric Navier-Stokes Equations. *Journal of Differential Equations*, **262**, 1341-1358. https://doi.org/10.1016/j.jde.2016.10.013

[8] Cho, Y. and Kim, H. (2006) On Classical Solutions of the Compressible Navier-Stokes Equations with Nonnegative Initial Densities. *Manuscripta Mathematica*, **120**, 91-129. https://doi.org/10.1007/s00229-006-0637-y

[9] Huang, X.D., Li, J. and Xin, Z.P. (2012) Global Well-Posedness of Classical Solutions with Large Oscillations and Vacuum to the Three-Dimensional Isentropic Compressible Navier-Stokes Equations. *Communications on Pure and Applied Mathematics*, **65**, 549-585. https://doi.org/10.1002/cpa.21382

[10] Kobayashi, T. and Suzuki, T. (2004) Weak Solutions to the Navier-Stokes-Poisson Equation. *Advances in Mathematical Sciences and Applications*,**18**, 141-168.

[11] Zhang, Y.H. and Tan, Z. (2007) On the Existence of Solutions to the Navier-Stokes-Poisson Equations of a Two-Dimensional Compressible Flow. *Mathematical Methods in the Applied Sciences*, **30**, 305-329. https://doi.org/10.1002/mma.786

[12] Huang, X.D. and Wei, Y. (2020) Local Weak Solution of the Isentropic Compressible Navier-Stokes Equations. Preprint.

[13] Desjardins, B. (1997) Regularity of Weak Solution of Compressible Isentropic Navier-Stokes Equations. *Communications in Partial Differential Equations*, **22**, 997-1008.
https://doi.org/10.1080/03605309708821291