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Hans汉斯

$BiHom$-双代数上的 $Radford's$ 余积

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摘 要

为了研究 $BiHom$-双代数上的 $Radford's$ 双积，通过运用类比的思想方法，给出了 $BiHom$-余模余代数以及 $BiHom$-$Smash$ 余积的定义，并得到了 $BiHom$-$Smash$ 积和 $BiHom$-$Smash$ 余积形成 $BiHom$-双代数的充分必要条件。

关键词

$BiHom$-双代数，$BiHom$-$Smash$ 余积，$BiHom$-余模余代数，$BiHom$-$Hopf$代数

$Radford's$ Biproduct on $BiHom$-Bialgebras

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Abstract

It was aimed to study the *Radford's* biproduct over *BiHom*-bialgebras. By applying the thought of analogy, the notion of *BiHom*-comodule coalgebra and *BiHom-Smash* coproduct over *BiHom*-bialgebras was defined. Further, a necessary and sufficient condition for the *BiHom-Smash* product and *BiHom-Smash* coproduct to form a *BiHom*-bialgebra was obtained.

Keywords

***BiHom*-Bialgebra, *BiHom-Smash* Coproduct, *BiHom*-Comodule Coalgebra, *BiHom-Hopf* Algebra**

1. 介绍

在变形量子化理论的基础上，文献 [1,2]对 *Smash* 余积进行了介绍和研究,其定义如下: 设 H 是一个双代数, B 是左 H-余模余代数, 则 *Smash* 余积 $B \times H$ 是在 $B \otimes H$ 上定义的代数, 对任意的 $b \in B, h \in H$, 它的余乘法是

$$(b \times h) = (b_1 \otimes b_{2(-1)}h_1) \otimes (b_{2(0)} \otimes h_2).$$

关于 *Smash* 余积有诸多形式的推广, 文献 [3–6]分别给出了 *Hom-Smash* 余积和张量 *Hom-Smash* 余积的概念及一些重要的结论. 而作为 *Hom*-代数和张量 *Hom*-代数的推广, *Graziani* 等 [7]提出了 *BiHom*-结合代数, *BiHom*-余结合余代数和 *BiHom*-双代数等概念, 并得到若干的例子, 同时研究了 *BiHom*-双代数上的 *Smash* 积的若干性质. 随着对 *BiHom*-结构的深入探究, 许多相关的概念得到了推广, 比如已经定义了 *BiHom-Lie* 超代数、*BiHom-Novikov* 代数和无限小的 *BiHom*-双代数等等 [8,9].

本文借助 *BiHom*-双代数, 定义了 *BiHom*-双代数上的 *Smash* 余积, 并讨论其性质, 给出了 *BiHom-Smash* 积和 *BiHom-Smash* 余积形成 *BiHom*-双代数的充分必要条件.

2. 预备知识

首先回顾一些基本概念. 本文中所有的向量空间、张量积及同态映射都在固定的域 k 上进行. 对于余代数 C, 使用 $Sweedler$ 型记号来表示余乘: 对任意的 $c \in C$, $\Delta(B) = c_1 \otimes c_2$.

定义 1 [7] 设 A 是一个线性空间, $\alpha_A : A \to A, \beta_A : A \to A$ 是线性映射. 如果存在线性映射 $\mu : A \otimes A \to A, a \otimes b \mapsto ab$, 使得对任意的 $a, b, c \in A$, 满足

$$\alpha_A \circ \beta_A = \beta_A \circ \alpha_A, \alpha_A(ab) = \alpha_A(a)\alpha_A(b),$$
$$\beta_A(ab) = \beta_A(a)\beta_A(b), \alpha_A(a)(bc) = (ab)\beta_A(c),$$

那么称 $(A, \mu, \alpha_A, \beta_A)$ 为 $BiHom$-结合代数. 如果还存在元素 $1_A \in A$, 使得对任意的 $a \in A$, 满足

$$\alpha_A(1_A) = 1_A, \beta_A(1_A) = 1_A,$$
$$a1_A = \alpha_A(a), 1_A a = \beta_A(a),$$

那么称 $(A, \mu, \alpha_A, \beta_A)$ 为有单位元的 $BiHom$-结合代数. 简记为 (A, α_A, β_A).

定义 2 [7] 设 C 是一个线性空间, $\psi_C : C \to C, \omega_C : C \to C$ 是线性映射. 如果存在线性映射 $\Delta : C \to C \otimes C$, 满足

$$\psi_C \circ \omega_C = \omega_C \circ \psi_C, (\psi_C \otimes \psi_C) \circ \Delta = \Delta \circ \psi_C,$$
$$(\omega_C \otimes \omega_C) \circ \Delta = \Delta \circ \omega_C, (\Delta \otimes \psi_C) \circ \Delta = (\omega_C \otimes \Delta) \circ \Delta,$$

那么称 $(C, \Delta, \psi_C, \omega_C)$ 为 $BiHom$-余结合余代数. 如果还存在线性映射 $\varepsilon : C \to k$, 满足

$$\varepsilon \circ \psi_C = \varepsilon, \varepsilon \circ \omega_C = \varepsilon,$$
$$(id_C \otimes \varepsilon) \circ \Delta = \omega_C, (\varepsilon \otimes id_C) \circ \Delta = \psi_C,$$

那么称 $(C, \Delta, \psi_C, \omega_C)$ 为有余单位的 $BiHom$-余结合余代数. 简记为 (C, ψ_C, ω_C).

定义 3 [7] 设 $(H, \mu, \alpha_H, \beta_H)$ 是 $BiHom$-结合代数, $(H, \Delta, \psi_H, \omega_H)$ 是 $BiHom$-余结合余代数. 如果对于任意的 $h, g \in H$, 满足

$$\Delta(hg) = (h_1 g_1) \otimes (h_2 g_2),$$
$$\alpha_H \circ \psi_H = \psi_H \circ \alpha_H, \alpha_H \circ \omega_H = \omega_H \circ \alpha_H,$$
$$\beta_H \circ \psi_H = \psi_H \circ \beta_H, \beta_H \circ \omega_H = \omega_H \circ \beta_H,$$
$$(\alpha_H \otimes \alpha_H) \circ \Delta = \Delta \circ \alpha_H, (\beta_H \otimes \beta_H) \circ \Delta = \Delta \circ \beta_H,$$
$$\psi_H(hg) = \psi_H(h)\psi_H(g), \omega_H(hg) = \omega_H(h)\omega_H(g),$$

那么称 $(H, \mu, \Delta, \alpha_H, \beta_H, \psi_H, \omega_H)$ 为 $BiHom$-双代数. 如果存在元素 $1_A \in A$ 和线性映射 $\varepsilon_H : C \to$

k, 满足

$$\Delta(1_H) = 1_H \otimes 1_H, \varepsilon_H(1_H) = 1,$$
$$\psi_H(1_H) = 1_H, \omega_H(1_H) = 1_H,$$
$$\varepsilon_H \circ \alpha_H = \varepsilon_H, \varepsilon_H \circ \beta_H = \varepsilon_H,$$
$$\varepsilon_H(hg) = \varepsilon_H(h)\varepsilon_H(g),$$

那么称 $(H, \mu, \Delta, \alpha_H, \beta_H, \psi_H, \omega_H)$ 为有单位元和余单位的 $BiHom$-双代数. 简记为 $(H, \alpha_H, \beta_H, \psi_H, \omega_H)$.

如果还存在线性映射 S_H(对极) : $H \to H$, 使得对任意的 $h \in H$, 满足

$$\beta_H \psi_H S_H(h_1)\alpha_H \omega_H(h_2) = \varepsilon_H(h)1_H = \beta_H \psi_H(h_1)\alpha_H \omega_H S_H(h_2),$$

那么称 $(H, \alpha_H, \beta_H, \psi_H, \omega_H)$ 为 $BiHom$-$Hopf$ 代数.

定义 4 [7] 设 $(H, \alpha_H, \beta_H, \psi_H, \omega_H)$ 是 $BiHom$-双代数, (A, α_A, β_A) 是 $BiHom$-结合代数, $\alpha_H, \beta_H, \psi_H, \omega_H$ 是双射. 如果 (A, α_A, β_A) 是左 $(H, \alpha_H, \beta_H, \psi_H, \omega_H)$-模 (模作用定义为 $H \otimes A \to A$, $h \otimes a \mapsto (h \cdot a)$), 使得对任意的 $a, b \in A, h, g \in H$, 满足

$$(hg) \cdot \beta_A(a) = \alpha_H(h) \cdot (g \cdot a),$$
$$\alpha_A(h \cdot a) = \alpha_H(h) \cdot \alpha_A(a),$$
$$\beta_A(h \cdot a) = \beta_H(h) \cdot \beta_A(a),$$
$$1_H \cdot a = \beta_A(a), h \cdot 1_A = \varepsilon_H(h)1_A,$$
$$h \cdot (ab) = (\alpha_H^{-1}\omega_H^{-1}(h_1) \cdot a)(\beta_H^{-1}\psi_H^{-1}(h_2) \cdot b),$$

那么称 (A, α_A, β_A) 是左 $(H, \alpha_H, \beta_H, \psi_H, \omega_H)$-$BiHom$-模代数.

定义 5 [7] 设 $(H, \alpha_H, \beta_H, \psi_H, \omega_H)$ 是 $BiHom$-双代数, (A, α_A, β_A) 是左 $(H, \alpha_H, \beta_H, \psi_H, \omega_H)$-$BiHom$- 模代数, $\alpha_H, \beta_H, \psi_H, \omega_H, \alpha_A, \beta_A$ 是双射. 如果把 $BiHom$-结合代数 $A \otimes H$ 定义为 $A \sharp H$, 且对任意的 $a, b \in A, h, g \in H$, 满足

$$(a\sharp h)(b\sharp g) = (a(\beta_H^{-1}\omega_H^{-1}(h_1) \cdot \beta_A^{-1}(b)))\sharp(\psi_H^{-1}(h_2)g),$$

那么称 $(A\sharp H, \alpha_A \otimes \alpha_H, \beta_A \otimes \beta_H)$ 为 (A, α_A, β_A) 和 $(H, \alpha_H, \beta_H, \psi_H, \omega_H)$ 的 $BiHom$-$Smash$ 积.

3. $BiHom$-$Radford's$ 双积

主要介绍了 $BiHom$-$Smash$ 余积的定义, 并且给出了 $BiHom$-$Smash$ 积和 $BiHom$-$Smash$ 余积形成 $BiHom$-双代数的充分必要条件.

定义 6 设 $(H,\alpha_H,\beta_H,\psi_H,\omega_H)$ 是 $BiHom$-双代数,$(B,\Delta_B,\psi_B,\omega_B)$ 是 $BiHom$-余代数.若 (B,ρ,ψ_B,ω_B) 是左 H-$BiHom$-余模,其中余模作用定义为 $\rho: B\to H\otimes B$,$b\mapsto b_{(-1)}\otimes b_0=\omega_H(b_{(-1)})\otimes\psi_B(b_{(0)})$.使得对任意的 $b\in B,h\in H$, 满足

$$\psi_H(b_{(-1)})\otimes\psi_B(b_{(0)})=(\psi_B(b))_{(-1)}\otimes(\psi_B(b))_{(0)},$$
$$\omega_H(b_{(-1)})\otimes\omega_B(b_{(0)})=(\omega_B(b))_{(-1)}\otimes(\omega_B(b))_{(0)},$$
$$\omega_H(b_1)\otimes b_{21}\otimes b_{22}=b_{11}\otimes b_{12}\otimes\psi_H(b_2),$$
$$\beta_H\psi_H(b_{1(-1)})\alpha_H\omega_H(b_{2(-1)})\otimes b_{1(0)}\otimes b_{2(0)}$$
$$=\alpha_H\beta_H\omega_H\psi_H(b_{(-1)})\otimes b_{(0)1}\otimes b_{(0)2},$$
$$b_{(-1)}\varepsilon(b_{(0)})=\varepsilon(b)1_H, b_1\varepsilon(b_2)=\omega(b),\varepsilon(b_1)b_2=\psi(b).$$

那么 $(B,\Delta_B,\psi_B,\omega_B)$ 被称为左 $(H,\alpha_H,\beta_H,\psi_H,\omega_H)$-$BiHom$-余模余代数.

命题 1 设 $(H,\alpha_H,\beta_H,\psi_H,\omega_H)$ 是 $BiHom$-双代数, $(B,\Delta_B,\psi_B,\omega_B)$ 是左 $(H,\alpha_H,\beta_H,\psi_H,\omega_H)$-$BiHom$-余模余代数, $\alpha_H,\beta_H,\psi_H,\omega_H,\psi_B,\omega_B$ 是双射. 如果对任意的 $a,b\in B,h,g\in H$, 满足

1) 作为 k-空间, $B\times H{=}B\otimes H$,
2) BiHom-余乘法, 即
$\Delta(b\times h)=(b_1\times\alpha_H^{-1}\psi_H^{-1}(b_{2(-1)})\beta_H^{-1}(h_1))\otimes(\psi_B^{-1}(b_{2(0)})\times h_2).$,

那么 $(B\times H,\alpha_B\times\alpha_H,\beta_B\times\beta_H)$ 是 $BiHom$-余结合余代数. 称其为 $BiHom$-$Smash$ 余积.余单位为 $\varepsilon_B\times\varepsilon_H$.

证明 由 $BiHom$-余模余代数和 $BiHom$-$Smash$-余积的定义得:

$$\begin{aligned}
&(\Delta\otimes\psi_B)\Delta(b\times h)\\
=\;&(\Delta\otimes\psi_B)(b_1\otimes\alpha_H^{-1}\psi_H^{-1}(b_{2(-1)})\beta_H^{-1}(h_1)\otimes\psi_B^{-1}(b_{2(0)})\otimes h_2)\\
=\;&[b_{11}\otimes\alpha_H^{-1}\psi_H^{-1}(b_{12(-1)})(\beta_H^{-1}\alpha_H^{-1}\psi_H^{-1}(b_{2(-1)1})\beta_H^{-2}(h_{11}))]\\
&\otimes[\psi_B^{-1}(b_{12(0)})\otimes\alpha_H^{-1}\psi_H^{-1}(b_{2(-1)2})\beta_H^{-1}(h_{12})\otimes(b_{2(0)}\otimes\psi_H(h_2))]\\
=\;&\omega_B(b_1)\otimes\alpha_H^{-1}\psi_H^{-1}(b_{21(-1)})(\beta_H^{-1}\alpha_H^{-1}\psi_H^{-2}(b_{22(-1)1})\omega_H\beta_H^{-2}(h_1))\\
&\otimes\psi_B^{-1}(b_{21(0)})\otimes\alpha_H^{-1}\psi_H^{-2}(b_{22(-1)2})\beta_H^{-1}(h_{21})\otimes(\psi_B^{-1}(b_{22(0)})\otimes h_{22})\\
=\;&\omega_B(b_1)\otimes(\alpha_H^{-2}\psi_H^{-1}(b_{21(-1)})\beta_H^{-1}\alpha_H^{-1}\psi_H^{-2}(b_{22(-1)1})\omega_H\beta_H^{-1}(h_1)\\
&\otimes\psi_B^{-1}(b_{21(0)})\otimes\alpha_H^{-1}\psi_H^{-2}(b_{22(-1)2})\beta_H^{-1}(h_{21})\otimes(\psi_B^{-1}(b_{22(0)})\otimes h_{22})\\
=\;&\omega_B(b_1)\otimes(\alpha_H^{-2}\psi_H^{-1}(b_{21(-1)})\beta_H^{-1}\alpha_H^{-1}\psi_H^{-2}\omega_H(b_{22(-1)}))\omega_H\beta_H^{-1}(h_1)\\
&\otimes\psi_B^{-1}(b_{21(0)})\otimes\alpha_H^{-1}\psi_H^{-2}(b_{22(0)(-1)})\beta_H^{-1}(h_{21})\otimes(\psi_B^{-2}(b_{22(0)(0)})\otimes h_{22})\\
=\;&\omega_B(b_1)\otimes\omega_H\alpha_H^{-1}\psi_H^{-1}(b_{2(-1)})\omega_H\beta_H^{-1}(h_1)\otimes\psi_B^{-1}(b_{2(0)1})\\
&\otimes\alpha_H^{-1}\psi_H^{-2}(b_{2(0)2(-1)})\beta_H^{-1}(h_{21})\otimes(\psi_B^{-2}(b_{2(0)2(0)})\otimes h_{22})
\end{aligned}$$

$$
\begin{aligned}
&= (\omega_H \otimes \Delta)(b_1 \otimes \alpha_H^{-1}\psi_H^{-1}(b_{2(-1)})\beta_H^{-1}(h_1) \otimes \psi_B^{-1}(b_{2(0)}) \otimes h_2) \\
&= (\omega_H \otimes \Delta)\Delta(b \times h).
\end{aligned}
$$

说明 $(\Delta \otimes \psi_B)\Delta(b \times h) = (\omega_H \otimes \Delta)\Delta(b \times h)$. 命题 1 证毕.

引理 1 设 $(H, \alpha_H, \beta_H, \psi_H, \omega_H)$ 是 $BiHom$-双代数, (B, α_B, β_B) 是 $BiHom$-代数,若 (B, α_B, β_B) 是左 H-$BiHom$-余模,则 (B, α_B, β_B) 被称为左 H-$BiHom$-余模代数,其中余模作用为:

$$
\rho(ab) = a_{(-1)}b_{(-1)} \otimes a_{(0)}b_{(0)}, \rho(1_B) = 1_H \otimes 1_B.
$$

引理 2 设 $(H, \alpha_H, \beta_H, \psi_H, \omega_H)$ 是 $BiHom$-双代数, (B, ψ_B, ω_B) 是 $BiHom$-余代数,若 (B, α_B, β_B) 是左 H-$BiHom$-模,则 (B, α_B, β_B) 被称为左 H-$BiHom$-模余代数,其中模作用为:

$$
\Delta(h \cdot b) = h_1 \cdot b_1 \otimes h_2 \cdot b_2, \varepsilon_B(h \cdot b) = \varepsilon_H(h)\varepsilon_B(b).
$$

定理 1 设 $(H, \alpha_H, \beta_H, \psi_H, \omega_H)$, $(B, \alpha_B, \beta_B, \psi_B, \omega_B)$ 是 $BiHom$-双代数, $\alpha_H, \beta_H, \psi_H,$ $\omega_H, \alpha_B, \beta_B, \psi_B, \omega_B$ 是双射. 则 $BiHom$-$Smash$ 积 $(B\sharp H, \alpha_B\sharp\alpha_H, \beta_B\sharp\beta_H)$ 和 $BiHom$-$Smash$ 余积 $(B \times H, \alpha_B \times \alpha_H, \beta_B \times \beta_H)$ 形成 $BiHom$-双代数, 当且仅当对任意的 $a \in A$, $h \in H$, 下列条件等价:

(I) $(B_\sharp^\times H, \alpha_B \times \alpha_H, \beta_B \times \beta_H)$ 是 $BiHom$-双代数.

(II)以下条件成立,

$c_1)\varepsilon_B$ 是代数同态且 $\Delta_B(1_B) = 1_B \otimes 1_B$.

$c_2)(B, \alpha_B, \beta_B, \psi_B, \omega_B)$ 是左 $(H, \alpha_H, \beta_H, \psi_H, \omega_H)$ $BiHom$-模余代数,

$c_3)(B, \alpha_B, \beta_B, \psi_B, \omega_B)$ 是左 $(H, \alpha_H, \beta_H, \psi_H, \omega_H)$ $BiHom$-余模代数,

$c_4)\Delta_B(ab) = a_1(a_{2(-1)} \cdot \beta^{-1}(b_1)) \otimes \beta(a_{2(0)})b_2, (4)$

$c_5)h_1 \cdot b_{(-1)} \otimes h_2 \cdot b_{(0)} = \alpha_H^{-1}((\omega_H^{-1}(h_1) \cdot b)_{(-1)})$
$\alpha_H\omega_H\beta_H^{(-1)}\psi_H^{(-1)}(h_2) \otimes ((\omega_H^{-1}(h_1) \cdot b)_{(-1)})$.

证明 首先,我们先证明 $(I) \Rightarrow (II)$.
因为 $\Delta_{B\times H}$ 和 $\varepsilon_{B\times H}$ 都是代数同态,对任意的 $a, b \in B.h, k \in H$. 我们有

$$
\begin{aligned}
\varepsilon((a \times h)(b \times k)) &= \varepsilon(a(\beta_H^{-1}\omega_H^{-1}(h_1) \cdot \beta_B^{-1}(b)) \times (\psi_H^{-1}(h_2)k)) \\
&= \varepsilon_B(a(\beta_H^{-1}\omega_H^{-1}(h_1) \cdot \beta_B^{-1}(b)))\varepsilon_H(h_2)\varepsilon_H(k) \\
&= \varepsilon_B(a(\beta_H^{-1}(h) \cdot \beta_B^{-1}(b)))\varepsilon_H(k) \\
\varepsilon(a \times h)\varepsilon(b \times k) &= \varepsilon_B(a)\varepsilon_H(h)\varepsilon_B(b)\varepsilon_H(k).
\end{aligned}
$$

令 $h = k = 1_H$. 可以得到 $\varepsilon_B(ab) = \varepsilon_B(a)\varepsilon_B(b)$. 令 $a = 1_B, k = 1_H$, 可以得到

$$
\varepsilon_B(1_B(\beta_H^{-1}(h) \cdot \beta_B^{-1}(b)))\varepsilon_H(1_H) = \varepsilon_B(h \cdot b) = \varepsilon_H(h)\varepsilon_B(h).
$$

因为

$$\begin{aligned}\Delta(1_B\times 1_H) &= (1_{B1}\otimes\alpha_H^{-1}\psi_H^{-1}(1_{B2(-1)})\beta_H^{-1}(1_H))\otimes(\psi_B^{-1}(1_{B2(0)})\otimes 1_H)\\ &= (1_{B1}\times\psi_H^{-1}(1_{B2(-1)}))\otimes(\psi_B^{-1}(1_{B2(0)})\times 1_H)\\ &= (1_B\times 1_H)\otimes(1_B\times 1_H).\end{aligned}$$

左右两边同时作用 $Id\otimes\varepsilon_H\otimes Id\otimes\varepsilon_H$, 可以得到 $\Delta(1_B)=1_B\otimes 1_B$.

左右两边同时作用 $\varepsilon_B\otimes Id\otimes Id\otimes\varepsilon_H$, 可以得到

$$\varepsilon_B(1_{B1})\psi_H^{-1}(1_{B2(-1)})\otimes\psi_B^{-1}(1_{B2(0)})\varepsilon_H(1_H)=1_{B(-1)}\otimes 1_{B(0)}=\rho(1_B)=1_H\otimes 1_B.$$

令 $h=k=1_H$. 因为 $\Delta((a\times 1_H)(b\times 1_H))=\Delta(a\times 1_H)\Delta(b\times 1_H)$, 所以我们可以计算.

$$\begin{aligned}&\Delta((a\times 1_H)(b\times 1_H))\\ =\ &\Delta(a(\beta_H^{-1}\omega_H^{-1}(1_H)\cdot\beta_B^{-1}(b))\times\psi_H^{-1}(1_H)\cdot 1_H)=\Delta(ab\times 1_H)\\ =\ &((ab)_1\times\alpha_H^{-1}\psi_H^{-1}((ab)_{2(-1)})\cdot\beta_H^{-1}(1_H))\otimes(\psi_B^{-1}((ab)_{2(0)})\times 1_H)\\ =\ &((ab)_1\times\psi_H^{-1}((ab)_{2(-1)}))\otimes(\psi_B^{-1}((ab)_{2(0)})\times 1_H).\end{aligned}$$

$$\begin{aligned}&\Delta(a\times 1_H)\Delta(b\times 1_H)\\ =\ &[(a_1\times\alpha_H^{-1}\psi_H^{-1}(a_{2(-1)})\cdot\beta_H^{-1}(1_H))\otimes(\psi_B^{-1}(a_{2(0)})\times 1_H)]\\ &[(b_1\times\alpha_H^{-1}\psi_H^{-1}(b_{2(-1)})\cdot\beta_H^{-1}(1_H))\otimes(\psi_B^{-1}(b_{2(0)})\times 1_H)]\\ =\ &[(a_1\times\psi_H^{-1}(a_{2(-1)}))(b_1\times\psi_H^{-1}(b_{2(-1)}))]\otimes[(\psi_B^{-1}(a_{2(0)})\\ &\times 1_H)(\psi_B^{-1}(b_{2(0)})\times 1_H)]\\ =\ &(a_1(\beta_H^{-1}\omega_H^{-1}\psi_H^{-1}(a_{2(-1)1})\cdot\beta_B^{-1}(b_1))\times\psi_H^{-2}(a_{2(-1)2})\cdot\psi_H^{-1}(b_{2(-1)}))\\ &\otimes(\psi_B^{-1}(a_{2(0)})(\beta_H^{-1}\omega_H^{-1}(1_H)\cdot\beta_B^{-1}(\psi_B^{-1}(b_{2(0)}))\times\psi_H^{-1}(1_H)\cdot 1_H))\\ =\ &(a_1(\beta_H^{-1}\omega_H^{-1}\psi_H^{-1}(a_{2(-1)1})\cdot\beta_B^{-1}(b_1))\times\psi_H^{-2}(a_{2(-1)2})\cdot\psi_H^{-1}(b_{2(-1)}))\\ &\otimes\psi_B^{-1}(a_{2(0)})(\psi_B^{-1}(b_{2(0)})\times 1_H)).\end{aligned}$$

因此,

$$\begin{aligned}&((ab)_1\times\psi_H^{-1}((ab)_{2(-1)}))\otimes(\psi_B^{-1}((ab)_{2(0)})\times 1_H)\\ =\ &(a_1(\beta_H^{-1}\omega_H^{-1}\psi_H^{-1}(a_{2(-1)1})\cdot\beta_B^{-1}(b_1))\\ &\times\psi_H^{-2}(a_{2(-1)2})\cdot\psi_H^{-1}(b_{2(-1)}))\otimes(\psi_B^{-1}(a_{2(0)})(\psi_B^{-1}(b_{2(0)})\times 1_H)).\end{aligned}$$

两边同时作用 $\varepsilon_B\otimes Id\otimes Id\otimes\varepsilon_H$.

$$\begin{aligned}
&\varepsilon_B((ab)_1)\psi_H^{-1}((ab)_{2(-1)})\otimes\psi_B^{-1}((ab)_{2(0)})\varepsilon_H(1_H)=(ab)_{(-1)}\otimes(ab)_{(0)}.\\
&\varepsilon_B(a_1(\beta_H^{-1}\omega_H^{-1}\psi_H^{-1}(a_{2(-1)1})\cdot\beta_B^{-1}(b_1)))(\psi_H^{-2}(a_{2(-1)2})\psi_H^{-1}(b_{2(-1)}))\\
&\otimes(\psi_B^{-1}(a_{2(0)})\psi_B^{-1}(b_{2(0)}))\varepsilon_H(1_H)\\
=\;&\varepsilon_B(a_1)\varepsilon_B(a_{2(-1)1})\varepsilon_B(b_1)\psi_H^{-2}(a_{2(-1)2})\psi_H^{-1}(b_{2(-1)})\\
&\otimes\psi_B^{-1}(a_{2(0)})\psi_H^{-1}(b_{2(-1)}\psi_B^{-1}(b_{2(0)})\\
=\;&\varepsilon_B(a_1)\psi_H^{-1}(a_{2(-1)})b_{(-1)}\otimes\psi_B^{-1}(a_{2(0)})b_{(0)}\\
=\;&a_{(-1)}b_{(-1)}\otimes a_{(0)}b_{(0)}.
\end{aligned}$$

可以得到 $\rho(ab)=(ab)_{(-1)}\otimes(ab)_{(0)}=a_{(-1)}b_{(-1)}\otimes a_{(0)}b_{(0)}$.

两边再次同时作用 $Id\otimes\varepsilon_H\otimes Id\otimes\varepsilon_H$.

$$\begin{aligned}
&(ab)_1\varepsilon_H(\psi_H^{-1}((ab)_{2(-1)}))\otimes\psi_B^{-1}((ab)_{2(0)})\varepsilon_H(1_H)\\
=\;&(ab)_1\otimes(ab)_2=\Delta(ab).
\end{aligned}$$

$$\begin{aligned}
&(a_1(\beta_H^{-1}\omega_H^{-1}\psi_H^{-1}(a_{2(-1)1})\cdot\beta_B^{-1}(b_1)))\varepsilon_H(\psi_H^{-1}(a_{2(-1)2})\\
&\psi_H^{-1}(b_{2(-1)}))\otimes\psi_B^{-1}(a_{2(0)})\psi_B^{-1}(b_{2(0)})\varepsilon_H(1_H)\\
=\;&(a_1(\beta_H^{-1}\omega_H^{-1}\psi_H^{-1}(a_{2(-1)1})\cdot\beta_B^{-1}(b_1)))\varepsilon_H(a_{2(-1)2})\varepsilon_H(b_{2(-1)})\\
&\otimes\psi_B^{-1}(a_{2(0)})\psi_B^{-1}(b_{2(0)})\\
=\;&(a_1(\beta_H^{-1}\psi_H^{-1}(a_{2(-1)})\cdot\beta_B^{-1}(b_1)))\otimes\psi_B^{-1}(a_{2(0)})b_2.
\end{aligned}$$

可以得到 $\Delta(ab)=(ab)_1\otimes(ab)_2=(a_1(\beta_H^{-1}\psi_H^{-1}(a_{2(-1)})\cdot\beta_B^{-1}(b_1)))\otimes\psi_B^{-1}(a_{2(0)})b_2$, 即 $c4)$ 成立.

令 $a=1_B, k=1_H$. 因为 $\Delta((1_B\times h)(b\times 1_H))=\Delta(1_B\times h)\Delta(b\times 1_H)$, 所以我们可以计算.

$$\begin{aligned}
&\Delta((1_B\times h)(b\times 1_H))\\
=\;&\Delta(1_B(\beta_H^{-1}\omega_H^{-1}(h_1)\cdot\beta_B^{-1}(b))\times\psi_H^{-1}(h_2)\cdot 1_H)\\
=\;&\Delta(\omega_H^{-1}(h_1)\cdot b\times\alpha_H\psi_H^{-1}(h_2))\\
=\;&(\omega_H^{-1}(h_1)\cdot b)_1\times\alpha_H^{-1}\psi_H^{-1}((\omega_H^{-1}(h_1)\cdot b)_{2(-1)})\cdot\beta_H^{-1}\alpha_H\psi_H^{-1}(h_{21})\\
&\otimes\psi_B^{-1}((\omega_H^{-1}(h_1)\cdot b)_{2(0)})\times\alpha_H\psi_H^{-1}(h_{22}).
\end{aligned}$$

$$\begin{aligned}
&\Delta(1_B\times h)\Delta(b\times 1_H)\\
=\;&[1_{B1}\times\alpha_H^{-1}\psi_H^{-1}(1_{B2(-1)})\cdot\beta_H^{-1}(h_1)\otimes\psi_B^{-1}(1_{B2(0)})\times h_2][b_1\\
&\times\alpha_H^{-1}\psi_H^{-1}(b_{2(-1)})\cdot\beta_H^{-1}(1_H)\otimes\psi_B^{-1}(b_{2(0)})\times 1_H]\\
=\;&[(1_B\times h_1)\otimes(1_B\times h_2)][(b_1\times\psi_H^{-1}(b_{2(-1)}))\otimes(\psi_B^{-1}(b_{2(0)})\times 1_H)]\\
=\;&[(1_B\times h_1)(b_1\times\psi_H^{-1}(b_{2(-1)}))]\otimes[(1_B\times h_2)(\psi_B^{-1}(b_{2(0)})\times 1_H)]\\
=\;&[1_B(\beta_H^{-1}\omega_H^{-1}(h_{11})\cdot\beta_B^{-1}(b_1))\times\psi_H^{-1}(h_{12})\cdot\psi_H^{-1}(b_{2(-1)})]\\
&[1_B(\beta_H^{-1}\omega_H^{-1}(h_{21})\cdot\beta_B^{-1}\psi_B^{-1}(b_{2(0)}))\times\psi_H^{-1}(h_{22})\cdot 1_H]\\
=\;&[\omega_H^{-1}(h_{11})\cdot b_1\times\psi_H^{-1}(h_{12})\cdot\psi_H^{-1}(b_{2(-1)})][\omega_H^{-1}(h_{21})\cdot\psi_B^{-1}(b_{2(0)})\\
&\times\alpha_H\psi_H^{-1}(h_{22})].
\end{aligned}$$

可以得到:

$$\begin{aligned}
&(\omega_H^{-1}(h_1)\cdot b)_1\times\alpha_H^{-1}\psi_H^{-1}((\omega_H^{-1}(h_1)\cdot b)_{2(-1)})\cdot\beta_H^{-1}\alpha_H\psi_H^{-1}(h_{21})\\
&\otimes\psi_B^{-1}((\omega_H^{-1}(h_1)\cdot b)_{2(0)})\times\alpha_H\psi_H^{-1}(h_{22})\\
=\;&[\omega_H^{-1}(h_{11})\cdot b_1\times\psi_H^{-1}(h_{12})\cdot\psi_H^{-1}(b_{2(-1)})][\omega_H^{-1}(h_{21})\cdot\psi_B^{-1}(b_{2(0)})\\
&\times\alpha_H\psi_H^{-1}(h_{22})]
\end{aligned}$$

两边同时作用 $Id\otimes\varepsilon_H\otimes Id\otimes\varepsilon_H$.

$$\begin{aligned}
&(\omega_H^{-1}(h_1)\cdot b)_1\varepsilon_H(\alpha_H^{-1}\psi_H^{-1}((\omega_H^{-1}(h_1)\cdot b)_{2(-1)}))\varepsilon_H(\beta_H^{-1}\alpha_H\psi_H^{-1}(h_{21}))\\
&\otimes\psi_B^{-1}((\omega_H^{-1}(h_1)\cdot b)_{2(0)})\varepsilon_H(\alpha_H\psi_H^{-1}(h_{22}))\\
=\;&(\omega_H^{-1}(h_1)\cdot b)_1\varepsilon_H((\omega_H^{-1}(h_1)\cdot b)_{2(-1)}))\varepsilon_H(h_{21})\\
&\otimes\psi_B^{-1}((\omega_H^{-1}(h_1)\cdot b)_{2(0)})\varepsilon_H(h_{22})\\
=\;&(\omega_H^{-1}(h_1)\cdot b)_1\otimes(\omega_H^{-1}(h_1)\cdot b)_2\varepsilon_H(h_2)\\
=\;&(h\cdot b)_1\otimes(h\cdot b)_2.
\end{aligned}$$

$$\begin{aligned}
&\omega_H^{-1}(h_{11})\cdot b_1\varepsilon_H(\psi_H^{-1}(h_{12}))\varepsilon_H(\psi_H^{-1}(b_{2(-1)}))\\
&\otimes\omega_H^{-1}(h_{21})\psi_B^{-1}(b_{2(0)})\varepsilon_H(\alpha_H\psi_H^{-1}(h_{22}))\\
=\;&\omega_H^{-1}(h_{11})\cdot b_1\varepsilon_H(h_{12})\varepsilon_H(b_{2(-1)})\otimes\omega_H^{-1}(h_{21})\psi_B^{-1}(b_{2(0)})\varepsilon_H(h_{22})\\
=\;&h_1\cdot b_1\otimes h_2\cdot b_2.
\end{aligned}$$

可以得到 $\Delta(h\cdot b)=(h\cdot b)_1\otimes(h\cdot b)_2=h_1\cdot b_1\otimes h_2\cdot b_2$.

两边再同时作用 $\varepsilon_B\otimes Id\otimes Id\otimes\varepsilon_H$.

$$
\begin{aligned}
&\varepsilon_B((\omega_H^{-1}(h_1)\cdot b)_1)(\alpha_H^{-1}\psi_H^{-1}((\omega_H^{-1}(h_1)\cdot b)_{2(-1)}))\varepsilon_H(\beta_H^{-1}\alpha_H\psi_H^{-1}(h_{21}))\\
&\otimes\psi_B^{-1}((\omega_H^{-1}(h_1)\cdot b)_{2(0)})\varepsilon_H(\alpha_H\psi_H^{-1}(h_{22}))\\
=\;&\alpha_H^{-1}((\omega_H^{-1}(h_1)\cdot b)_{(-1)})\alpha_H\beta_H^{-1}\psi_H^{-1}\omega_H(h_2)\otimes(\omega_H^{-1}(h_1)\cdot b)_{(0)}.\\
&\varepsilon_B(\omega_H^{-1}(h_{11})\cdot b_1)\psi_H^{-1}(h_{12})\psi_H^{-1}(b_{2(-1)})\otimes\omega_H^{-1}(h_{21})\cdot\psi_B^{-1}(b_{2(0)})\varepsilon_H(\alpha_H\psi_H^{-1}(h_{22}))\\
=\;&\varepsilon_B(h_{11})\varepsilon_B(b_1)\psi_H^{-1}(h_{12})\psi_H^{-1}(b_{2(-1)})\otimes\omega_H^{-1}(h_{21})\psi_B^{-1}(b_{2(0)})\varepsilon_H(h_{22})\\
=\;&h_1b_{(-1)}\otimes h_2b_{(0)}.
\end{aligned}
$$

可以得到 $h_1b_{(-1)}\otimes h_2b_{(0)}=\alpha_H^{-1}((\omega_H^{-1}(h_1)\cdot b)_{(-1)})\alpha_H\beta_H^{-1}\psi_H^{-1}\omega_H(h_2)\otimes(\omega_H^{-1}(h_1)\cdot b)_{(0)}$. 即 $c5)$ 成立. 所以,$c1)-c5)$ 都成立.

现在,我们来证明 $(II)\Rightarrow(I)$.很容易验证 $\varepsilon((a\times h)(b\times k))=\varepsilon(a\times h)\varepsilon(b\times k),\varepsilon(1_B\times 1_H)=\varepsilon(1_k),\Delta(1_B\times 1_H)=(1_B\times 1_H)\otimes(1_B\times 1_H)$. 根据 $BiHom$-余结合余代数的定义可知, 对任意的 $h\in H$, 我们有

$$\omega_H(a_{(-1)})\otimes a_{(0)(-1)}\otimes a_{(0)(0)}=a_{(-1)1}\otimes a_{(-1)2}\otimes\psi_B^{-1}(a_{(0)}). \tag{1}$$

$$h_{11}\otimes h_{12}\otimes h_{21}\otimes h_{22}=h_{11}\otimes\omega_H^{-1}(h_{121})\otimes\psi_H^{-1}(h_{122})\otimes\psi_H(h_2). \tag{2}$$

现在我们计算

$$
\begin{aligned}
&\Delta((a\times h)(b\times k))\\
=\;&\Delta(a(\beta_H^{-1}\omega_H^{-1}(h_1)\cdot\beta_B^{-1}(b)\times\psi_H^{-1}(h_2)k)\\
=\;&[a(\beta_H^{-1}\omega_H^{-1}(h_1)\cdot\beta_B^{-1}(b))]_1\times\alpha_H^{-1}\psi_H^{-1}([a(\beta_H^{-1}\omega_H^{-1}(h_1)\cdot\beta_B^{-1}(b))]_{2(-1)})\beta_H^{-1}(\psi_H^{-1}(h_{21})k_1)\\
&\otimes\psi_B^{-1}([a(\beta_H^{-1}\omega_H^{-1}(h_1)\cdot\beta_B^{-1}(b))]_{2(0)})\times\psi_H^{-1}(h_{22})k_2\\
\overset{((1),c2)}{=}\;&a_1(\beta_H^{-1}\psi_H^{-1}(a_{2(-1)})\cdot\beta_B^{-1}(\beta_H^{-1}\omega_H^{-1}(h_{11})\cdot\beta_B^{-1}(b_1)))\\
&\times\alpha_H^{-1}\psi_H^{-1}((\psi_B^{-1}(a_{2(0)})(\beta_H^{-1}\omega_H^{-1}(h_{12})\cdot\beta_B^{-1}(b_2)))_{(-1)})\beta_H^{-1}(\psi_H^{-1}(h_{21})k_1)\\
&\otimes\psi_B^{-1}((\psi_B^{-1}(a_{2(0)})(\beta_H^{-1}\omega_H^{-1}(h_{12})\cdot\beta_B^{-1}(b_2)))_{(0)})\times\psi_H^{-1}(h_{22})k_2\\
\overset{(c3)}{=}\;&a_1(\beta_H^{-1}\psi_H^{-1}(a_{2(-1)})\cdot(\beta_H^{-2}\omega_H^{-1}(h_{11})\cdot\beta_B^{-2}(b_1)))\\
&\times(\alpha_H^{-1}\psi_H^{-2}(a_{2(0)(-1)})\alpha_H^{-1}\psi_H^{-1}((\beta_H^{-1}\omega_H^{-1}(h_{12})\cdot\beta_B^{-1}(b_2))_{(-1)}))(\beta_H^{-1}\psi_H^{-1}(h_{21})\beta_H^{-1}(k_1))\\
&\otimes(\psi_B^{-2}(a_{2(0)(0)})\psi_B^{-1}((\beta_H^{-1}\omega_H^{-1}(h_{12})\cdot\beta_B^{-1}(b_2))_{(0)})\times\psi_H^{-1}(h_{22})k_2\\
=\;&a_1(\alpha_H^{-1}\beta_H^{-1}\psi_H^{-1}(a_{2(-1)})\beta_H^{-2}\omega_H^{-1}(h_{11}))\cdot\beta_B^{-1}(b_1))\\
&\times[(\alpha_H^{-2}\psi_H^{-2}(a_{2(0)(-1)})\alpha_H^{-2}\psi_H^{-1}((\beta_H^{-1}\omega_H^{-1}(h_{12})\cdot\beta_B^{-1}(b_2))_{(-1)}))\beta_H^{-1}\psi_H^{-1}(h_{21})]k_1\\
&\otimes(\psi_B^{-2}(a_{2(0)(0)})\psi_B^{-1}((\beta_H^{-1}\omega_H^{-1}(h_{12})\cdot\beta_B^{-1}(b_2))_{(0)})\times\psi_H^{-1}(h_{22})k_2\\
=\;&a_1((\alpha_H^{-1}\beta_H^{-1}\psi_H^{-1}(a_{2(-1)})\beta_H^{-2}\omega_H^{-1}(h_{11}))\cdot\beta_B^{-1}(b_1))\\
&\times[\alpha_H^{-1}\psi_H^{-2}(a_{2(0)(-1)})(\alpha_H^{-2}\psi_H^{-1}((\beta_H^{-1}\omega_H^{-1}(h_{12})\cdot\beta_B^{-1}(b_2))_{(-1)})\beta_H^{-2}\psi_H^{-1}(h_{21}))]k_1
\end{aligned}
$$

$$
\begin{aligned}
&\otimes\psi_B^{-2}(a_{2(0)(0)})\psi_B^{-1}((\beta_H^{-1}\omega_H^{-1}(h_{12})\cdot\beta_B^{-1}(b_2))_{(0)})\times\psi_H^{-1}(h_{22})k_2\\
\overset{(2)}{=}\ &a_1((\alpha_H^{-1}\beta_H^{-1}\psi_H^{-1}(a_{2(-1)})\beta_H^{-2}\omega_H^{-1}(h_{11}))\cdot\beta_B^{-1}(b_1))\\
&\times[\alpha_H^{-1}\psi_H^{-2}(a_{2(0)(-1)})\alpha_H^{-1}\psi_H^{-1}(\alpha_H^{-1}((\beta_H^{-1}\omega_H^{-2}(h_{121})\cdot\beta_B^{-1}(b_2))_{(-1)})\alpha_H^{-1}\beta_H^{-2}\psi_H^{-1}(h_{122}))]k_1\\
&\otimes\psi_B^{-2}(a_{2(0)(0)})\psi_B^{-1}((\beta_H^{-1}\omega_H^{-2}(h_{121})\cdot\beta_B^{-1}(b_2))_{(0)})\times h_2k_2\\
\overset{(c5)}{=}\ &a_1((\alpha_H^{-1}\beta_H^{-1}\psi_H^{-1}(a_{2(-1)})\beta_H^{-2}\omega_H^{-1}(h_{11}))\cdot\beta_B^{-1}(b_1))\\
&\times[\alpha_H^{-1}\psi_H^{-2}(a_{2(0)(-1)})(\alpha_H^{-1}\psi_H^{-1}\beta_H^{-1}\omega_H^{-1}(h_{121})\alpha_H^{-1}\psi_H^{-1}\beta_B^{-1}(b_{2(-1)}))]k_1\\
&\otimes\psi_B^{-2}(a_{2(0)(0)})\psi_B^{-1}(\beta_H^{-1}\omega_H^{-1}(h_{122})\cdot\beta_B^{-1}(b_{2(0)}))\times h_2k_2\\
\overset{((1),(2))}{=}\ &a_1((\alpha_H^{-1}\beta_H^{-1}\psi_H^{-1}\omega_H^{-1}(a_{2(-1)})\beta_H^{-2}\omega_H^{-1}(h_{11}))\cdot\beta_B^{-1}(b_1))\\
&\times[(\alpha_H^{-2}\psi_H^{-2}(a_{2(-1)2})\alpha_H^{-1}\psi_H^{-1}\beta_H^{-1}(h_{12}))(\alpha_H^{-1}\psi_H^{-1}(b_{2(-1)}))]k_1\\
&\otimes\psi_B^{-2}(a_{2(0)})(\beta_H^{-1}\omega_H^{-1}(h_{21})\cdot\beta_B^{-1}\psi_B^{-1}(b_{2(0)}))\times\psi_H^{-1}h_{22}k_2\\
=\ &a_1((\alpha_H^{-1}\beta_H^{-1}\psi_H^{-1}\omega_H^{-1}(a_{2(-1)})\beta_H^{-2}\omega_H^{-1}(h_{11}))\cdot\beta_B^{-1}(b_1))\\
&\times(\alpha_H^{-1}\psi_H^{-2}(a_{2(-1)2})\psi_H^{-1}\beta_H^{-1}(h_{12}))(\alpha_H^{-1}\psi_H^{-1}(b_{2(-1)})\beta_H^{-1}(k_1))\\
&\otimes\psi_B^{-1}(a_{2(0)})(\beta_H^{-1}\omega_H^{-1}(h_{21})\cdot\beta_B^{-1}\psi_B^{-1}(b_{2(0)}))\times\psi_H^{-1}h_{22}k_2.\\
&\Delta(a\times h)\Delta(b\times h)\\
=\ &[a_1\times(\alpha_H^{-1}\psi_H^{-1}(a_{2(-1)})\beta_H^{-1}(h_1))\otimes\psi_B^{-1}(a_{2(0)})\times h_2][b_1\times(\alpha_H^{-1}\psi_H^{-1}(b_{2(-1)})\beta_H^{-1}(k_1))\\
&\otimes\psi_B^{-1}(b_{2(0)})\times k_2]\\
=\ &[(a_1\times\alpha_H^{-1}\psi_H^{-1}(a_{2(-1)})\cdot\beta_H^{-1}(h_1))(b_1\times\alpha_H^{-1}\psi_H^{-1}(b_{2(-1)})\cdot\beta_H^{-1}(k_1))]\\
&\otimes[(\psi_B^{-1}(a_{2(0)})\times h_2)(\psi_B^{-1}(b_{2(0)})\times k_2)]\\
=\ &a_1((\beta_H^{-1}\omega_H^{-1}(\alpha_H^{-1}\psi_H^{-1}(a_{2(-1)})\cdot\beta_H^{-1})_1)\cdot\beta_B^{-1}(b_1))\\
&\times\psi_H^{-1}((\alpha_H^{-1}\psi_H^{-1}(a_{2(-1)})\cdot\beta_H^{-1}(h_1))_2)(\alpha_H^{-1}\psi_H^{-1}(b_{2(-1)})\cdot\beta_H^{-1}(k_1))\\
&\otimes\psi_B^{-1}(a_{2(0)})(\beta_H^{-1}\omega_H^{-1}(h_{21})\cdot\beta_B^{-1}(\psi_B^{-1}(b_{2(0)})))\times\psi_H^{-1}(h_{22})\cdot k_2\\
=\ &a_1((\alpha_H^{-1}\beta_H^{-1}\psi_H^{-1}\omega_H^{-1}(a_{2(-1)})\beta_H^{-2}\omega_H^{-1}(h_{11}))\cdot\beta_B^{-1}(b_1))\\
&\times(\alpha_H^{-1}\psi_H^{-2}(a_{2(-1)2})\psi_H^{-1}\beta_H^{-1}(h_{12}))(\alpha_H^{-1}\psi_H^{-1}(b_{2(-1)})\beta_H^{-1}(k_1))\\
&\otimes\psi_B^{-1}(a_{2(0)})(\beta_H^{-1}\omega_H^{-1}(h_{21})\cdot\beta_B^{-1}\psi_B^{-1}(b_{2(0)}))\times\psi_H^{-1}h_{22}k_2.
\end{aligned}
$$

因此我们有 $\Delta((a\times h)(b\times k))=\Delta(a\times h)\Delta(b\times h)$, 且 $(B_\sharp^\times H,\alpha_B\times\alpha_H,\beta_B\times\beta_H)$ 是 $BiHom$-双代数, 证明完毕.

若定理 1中的 $(B_\sharp^\times H,\alpha_B\times\alpha_H,\beta_B\times\beta_H)$ 是 $BiHom$-双代数,则我们称之为 $BiHom$-双代数的 $Radford's$- 双积, 或简称为 $Radford's$-$BiHom$-双代数,下面我们讨论什么情况下 $Radford's$-$BiHom$-双代数 $(B_\sharp^\times H,\alpha_B\times\alpha_H,\beta_B\times\beta_H)$ 是 $BiHom$-$Hopf$代数.

命题 2 设 $(B_{\sharp}^{\times}H, \alpha_B\times\alpha_H, \beta_B\times\beta_H)$ 是 *Radford's-BiHom*-双代数, 若 H 是 *BiHom-Hopf*-代数,其对极分别是 $S_H, S_B: H\rightarrow H.(S_B\circ\alpha_B=\alpha_B\circ S_B, S_B\circ\beta_B=\beta_B\circ S_B)$, 则 $(B_{\sharp}^{\times}H, \alpha_B\times\alpha_H, \beta_B\times\beta_H)$ 是 *BiHom-Hopf*-代数,其对极 S 为

$$S(b\times h)=(1_B\times S_H(\alpha_H^{-1}\beta_H^{-1}\psi_H^{-1}(b_{(-1)})\beta_H^{-2}\psi_H^{-1}\omega_H(h)))(S_B(\alpha_B\beta_B^{-2}\psi_B^{-2}\omega_B(b_{(0)})\times 1_H).$$

证明：对任意的 $b\in B, h\in H$, 有

$$\begin{aligned}
& (Id*S)(b\times h)\\
= & (b_1\times\alpha_H^{-1}\psi_H^{-1}(b_{2(-1)})\beta_H^{-1}(h_1))S_H(\psi_B^{-1}(b_{2(0)})\times h_2)\\
= & (b_1\times\alpha_H^{-1}\psi_H^{-1}(b_{2(-1)})\beta_H^{-1})[(1_B\times S_H(\alpha_H^{-1}\beta_H^{-1}\psi_H^{-2}(b_{2(0)(-1)})\beta_H^{-2}\psi_H^{-1}\omega_H(h_2)))\\
& (S_B(\alpha_B\beta_B^{-2}\psi_B^{-3}\omega_B(b_{2(0)(0)}))\times 1_H)]\\
= & [(\alpha_H^{-1}(b_1)\times\alpha_H^{-2}\psi_H^{-1}(b_{2(-1)})\alpha_H^{-1}\beta_H^{-1}(h_1))(1_B\times S_H(\alpha_H^{-1}\beta_H^{-1}\psi_H^{-2}(b_{2(0)(-1)})\\
& \beta_H^{-2}\psi_H^{-1}\omega_H(h_2)))](S_B\alpha_B\beta_B^{-1}\psi_B^{-3}\omega_B(b_{2(0)(0)})\times 1_H)\\
= & [\alpha_H^{-1}(b_1)(\beta_H^{-1}\omega_H^{-1}((\alpha_H^{-2}\psi_H^{-1}(b_{2(-1)})\alpha_H^{-1}\beta_H^{-1}(h_1))_1)\cdot\beta_B^{-1}(1_B))\times\psi_H^{-1}((\alpha_H^{-2}\psi_H^{-1}(b_{2(-1)})\\
& \alpha_H^{-1}\beta_H^{-1}(h_1))_2)S_H(\alpha_H^{-1}\beta_H^{-1}\psi_H^{-2}(b_{2(0)(-1)})\beta_H^{-2}\psi_H^{-1}\omega_H(h_2))]\\
& (S_B\alpha_B\beta_B^{-1}\psi_B^{-3}\omega_B(b_{2(0)(0)})\times 1_H)\\
= & [b_1\times(\alpha_H^{-2}\psi_H^{-1}(b_{2(-1)})\alpha_H^{-1}\beta_H^{-1}(h_1))(S_H(\alpha_H^{-1}\beta_H^{-1}\psi_H^{-2}(b_{2(0)(-1)})\beta_H^{-2}\psi_H^{-1}\omega_H(h_2))]\\
& (S_B\alpha_B\beta_B^{-1}\psi_B^{-3}\omega_B(b_{2(0)(0)})\times 1_H)\\
= & [b_1\times(\alpha_H^{-2}\psi_H^{-1}(b_{2(-1)}))((\alpha_H^{-1}\beta_H^{-1}(h_1)(S_H\alpha_H^{-1}\beta_H^{-2}\psi_H^{-1}\omega_H(h_2)))S_H\beta_H^{-2}\psi_H^{-2}(b_{2(0)(-1)})]\\
& (S_B\alpha_B\beta_B^{-1}\psi_B^{-3}\omega_B(b_{2(0)(0)})\times 1_H)\\
= & (b_1\times(\alpha_H^{-2}\psi_H^{-1}(b_{2(-1)}))S_H\beta_H^{-1}\psi_H^{-2}(b_{2(0)(-1)}))(S_B\alpha_B\beta_B^{-1}\psi_B^{-3}\omega_B(b_{2(0)(0)})\times 1_H)\varepsilon_H(h)\\
= & (b_1\times\alpha_H^{-2}\psi_H^{-1}(b_{2(-1)1})S_H\beta_H^{-1}\psi_H^{-2}(b_{2(-1)2}))(S_B\alpha_B\beta_B^{-1}\psi_B^{-2}\omega_B(b_{2(0)})\times 1_H)\varepsilon_H(h)\\
= & (b_1\times 1_H)(S_B\alpha_B\beta_B^{-1}\psi_B^{-1}\omega_B(b_2)\times 1_H)\varepsilon_H(h)\\
= & (b_1 S_B\alpha_B\beta_B^{-1}\psi_B^{-1}\omega_B(b_2)\times 1_H)\varepsilon_H(h)\\
= & 1_B\times 1_H\varepsilon_B(b)\varepsilon_H(h)\\
= & 1_B\times 1_H\varepsilon(b\times h).
\end{aligned}$$

说明 S 是 $(B_{\sharp}^{\times}H, \alpha_B\times\alpha_H, \beta_B\times\beta_H)$ 的对极.

4. 结语

在前人研究的基础上, 运用类比的思想方法, 经过大量的计算, 本文将 *Smash* 余积推广到 *BiHom*-代数上, 给出了 *BiHom-Smash* 余积的定义, 得到了 *BiHom-Smash* 积和 *BiHom-Smash* 余积形成 *Radford's-BiHom*-双代数的充分必要条件. 在本文的研究基础上, 将来可进一步研究 Yetter-Drinfeld 模以及拟三角等问题.

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