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Hom-Hopf代数上的交叉余积

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摘 要

为了研究 **Hom**-交叉余积，通过运用类比的思想方法，定义了 **Hom**-交叉余积，并通过计算给出了若干 **Hom**-交叉余积的相关性质。作为应用，得到了 **Hom**-交叉余积构成 **Hom**-余代数的充要条件，还有 **Hom**-交叉余积和 **Hom smash** 积形成 **Hom**-双代数的充分必要条件。

关键词

Hom-余代数，Hom-双代数，Hom Smash 积，Hom-双模代数

Crossed Coproducts over Hom-Hopf Algebras

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Abstract

In order to study the Hom-crossed-coproduct, we define the Hom-crossed-coproduct by analogy, and give some properties of Hom-crossed-coproduct by calculation. As an application, we obtain the necessary and sufficient conditions for Hom-crossed-coproduct to form Hom-coalgebra, and the necessary and sufficient conditions for Hom-crossed-coproduct and Hom-smash-product to form Hom-bialgebra.

Keywords

Hom-Coalgebra, Hom-Bialgebra, Hom Smash Product, Hom-Bimodule Algebra

1. 介绍

Hom-型代数最早与 q-形变的 Witt 代数 [1]和 q-形变的 Virasoro 代数 [2]相关, 主要应用于物理学的共场理论. 2008 年, Makhlouf 和 Silvestrov 在文献 [3]中研究拟李代数时引入了 Hom-代数的概念. Hom-代数的引入实际上是推广了代数的概念, 把代数中的结合性法则变成了 Hom-结合性条件, 即 $\alpha(a)(bc) = (ab)\alpha(c)$. 随着 Hom-代数研究的深入, Hom-代数及其相关结构变得相当受欢迎. 一些学者 [4–6]继续引入了 Hom-余代数, Hom-双代数和 Hom-Hopf 代数等. 此外, Yau [7]把一些作用和余作用考虑到这些 Hom-结构中: Hom-模, Hom-余模, Hom-Hopf 模和 Hom-模代数, 并在文献 [5]中研究了 Hom-Hopf 模的基本结构定理.

交叉余积一直是 Hopf 代数中重要的研究对象, 起源于群论中. 1997 年, Wang S.H., Jiao Z.M. 和 Zhao W. Z.在文献 [8]中独立地把群交叉余积的理论推广到了 Hopf 代数上, 给出了交叉余积的定义并研究了其性质. 并且研究了交叉余积是余代数的充要条件. 交叉余积是 smash 积的推广, 其定义如下: 设 H 是 Hopf 代数和 C 是余代数, H 在 C上有左弱余作用, $\rho: C \to H \otimes C, \rho(c) = \sum c_{(-1)} \otimes c_{(0)}$.$\psi: C \to H \otimes H$是一个线性映射. $\psi(c) = \sum c' \otimes c''$.则 C 与 H 的交叉余积定义为向量空间 $C \otimes_\psi H$及其余乘法

$$\Delta(c \otimes h) = \sum c_1 \otimes c_{2(-1)} c_3{}' h_1 \otimes c_{2(0)} \otimes c_3{}'' h_2$$

其中, 对任意 $a \in C,\ h \in H$.

本文借助 Hom-双代数, 定义了 Hom-双代数上的交叉余积, 并讨论其性质, 给出了 Hom 交叉余积和 Hom smash 积形成 Hom-双代数的充分必要条件.

2. 预备知识

本文所涉及到的向量空间、张量积、模、线性映射都是在数域 k 上进行研究. 文中沿用 Sweedler 关于余代数余乘和余模的记法. 对于余代数 C, 任意的 $c \in C$, 我们记它的余乘为 $\Delta(c) = c_1 \otimes c_2$. 关于右 C-余模 M, 任意的 $m \in M$, 余作用记为 $\rho(m) = m_0 \otimes m_1$. 另外, 设 I 是线性空间 V 上的恒等映射, α, β 是可逆映射.

定义 1 [7] 设 V 是线性空间, $\mu : V \otimes V \to V, x \otimes y \mapsto xy, \alpha : V \to V$ 都是线性映射. 如果对任意的 $x, y, z \in V$, 满足 Hom-结合条件:

$$\mu(\alpha(x) \otimes \mu(y \otimes z)) = \mu(\mu(x \otimes y) \otimes \alpha(z)),$$

那么称三元组 (V, μ, α) 是 Hom-代数. 如果有线性映射 $\eta : k \to V$ 满足

$$\mu(\eta(1) \otimes I(x)) = \alpha(x) = \mu(I(x) \otimes \eta(1)),$$

那么称 V 是有单位元的 Hom-代数.

设 (V, μ, α) 和 (V', μ', α') 都是 Hom-代数. 如果线性映射 $f : V \to V'$ 满足

$$f \circ \alpha = \alpha' \circ f, \quad \mu' \circ (f \otimes f) = f \circ \mu,$$

那么称线性映射 $f : V \to V'$ 是 Hom-代数同态.

定义 2 [7] 设 V 是线性空间, $\Delta : V \to V \otimes V, \beta : V \to V$ 都是线性映射. 如果

$$(\beta \otimes \Delta) \circ \Delta = (\Delta \otimes \beta) \circ \Delta,$$

那么称三元组 (V, Δ, β) 是 Hom-余代数. 如果有线性映射 $\varepsilon : V \to k$, 满足

$$(I \otimes \varepsilon) \circ \Delta = \beta = (\varepsilon \otimes I) \circ \Delta,$$

那么称 V 是有余单位元的 Hom-余代数.

设 (V, Δ, β) 和 (V', Δ', β') 都是 Hom-余代数. 如果线性映射 $f : V \to V'$ 满足

$$f \circ \beta = \beta' \circ f, \quad \Delta' \circ f = (f \otimes f) \circ \Delta,$$

那么称线性映射 $f : V \to V'$ 是 Hom-余代数同态.

定义 3 [7] 设

1) (V,μ,α,η) 是有单位元 η 的 Hom-代数;

2) $(V,\Delta,\alpha,\varepsilon)$ 是有余单位元 ε 的 Hom-余代数;

3) 线性映射 Δ 和 ε 都是 Hom-代数同态, 即

$$\begin{cases} \Delta(e_1) &= e_1\otimes e_1, \quad e_1=\eta(1), \\ \Delta(\mu(x\otimes y)) &= \Delta(x)\Delta(y), \\ \varepsilon(e_1) &= 1, \\ \varepsilon(\mu(x\otimes y)) &= \varepsilon(x)\varepsilon(y), \\ \Delta(\alpha(x)) &= \alpha(x_1)\alpha(x_2), \\ \varepsilon\circ\alpha(x) &= \varepsilon(x). \end{cases}$$

则称六元组 $(V,\mu,\alpha,\eta,\Delta,\varepsilon)$ 是 Hom-双代数.

定义 4 [7] 设 (A,α) 是 Hom-代数, M 是线性空间, 且 $\beta: M\to M$ 是线性映射. 如果有线性映射 $\varphi: A\otimes M\to M$, $a\otimes m\mapsto a\cdot m$, 对任意的 $a,b\in A$, $m\in M$, 满足

$$\alpha(a)\cdot(b\cdot m)=(ab)\cdot\beta(m), \quad 1\cdot m=\beta(m),$$

$$\beta(a\cdot m)=\alpha(a)\cdot\beta(m),$$

那么称 (M,β) 是左 (A,α)-Hom-模.

类似地, 我们可以定义右 (A,α)-Hom-模. 设 (M,β_M) 和 (N,β_N) 是两个左 (A,α)-Hom-模. 如果对任意的 $a\in A, m\in M$, 线性映射 $f: M\to N$ 满足

$$f(a\cdot m)=a\cdot f(m), \quad f\circ\beta_M=\beta_N\circ f,$$

那么称线性映射 $f: M\to N$ 是左 A-模同态.

定义 5 [7] 设 (H,β) 是 Hom-双代数, (A,α) 是 Hom-代数. 如果有线性映射 $\rho: H\otimes A\to A$, $h\otimes a\otimes\mapsto h\cdot a$, 对任意的 $h,g\in H$, $a\in A$, 满足

$$(hg)\cdot\alpha(a)=\beta(h)\cdot(g\cdot a), \quad 1\cdot a=\alpha(a),$$

$$\alpha(h\cdot a)=\beta(h)\cdot\alpha(a),$$

$$\beta^2(h)\cdot(ab)=(h_1\cdot a)(h_2\cdot b), \quad h\cdot 1=\varepsilon_H(h)1,$$

那么称 (A,α) 是左 (H,β)-Hom-模代数.

3. Hom-Hopf 代数上的交叉余积

主要介绍了 Hom-交叉余积的定义, 给出了 Hom-交叉余积构成余代数的充要条件, 还给出了

Hom-交叉余积和 Hom smash 积形成 Hom-双代数的充分必要条件.

定义 1 设 (H,β) 是 Hom-双代数, (C,α) 是 Hom-余代数, 如果存在线性映射 $\rho: C\to H\otimes C, \rho(c)=\sum c_{(-1)}\otimes c_{(0)}$, 对任意的 $c\in C$, 满足

1) $\sum\epsilon(c_{(-1)})c_{(0)}=\alpha(c)$,

2) $\sum c_{(-1)}\epsilon(c_{(0)})=\epsilon(c)1_H$,

3)

$$\sum\beta^2(c_{(-1)})\otimes c_{(0)1}\otimes c_{(0)2}=\sum c_{1(-1)}c_{2(-1)}\otimes c_{1(0)}\otimes c_{2(0)}, \tag{1}$$

其中 $\Delta(c)=\sum c_1\otimes c_2$.

那么称为 (H,β)在 (C,α)上的左弱 Hom-余作用.

定义 2 设 (H,β)是 Hom-双代数, (C,α)是 Hom-余代数. (H,β)在 (C,α)上的左弱 Hom-余作用 $\rho: C\to H\otimes C$, 记 $\rho(c)=\sum c_{(-1)}\otimes c_{(0)}$, (C,α)为左 H-模 Hom-余代数. $\psi: C\to H\otimes H$, 是一个线性映射. $\psi(c)=\sum c'\otimes c''$. 作为向量空间 $C\otimes_\psi H=C\otimes H$. 对 $c\in C, h\in H$, 定义 Hom-余乘和 Hom-余单位如下:

$$\Delta(c\otimes h)=\sum\alpha^{-2}(c_{11})\otimes(\beta^{-4}(c_{12(-1)})\beta^{-2}(c_2{}'))\beta^{-1}(h_1)\otimes\alpha^{-2}(c_{12(0)})\otimes\beta^{-1}(c_2{}'')\beta^{-1}(h_2)$$

$$\epsilon(c\otimes h)=\epsilon(c)\epsilon(h)$$

定理 1 $C\otimes_\psi H$ 构成 Hom-余代数的充要条件是 ψ 满足下列条件:

1) $\sum\epsilon(c')c''=\sum c'\epsilon(c'')=\epsilon(c)1_H$

2)

$$\begin{aligned}&\sum c_{1(-1)}\beta(c_2{}')\otimes\beta^{-1}(c_{1(0)(-1)})\beta(c_2{}'')\otimes c_{1(0)(0)}\\ =&\sum\beta(c_1{}')\beta^{-1}(c_{2(-1)1})\otimes\beta(c_1{}'')\beta^{-1}(c_{2(-1)2})\otimes\alpha(c_{2(0)})\end{aligned} \tag{2}$$

3)

$$\begin{aligned}&\sum c_{1(-1)}\beta(c_2{}')\otimes c_{1(0)}{}'c_2{}''{}_1\otimes c_{1(0)}{}''c_2{}''{}_2\\ =&\sum\beta(c_1{}')c_2{}'{}_1\otimes\beta(c_1{}'')c_2{}'{}_2\otimes\beta^2(c_2{}'')\end{aligned} \tag{3}$$

证明 首先, 假设上式都成立, 先证明余单位, 对任意的 $c\in C, h\in H$, 有

$$
\begin{aligned}
&(id\otimes\epsilon)\Delta(c\otimes h)\\
=\ &(id\otimes\epsilon)(\sum\alpha^{-1}(c_{11})\otimes(\beta^{-4}(c_{12(-1)})\beta^{-2}(c_2{}'))\beta^{-1}(h_1)\otimes\alpha^{-2}(c_{12(0)})\otimes\beta^{-1}(c_2{}'')\beta^{-1}(h_2))\\
=\ &\sum\alpha^{-1}(c_{11})\otimes(\beta^{-4}(c_{12(-1)})\beta^{-2}(c_2{}'))\beta^{-1}(h_1)\otimes\epsilon(c_{12(0)})\otimes\epsilon(c_2{}'')\epsilon(h_2)\\
=\ &\sum\alpha^{-1}(c_{11})\otimes\epsilon(c_{12})1_H\epsilon(c_2)\beta(h)=\sum c_1\otimes\epsilon(c_2)\beta(h)=\alpha(c)\otimes\beta(h).
\end{aligned}
$$

$$
\begin{aligned}
&(\epsilon\otimes id)\Delta(c\otimes h)\\
=\ &(\epsilon\otimes id)(\sum\alpha^{-1}(c_{11})\otimes(\beta^{-4}(c_{12(-1)})\beta^{-2}(c_2{}'))\beta^{-1}(h_1)\otimes\alpha^{-2}(c_{12(0)})\otimes\beta^{-1}(c_2{}'')\beta^{-1}(h_2))\\
=\ &\sum\epsilon(c_{11})\otimes\epsilon(c_{12(-1)})\epsilon(c_2{}')\epsilon(h_1)\otimes\alpha^{-2}(c_{12(0)})\otimes\beta^{-2}(c_2{}'')\beta^{-1}(h_2)\\
=\ &\sum\epsilon(c_{11})\otimes\alpha^{-1}(c_{12})\otimes\epsilon(c_2)\beta(h)\\
=\ &\sum c_1\otimes\epsilon(c_2)\beta(h)\\
=\ &\alpha(c)\otimes\beta(h).
\end{aligned}
$$

再证明余结合性

$$
\begin{aligned}
&(\alpha\otimes\Delta)\Delta(c\otimes h)\\
=\ &(\alpha\otimes\Delta)(\sum\alpha^{-1}(c_{11})\otimes(\beta^{-4}(c_{12(-1)})\beta^{-2}(c_2{}'))\beta^{-1}(h_1)\otimes\alpha^{-2}(c_{12(0)})\otimes\beta^{-1}(c_2{}'')\beta^{-1}(h_2))\\
=\ &\sum c_{11}\otimes(\beta^{-3}(c_{12(-1)})\beta^{-1}(c_2{}'))h_1\otimes\alpha^{-3}(c_{12(0)11})\otimes(\beta^{-6}(c_{12(0)12(-1)})\beta^{-4}(c_{12(0)2}{}'))\\
&(\beta^{-2}(c_2{}''_1)\beta^{-2}(h_{21}))\otimes\alpha^{-4}(c_{12(0)12(0)})\otimes\beta^{-3}(c_{12(0)2}{}'')(\beta^{-2}(c_2{}''_2)\beta^{-2}(h_{22}))\\
=\ &\sum c_{11}\otimes(\beta^{-3}(c_{12(-1)})\beta^{-1}(c_2{}'))h_1\otimes\alpha^{-2}(c_{12(0)1})\otimes(\beta^{-6}(c_{12(0)21(-1)})\beta^{-5}(c_{12(0)22}{}'))\\
&(\beta^{-2}(c_2{}''_1)\beta^{-2}(h_{21}))\otimes\alpha^{-4}(c_{12(0)21(0)})\otimes\beta^{-4}(c_{12(0)22}{}'')(\beta^{-2}(c_2{}''_2)\beta^{-2}(h_{22}))\\
\overset{(1)}{=}\ &\sum c_{11}\otimes((\beta^{-5}(c_{121(-1)}\beta^{-5}(c_{122(-1)}))\beta^{-1}(c_2{}'))h_1\otimes\alpha^{-2}(c_{121(0)})\otimes(\beta^{-6}(c_{122(0)1(-1)})\\
&\beta^{-5}(c_{122(0)2}{}'))(\beta^{-2}(c_2{}''_1)\beta^{-2}(h_{21}))\otimes\alpha^{-4}(c_{122(0)1(0)})\otimes\beta^{-4}(c_{122(0)2}{}'')(\beta^{-2}(c_2{}''_2)\beta^{-2}(h_{22}))\\
\overset{(1)}{=}\ &\sum c_{11}\otimes((\beta^{-5}(c_{121(-1)}(\beta^{-7}(c_{1221(-1)})\beta^{-7}(c_{1222(-1)})))\beta^{-1}(c_2{}'))h_1\otimes\alpha^{-2}(c_{121(0)})\\
&\otimes(\beta^{-6}(c_{1221(0)(-1)})\beta^{-5}(c_{1222(0)}{}'))(\beta^{-2}(c_2{}''_1)\beta^{-2}(h_{21}))\otimes\alpha^{-4}(c_{1221(0)(0)})\otimes\beta^{-4}(c_{1222(0)}{}'')\\
&(\beta^{-2}(c_2{}''_2)\beta^{-2}(h_{22}))\\
=\ &\sum c_{11}\otimes((\beta^{-5}(c_{121(-1)}(\beta^{-7}(c_{1221(-1)})\beta^{-7}(c_{1222(-1)})))\beta^{-1}(c_2{}'))\beta^{-1}(h_{11})\otimes\alpha^{-2}(c_{121(0)})\\
&\otimes(\beta^{-6}(c_{1221(0)(-1)})\beta^{-5}(c_{1222(0)}{}'))(\beta^{-2}(c_2{}''_1)\beta^{-2}(h_{12}))\otimes\alpha^{-4}(c_{1221(0)(0)})\otimes\beta^{-4}(c_{1222(0)}{}'')\\
&(\beta^{-2}(c_2{}''_2)\beta^{-1}(h_2))\\
=\ &\sum c_{11}\otimes\beta^{-3}(c_{121(-1)}((\beta^{-6}(c_{1221(-1)})(\beta^{-7}(c_{1222(-1)})\beta^{-3}(c_2{}')))\beta^{-2}(h_{11}))\otimes\alpha^{-2}(c_{121(0)})\\
&\otimes(\beta^{-6}(c_{1221(0)(-1)})(\beta^{-6}(c_{1222(0)}{}')\beta^{-3}(c_2{}''_1)))\beta^{-1}(h_{12})\otimes\alpha^{-4}(c_{1221(0)(0)})\otimes\beta^{-5}(c_{1222(0)}{}'')\\
&(\beta^{-2}(c_2{}''_2))h_2)
\end{aligned}
$$

$$
\begin{aligned}
&(\Delta\otimes\alpha)\Delta(c\otimes h)\\
=\;&(\Delta\otimes\alpha)(\sum\alpha^{-1}(c_{11})\otimes(\beta^{-4}(c_{12(-1)})\beta^{-2}(c_2{}'))\beta^{-1}(h_1)\otimes\alpha^{-2}(c_{12(0)})\otimes\beta^{-1}(c_2{}'')\beta^{-1}(h_2))\\
=\;&\sum\alpha^{-2}(c_{1111})\otimes(\beta^{-5}(c_{1112(-1)})\beta^{-3}(c_{112}{}'))((\beta^{-5}(c_{12(-1)1})\beta^{-3}c_2{}'_1))\beta^{-2}(h_{11}))\\
&\otimes\alpha^{-3}(c_{1112(0)})\otimes\beta^{-2}(c_{112}{}'')((\beta^{-5}(c_{12(-1)2})\beta^{-3}(c_2{}'_2))\beta^{-2}(h_{12}))\otimes\alpha^{-1}(c_{12(0)})\otimes c_2{}''h_2\\
=\;&\sum c_{11}\otimes(\beta^{-4}(c_{121(-1)})\beta^{-4}(c_{1221}{}'))((\beta^{-7}(c_{1222(-1)1})\beta^{-3}c_2{}'_1))\beta^{-2}(h_{11}))\\
&\otimes\alpha^{-2}(c_{121(0)})\otimes\beta^{-3}(c_{1221}{}'')((\beta^{-7}(c_{1222(-1)2})\beta^{-3}(c_2{}'_2))\beta^{-2}(h_{12}))\otimes\alpha^{-3}(c_{1222(0)})\otimes c_2{}''h_2\\
=\;&\sum c_{11}\otimes\beta^{-3}(c_{121(-1)})(((\beta^{-6}(c_{1221}{}')\beta^{-8}(c_{1222(-1)1}))\beta^{-3}c_2{}'_1))\beta^{-2}(h_{11}))\\
&\otimes\alpha^{-2}(c_{121(0)})\otimes((\beta^{-5}(c_{1221}{}'')\beta^{-7}(c_{1222(-1)2}))\beta^{-2}(c_2{}'_2))\beta^{-1}(h_{12})\otimes\alpha^{-3}(c_{1222(0)})\otimes c_2{}''h_2\\
=\;&\sum c_{11}\otimes\beta^{-3}(c_{121(-1)})(((\beta^{-6}(c_{1221}{}')\beta^{-8}(c_{1222(-1)1}))\beta^{-3}(c_2{}'_1))\beta^{-2}(h_{11}))\\
&\otimes\alpha^{-2}(c_{121(0)})\otimes((\beta^{-5}(c_{1221}{}'')\beta^{-7}(c_{1222(-1)2}))\beta^{-2}(c_2{}'_2))\beta^{-1}(h_{12})\otimes\alpha^{-3}(c_{1222(0)})\otimes c_2{}''h_2\\
\overset{(2)}{=}\;&\sum c_{11}\otimes\beta^{-3}(c_{121(-1)})(((\beta^{-7}(c_{1221(-1)})\beta^{-6}(c_{1222}{}'))\beta^{-3}c_2{}'_1))\beta^{-2}(h_{11}))\\
&\otimes\alpha^{-2}(c_{121(0)})\otimes((\beta^{-7}(c_{1221(0)(-1)})\beta^{-5}(c_{1222}{}''))\beta^{-2}(c_2{}'_2))\beta^{-1}(h_{12})\otimes\alpha^{-4}(c_{1221(0)(0)})\otimes c_2{}''h_2\\
=\;&\sum\alpha(c_1)\otimes\beta^{-3}(c_{211(-1)})(((\beta^{-6}(c_{212(-1)})\beta^{-5}(c_{221}{}'))\beta^{-5}(c_{222}{}'_1))\beta^{-2}(h_{11}))\\
&\otimes\alpha^{-2}(c_{211(0)})\otimes((\beta^{-6}(c_{212(0)(-1)})\beta^{-4}(c_{221}{}''))\beta^{-4}(c_{222}{}'_2))\beta^{-1}(h_{12})\otimes\alpha^{-3}(c_{212(0)(0)})\\
&\otimes\beta^{-2}(c_{222}{}'')h_2\\
=\;&\sum\alpha(c_1)\otimes\beta^{-3}(c_{211(-1)})((\beta^{-5}(c_{212(-1)})(\beta^{-5}(c_{221}{}')\beta^{-6}(c_{222}{}'_1)))\beta^{-2}(h_{11}))\\
&\otimes\alpha^{-2}(c_{211(0)})\otimes(\beta^{-5}(c_{212(0)(-1)})(\beta^{-4}(c_{221}{}'')\beta^{-5}(c_{222}{}'_2)))\beta^{-1}(h_{12})\otimes\alpha^{-3}(c_{212(0)(0)})\\
&\otimes\beta^{-2}(c_{222}{}'')h_2\\
\overset{(3)}{=}\;&\sum\alpha(c_1)\otimes\beta^{-3}(c_{211(-1)})((\beta^{-5}(c_{212(-1)})(\beta^{-6}(c_{221(-1)}))\beta^{-5}(c_{222}{}'_1)))\beta^{-2}(h_{11}))\\
&\otimes\alpha^{-2}(c_{211(0)})\otimes(\beta^{-5}(c_{212(0)(-1)})(\beta^{-5}(c_{221(0)}{}')\beta^{-5}(c_{222}{}''_1)))\beta^{-1}(h_{12})\otimes\alpha^{-3}(c_{212(0)(0)})\\
&\otimes(\beta^{-4}(c_{221(0)}{}'')\beta^{-4}(c_{222}{}''_2))h_2\\
=\;&\sum c_{11}\otimes\beta^{-3}(c_{121(-1)})((\beta^{-6}(c_{1221(-1)})(\beta^{-7}(c_{1222(-1)}))\beta^{-3}(c_2{}'_1)))\beta^{-2}(h_{11}))\\
&\otimes\alpha^{-2}(c_{121(0)})\otimes(\beta^{-6}(c_{1221(0)(-1)})(\beta^{-6}(c_{1222(0)}{}')\beta^{-3}(c_2{}''_1)))\beta^{-1}(h_{12})\otimes\alpha^{-4}(c_{1222(0)(0)})\\
&\otimes(\beta^{-5}(c_{1222(0)}{}'')\beta^{-2}(c_2{}''_2))h_2
\end{aligned}
$$

得出 $(\alpha\otimes\triangle)\triangle(c\otimes h)=(\triangle\otimes\alpha)\Delta(c\otimes h)$.
其次, 反之来证明.
由 $\alpha(c)\otimes\beta(h)=(id\otimes\epsilon)\Delta(c\otimes h)$

$$\begin{aligned}
& \alpha(c)\otimes 1_H \\
= & (id\otimes\epsilon)\Delta(c\otimes 1_H) \\
= & (id\otimes\epsilon)(\sum\alpha^{-1}(c_{11})\otimes(\beta^{-4}(c_{12(-1)})\beta^{-2}(c_2{}'))1_H\otimes\alpha^{-2}(c_{12(0)})\otimes\beta^{-1}(c_2{}'')1_H) \\
= & \sum\alpha^{-1}(c_{11})\otimes\beta^{-3}(c_{12(-1)})\beta^{-1}(c_2{}')\otimes\epsilon(c_{12(0)})\epsilon(c_2{}'') \\
= & \sum\alpha^{-1}(c_{11})\otimes\epsilon(c_{12})1_H\beta^{-1}(c_2{}')\otimes\epsilon(c_2{}'') \\
= & \sum\alpha^{-1}(c_{11})\otimes\epsilon(c_{12})c_2{}'\epsilon(c_2{}'' \\
= & \sum c_1\otimes c_2{}'\epsilon(c_2{}'').
\end{aligned}$$

左右同时作用 $(\epsilon\otimes id)$

$$(\epsilon\otimes id)(\alpha(c)\otimes 1_H)=\epsilon(c)\otimes 1_H$$

$$(\epsilon\otimes id)(\sum c_1\otimes c_2{}'\epsilon(c_2{}''))=\sum\beta(c')\epsilon(c'')$$

得到

$$\epsilon(c)\otimes 1_H=\sum(c')\epsilon(c'')$$

由 $\alpha(c)\otimes\beta(h)=(\epsilon\otimes id)\Delta(c\otimes h)$

$$\begin{aligned}
& \alpha(c)\otimes 1_H \\
= & (\epsilon\otimes id)\Delta(c\otimes 1_H) \\
= & (\epsilon\otimes id)(\sum\alpha^{-1}(c_{11})\otimes(\beta^{-4}(c_{12(-1)})\beta^{-2}(c_2{}'))1_H\otimes\alpha^{-2}(c_{12(0)})\otimes\beta^{-1}(c_2{}'')1_H) \\
= & \sum\epsilon(c_{11})\otimes\epsilon(c_{12(-1)})\epsilon(c_2{}')\otimes\alpha^{-2}(c_{12(0)})\otimes c_2{}'' \\
= & \sum\epsilon(c_{11})\otimes\alpha^{-1}(c_{12})\otimes\epsilon(c_2{}')c_2{}'' \\
= & \sum c_1\otimes\epsilon(c_2{}')c_2{}'
\end{aligned}$$

左右同时作用 $(\epsilon\otimes id)$

$$(\epsilon\otimes id)(\alpha(c)\otimes 1_H)=\epsilon(c)\otimes 1_H$$

$$(\epsilon\otimes id)(\sum c_1\otimes\epsilon(c_2{}')c_2{}'')=\sum\epsilon(c')\otimes\beta(c'')$$

得到

$$\epsilon(c)\otimes 1_H=\sum \epsilon(c')c''$$

若 $C\otimes_\psi H$是 Hom-余代数, 有 $(\alpha\otimes\triangle)\triangle=(\triangle\otimes\alpha)\Delta$

$$\begin{aligned}
&(\alpha\otimes\triangle)\triangle(c\otimes 1_H)\\
=\ &(\alpha\otimes\triangle)(\sum \alpha^{-1}(c_{11})\otimes\beta^{-3}(c_{12(-1)})\beta^{-1}(c_2{}')\otimes\alpha^{-2}(c_{12(0)})\otimes c_2{}'')\\
=\ &\sum c_{11}\otimes\beta^{-2}(c_{12(-1)})c_2{}'\otimes\alpha^{-3}(c_{12(0)11})\otimes(\beta^{-6}(c_{12(0)12(-1)})\beta^{-4}(c_{12(0)2}{}'))\beta^{-1}(c_2{}''{}_1)\\
&\otimes\alpha^{-4}(c_{12(0)12(0)})\otimes\beta^{-3}(c_{12(0)2}{}'')\beta^{-1}(c_2{}''{}_2)
\end{aligned}$$

$$\begin{aligned}
&(\triangle\otimes\alpha)\triangle(c\otimes 1_H)\\
=\ &(\triangle\otimes\alpha)(\sum \alpha^{-1}(c_{11})\otimes\beta^{-3}(c_{12(-1)})\beta^{-1}(c_2{}')\otimes\alpha^{-2}(c_{12(0)})\otimes c_2{}'')\\
=\ &\sum \alpha^{-2}(c_{1111})\otimes(\beta^{-5}(c_{1112(-1)})\beta^{-3}(c_{112}{}'))(\beta^{-4}(c_{12(-1)1})\beta^{-2}(c_2{}'{}_1))\otimes\alpha^{-3}(c_{1112(0)})\\
&\otimes\beta^{-2}(c_{112}{}'')(\beta^{-4}(c_{12(-1)2})\beta^{-2}(c_2{}'{}_2))\otimes\alpha^{-1}(c_{12(0)})\otimes\beta(c_2{}'')
\end{aligned}$$

对上面两个式子同时作用 $(\epsilon_c\otimes id)\otimes(\epsilon_c\otimes id)\otimes(id\otimes\epsilon_H)$

$$\begin{aligned}
&\sum \epsilon(c_{11})\otimes\beta^{-2}(c_{12(-1)})c_2{}'\otimes\epsilon(c_{12(0)11})\otimes(\beta^{-6}(c_{12(0)12(-1)})\beta^{-4}(c_{12(0)2}{}'))\\
&\beta^{-1}(c_2{}''{}_1)\otimes\alpha^{-4}(c_{12(0)12(0)})\otimes\epsilon(c_{12(0)2}{}'')\epsilon(c_2{}''{}_2)\\
=\ &\sum \epsilon(c_{11})\otimes\beta^{-2}(c_{12(-1)})c_2{}'\otimes(\beta^{-5}(c_{12(0)1(-1)})\epsilon(c_{12(0)2})1_H)c_2{}''\\
&\otimes\alpha^{-3}(c_{12(0)1(0)})\\
=\ &\sum \epsilon(c_{11})\otimes\beta^{-2}(c_{12(-1)})c_2{}'\otimes\beta^{-3}(c_{12(0)(-1)})c_2{}''\otimes\alpha^{-2}(c_{12(0)(0)})\\
=\ &\sum \beta^{-1}(c_{1(-1)})c_2{}'\otimes\beta^{-2}(c_{1(0)(-1)})c_2{}''\otimes\alpha^{-1}(c_{1(0)(0)})
\end{aligned}$$

$$\begin{aligned}
&\sum \epsilon(c_{1111})\otimes(\beta^{-5}(c_{1112(-1)})\beta^{-3}(c_{112}{}'))(\beta^{-4}(c_{12(-1)})\beta^{-2}(c_2{}'{}_1)\otimes\epsilon(c_{1112(0)})\\
&\otimes\beta^{-2}(c_{112}{}'')(\beta^{-4}(c_{12(-1)2})\beta^{-2}(c_2{}'{}_2))\otimes\alpha^{-1}(c_{12(0)})\otimes\epsilon(c_2{}'')\\
=\ &\sum(\epsilon(c_{111})\beta^{-2}(c_{112}{}'))\beta^{-3}(c_{12(-1)})\otimes\beta^{-2}(c_{112}{}'')(\beta^{-4}(c_{12(-1)2})(\epsilon(c_2)1_H))\otimes\alpha^{-1}(c_{12(0)})\\
=\ &\sum \beta^{-1}(c_{11}{}')\beta^{-3}(c_{12(-1)1})\otimes\beta^{-1}(c_{11}{}'')(\beta^{-4}(c_{12(-1)2})(\epsilon(c_2)1_H))\otimes\alpha^{-1}(c_{12(0)})\\
=\ &\sum c_1{}'\beta^{-2}(c_{2(-1)1})\otimes c_1{}''\beta^{-2}(c_{2(-1)2})\otimes c_{2(0)}
\end{aligned}$$

得到

$$\begin{aligned}&\sum \beta^{-1}(c_{1(-1)})c_2{}' \otimes \beta^{-2}(c_{1(0)(-1)})c_2{}'' \otimes \alpha^{-1}(c_{1(0)(0)})\\ =&\sum c_1{}'\beta^{-2}(c_{2(-1)1}) \otimes c_1{}''\beta^{-2}(c_{2(-1)2}) \otimes c_{2(0)}\end{aligned}$$

即为

$$\begin{aligned}&\sum c_{1(-1)}\beta(c_2{}') \otimes \beta^{-1}(c_{1(0)(-1)})\beta(c_2{}'') \otimes c_{1(0)(0)}\\ =&\sum \beta(c_1{}')\beta^{-1}(c_{2(-1)1}) \otimes \beta(c_1{}'')\beta^{-1}(c_{2(-1)2}) \otimes \alpha(c_{2(0)})\end{aligned}$$

对上面两个式子同时作用 $(\epsilon_c \otimes id) \otimes (\epsilon_c \otimes id) \otimes (\epsilon_c \otimes id)$

$$\begin{aligned}&\sum \epsilon(c_{11}) \otimes \beta^{-2}(c_{12(-1)})c_2{}' \otimes \epsilon(c_{12(0)11}) \otimes (\beta^{-6}(c_{12(0)12(-1)})\beta^{-4}(c_{12(0)2}{}'))\\ &\beta^{-1}(c_2{}''{}_1) \otimes \epsilon(c_{12(0)12(0)}) \otimes \beta^{-3}(c_{12(0)2}{}'')\beta^{-1}(c_2{}''{}_2)\\ =&\sum \epsilon(c_{11}) \otimes \beta^{-2}(c_{12(-1)})c_2{}' \otimes (\epsilon(c_{12(0)1})\beta^{-3}(c_{12(0)2}{}'))\beta^{-1}(c_2{}''{}_1) \otimes \beta^{-3}(c_{12(0)2}{}'')\beta^{-1}(c_2{}''{}_2)\\ =&\sum \epsilon(c_{11}) \otimes \beta^{-2}(c_{12(-1)})c_2{}' \otimes \beta^{-2}(c_{12(0)}{}')\beta^{-1}(c_2{}''{}_1) \otimes \beta^{-2}(c_{12(0)}{}'')\beta^{-1}(c_2{}''{}_2)\\ =&\sum \beta^{-1}(c_{1(-1)})c_2{}' \otimes \beta^{-1}(c_{1(0)}{}')\beta^{-1}(c_2{}''{}_1) \otimes \beta^{-1}(c_{1(0)}{}'')\beta^{-1}(c_2{}''{}_2)\end{aligned}$$

$$\begin{aligned}&\sum \epsilon(c_{1111}) \otimes (\beta^{-5}(c_{1112(-1)})\beta^{-3}(c_{112}{}'))(\beta^{-4}(c_{12(-1)})\beta^{-2}(c_2{}'{}_1)) \otimes \epsilon(c_{1112(0)}) \otimes \beta^{-2}(c_{112}{}'')\\ &(\beta^{-4}(c_{12(-1)2})\beta^{-2}(c_2{}'{}_2)) \otimes \epsilon(c_{12(0)}) \otimes \beta(c_2{}'')\\ =&\ \beta^{-1}(c_{11}{}')(\epsilon(c_{121})\beta^{-1}(c_2{}'{}_1)) \otimes \beta^{-1}(c_{11}{}'')(\epsilon(c_{122})\beta^{-1}(c_2{}'{}_2)) \otimes \beta(c_2{}'')\\ =&\ \beta^{-1}(c_{11}{}')\beta^{-1}(c_2{}'{}_1) \otimes \beta^{-1}(c_{11}{}'')(\epsilon(c_{12})\beta^{-1}(c_2{}'{}_2)) \otimes \beta(c_2{}'')\\ =&\ c_1{}'\beta^{-1}(c_2{}'{}_1) \otimes c_1{}''\beta^{-1}(c_2{}'{}_2) \otimes \beta(c_2{}'')\end{aligned}$$

得到

$$\begin{aligned}&\sum \beta^{-1}(c_{1(-1)})c_2{}' \otimes \beta^{-1}(c_{1(0)}{}'))\beta^{-1}(c_2{}''{}_1) \otimes \beta^{-1}(c_{1(0)}{}'')\beta^{-1}(c_2{}''{}_2)\\ =&\sum c_1{}'{}_1\beta^{-1}(c_2{}'{}_1) \otimes c_1{}''\beta^{-1}(c_2{}'{}_2)) \otimes \beta(c_2{}'')\end{aligned}$$

即为

$$\begin{aligned}&\sum c_{1(-1)}\beta(c_2{}') \otimes c_{1(0)}{}'c_2{}''{}_2 \otimes c_{1(0)}{}''c_2{}''{}_2\\ =&\sum \beta(c_1{}')c_2{}'{}_1 \otimes \beta(c_1{}'')c_2{}'{}_2 \otimes \beta^2(c_2{}'')\end{aligned}$$

定理 1 证毕. □

引理 1

1) 当 α, β 是恒等映射时, 这时 Hom-交叉余积 $C \otimes H$ 即为文献 [8]中的交叉余积

$$\triangle(C \otimes H) = \sum c_1 \otimes c_{2(-1)} {c_2}' h_1 \otimes c_{2(0)} \otimes {c_2}'' h_2$$

2) 当 α, β 是恒等映射时, 这时定理1的条件即为文献 [8]中的交叉余积 $C \otimes H$ 构成余代数的充要条件

(I)

$$\sum \epsilon(c')c'' = \sum c'\epsilon(c'') = \epsilon(c)1_H$$

(II)

$$\sum c_{1(-1)} {c_2}' \otimes c_{1(0)(-1)} {c_2}'' \otimes c_{1(0)(0)} \quad = \quad \sum {c_1}' c_{2(-1)1}) \otimes {c_1}'' c_{2(-1)2} \otimes c_{2(0)}$$

(III)

$$\sum c_{1(-1)} {c_2}' \otimes {c_{1(0)}}' {c_2''}_1 \otimes {c_{1(0)}}'' {c_2''}_2 \quad = \quad \sum {c_1}' {c_2'}_1 \otimes {c_1}'' {c_2'}_2 \otimes {c_2}''$$

4. 结语

在前人研究的基础上, 运用类比的思想方法, 经过大量的计算, 本文将 Hopf-交叉余积推广到 Hom-双代数上, 给出了 Hom-交叉余积的定义和一些性质, 得到了 Hom-交叉余积构成 Hom-余代数的充要条件. 在本文的研究基础上, 将来可进一步研究 Hom-交叉余积和 Hom smash 积形成 Hom-双代数的充要条件以及 Hom-交叉余积的对偶等问题.

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