Advances in Applied Mathematics 应用数学进展, 2023, 12(10), 4415-4436
Published Online October 2023 in Hans. https://www.hanspub.org/journal/aam
https://doi.org/10.12677/aam.2023.1210434

多层结构下非线性方程组的粘性消失极限

童林曦，肖 慧，戴冰清，李 昂

上海师范大学数理学院，上海

收稿日期：2023年9月25日；录用日期：2023年10月18日；发布日期：2023年10月25日

摘 要

本文研究多层结构下一维非线性粘性方程组的粘性消失极限问题，证明当两个不相互作用的激波满足熵条件时，粘性方程组的解与无粘方程组的解之间具有渐近等价性。该问题的证明使用了与粘性激波剖面稳定性理论相关的匹配渐近分析和能量估计。我们首先通过多尺度的匹配渐近展开方法构造粘性方程组的近似解，再通过能量估计的方法进行稳定性分析从而得出最终结论。

关键词

粘性激波层，匹配渐近展开，能量估计

The Viscosity Vanishing Limit of Systems of Nonlinear Equations under Multiple Structures

Linxi Tong, Hui Xiao, Bingqing Dai, Ang Li

College of Mathematics and Science, Shanghai Normal University, Shanghai

Received: Sep. 25^{th}, 2023; accepted: Oct. 18^{th}, 2023; published: Oct. 25^{th}, 2023

文章引用: 童林曦, 肖慧, 戴冰清, 李昂. 多层结构下非线性方程组的粘性消失极限[J]. 应用数学进展, 2023, 12(10): 4415-4436. DOI: 10.12677/aam.2023.1210434

Abstract

In this thesis, we study the vanishing viscous limit of one-dimensional nonlinear viscous system with multi-layer structure. It is proved that when two non-interacting shock waves satisfy the entropy condition for the inviscid system, the asymptotic equivalence can be achieved between the solution of the viscous system and the solution of the inviscid system. This is proved based on matched asymptotic analysis and energy estimate related to the stability theory of viscous shock profile. First, the approximate solution of the viscosity system is constructed by the multi-scale matched asymptotic expansion method, and then the final conclusion is obtained by the stability analysis with the method of energy estimate.

Keywords

Viscous Shock Layer, Matched Asymptotic Expansion, Energy Estimates

1. 引言

本文中, 我们研究多层结构下带有粘性项的双曲方程组

$$\partial_t u^\varepsilon + \partial_x\big(f(u^\varepsilon(x,t))\big) = \varepsilon\partial_x\big(B(u^\varepsilon)\partial_x u^\varepsilon\big), \quad u^\varepsilon \in \mathbb{R}^n,\ x\in\mathbb{R}^1,\ t\geq 0, \tag{1}$$

的解与相应无粘双曲方程组

$$\partial_t u + \partial_x\big(f(u(x,t))\big) = 0, \quad u\in\mathbb{R}^n,\ x\in\mathbb{R}^1,\ t\geq 0, \tag{2}$$

的解的渐近等价关系, 其中 f 为光滑函数, 且存在 $g(u(x,t))$, 使得 $\nabla_u g(u(x,t)) = B(u(x,t))$.

粘性是流体的一个十分重要的属性, 有关粘性消失极限的问题在流体力学研究中一直备受瞩目. 在实际应用中, 无粘方程的解通常是不连续的, 就像激波和接触间断一样, 这类问题的研究更为复杂, 随着科学技术水平的提高和发展, 人们对粘性消失极限问题的解决提出了更迫切的要求. 对于零耗散问题 Hoff [1] 于 19 世纪首先提出单激波存在时, 一维等熵可压缩流动的 Navier-Stokes 方程

的无粘极限问题, 证明了方程的解始终存在, 并且当粘性趋于零时, 它收敛于无粘 Euler 方程激波解. 在此基础上, Goodman 和 Z.P. Xin [2] 对一般拟线性粘性守恒律方程关于粘性激波剖面稳定性定理进行了推广, 得到了粘性方程组与无粘方程组之间的渐近等价性. 后来, 通过更深入的研究, 人们证明了具有更丰富结构并且更加一般的粘性守恒律方程的消失粘性极限问题 [3–10], Z.P. Xin [11] 证明了具有中心稀疏波初值的 Navier-Stokes 方程的解在远离初始间断的区域随着粘性的消失而收敛到中心稀疏波; Y. Wang [12] 研究了当粘性系数 ε 满足 $k = O(\varepsilon)$, 并且激波满足松弛熵条件时, 可压缩导热 Navier-Stokes 方程的零耗散极限; H.Y. Wang [13] 通过基本波结构的尺度变换和基本能量分析的方法研究了当有限强度的稀疏波存在时, 粘性 p-系统等熵流动模型的零耗散问题, 对于不光滑的无粘方程的解, 激波、接触间断和稀疏波有可能会组合出现 [14–22], 这就使问题变得更加复杂起来. 在包含 Euler 方程激波、稀疏波和接触间断的一般叠加的情况下, F.M. Huang [16] 证明了 Boltzmann 方程存在唯一解并收敛于由三种基本波型组成的 Riemann 解所确定的 Maxwell 解. 对于同阶的粘性激波与粘性接触波的组合波, H.H. Zeng [23] 证明了具有小初始扰动的粘性守恒律组的渐近稳定性. Hakho 和 T. Wang [24] 研究了当相应的 Euler 系统的解由两个稀疏波和接触间断的复合波组成时, Navier-Stokes 方程的解的零耗散极限和衰减速率.

本文主要研究对于粘性项是非线性的双曲方程组, 我们都知道, 对于相互作用的激波, 即使在最简单的情况下, 在两个激波相撞后, 可能会产生激波, 还可能会产生接触间断 [25, 26], 这使得问题变得复杂起来, 因此本文我们主要讨论两个不相互作用的激波, 但类似的结果在有限多个非相互作用激波下也同样成立. 当两个激波满足 Lax 熵条件时, 粘性方程组的解收敛于无粘方程组的解. 文章的证明主要分两个部分, 构造近似解和能量估计. 在第二章里, 通过匹配渐近展开的方法, 我们构造出粘性方程的近似解. 首先分区域分别展开, 在构造展开式的每一项时, 我们利用了中心流形定理和稳定流形定理 [27, 28] 的相关理论得到构造的近似解是成立的. 随后对远离激波的无粘方程解进行高阶修正, 最终得到近似解, 并找到粘性方程组和无粘方程组之间解的形式联系. 在第三章中, 我们用能量估计的方法进行稳定性分析. 首先使用与粘性激波剖面稳定性相关的理论对主波进行先验估计, 再对非主波进行估计, 使其能够被主波控制, 最后进行更高阶导数的估计, 从而得到粘性方程的近似解与无粘方程的真实解的渐近等价性的证明. 较之前的系列研究, 本文得到了精度更高的收敛阶, 这也使得本文的研究更有意义.

对于 (1) 和 (2) , 我们有以下假设成立

(1) 激波满足 Lax 熵条件

在 $x = s_i(t)$, $i = 1, 2$, 有

$$
\begin{aligned}
\lambda_1(u(s_i(t)-0,t)) < \cdots < \lambda_p(u(s_i(t)-0,t)),\\
\lambda_p(u(s_i(t)-0,t)) > \frac{d}{dt}\dot{s}(t) > \lambda_p(u(s_i(t)+0,t)),\\
-0.1cm\lambda_p(u(s_i(t)+0,t)) < \cdots < \lambda_n(u(s_i(t)+0,t)).
\end{aligned}
$$

且对于任意 $t \le T$ 和 $k = 0, 1, 2, 3, 4$ 下列极限存在

$$
\begin{aligned}
\partial_x^k u(s_i(t)-0,t) = \lim_{x\to s_i(t)^-} \partial_x^k(u(x,t)),\\
\partial_x^k u(s_i(t)+0,t) = \lim_{x\to s_i(t)^+} \partial_x^k(u(x,t)),
\end{aligned}
$$

其中 $i=1,\ 2$;

(2) 对任意 $u\in\mathcal{U}$, 其中 $\mathcal{U}\in\mathbb{R}^{\mathbb{N}}$, $0\in\mathbb{R}^{\mathbb{N}}$, 有 (i) - (iv) 成立

(i) 存在正定矩阵 $S(u)$, 使得 SA 对称, 其中 $A(u)=f'(u)$, $A(u)$ 可对角化, 即存在正定矩阵 $R(u)$, $L(u)$ 使得

$$A=R\Lambda L, RL=I,$$

其中 $\Lambda=\text{diag}(\lambda_1,\lambda_2,\cdots,\lambda_n)$, $\lambda_1<\lambda_2<\cdots<\lambda_p<0<\lambda_{p+1}<\cdots\lambda_n$;

(ii) SB 对称正定, 即存在 $c_0>0$, 使得 $(SB\xi,\xi)\geq c_0|\xi|^2$, 对于任意 $\xi\in\mathbb{R}^N$;

(iii) LBR 是正定的.

主要结果

定理 1. 假设无粘方程组 (2) 是严格双曲的, 若 $u^0(x,t)\in C^2([0,T];\ H^2(\mathbb{R}))$ 是无粘方程组 (2) 的解, 对 $\forall t\in[0,T]$, $i=1,2$, 存在正常数 η_i, $i=1,\ 2$, 满足

$$|u^0(s_i(t)+0,t)-u^0(s_i(t)-0,t)|<\eta_i, \tag{3}$$

则对足够小的 $\varepsilon>0$, 给定的 $\eta\in(0,1)$, 粘性方程组 (1) 存在唯一一个光滑解 $u^\varepsilon(x,t)\in C^1([0,T];H^2(\mathbb{R}))$ 满足

$$\sup_{0\leq t\leq T}\int_{\mathbb{R}}|u^\varepsilon(\cdot,t)-u^0(\cdot,t)|^2dx\leq C\varepsilon^\eta, \tag{4}$$

$$\sup_{\substack{0\leq t\leq T\\|x-s_i(t)|\geq\varepsilon^\gamma}}|u^\varepsilon(\cdot,t)-u^0(\cdot,t)|\leq C\varepsilon,\quad i=1,2, \tag{5}$$

其中 $\gamma\in(\frac{4}{5},1)$.

2. 近似解的构造

本节中, 我们首先通过分区域渐近展开再匹配的方法构造出方程组 (1) 的近似解 $v^\varepsilon(x,t)$. 再利用中心流形定理证明近似解构造的合理性.

2.1. 内外部展开与匹配条件

首先, 在远离激波的区域, 我们先将方程组 (1) 的解以级数的形式展开

$$u^{Out}(x,t)\sim u^0(x,t)+\varepsilon u^1(x,t)+\varepsilon^2u^2(x,t)+\cdots \tag{6}$$

其中 $u^0,u^1\cdots$ 在激波 $x=s_i(t)$ 处不连续, 在其他区域光滑. 将 u^{Out} 代入 (1) 并根据 ε 的幂次进行分类, 可以得到

$$O(1):\partial_t u^0+\partial_x f(u^0)=0, \tag{7}$$

$$O(\varepsilon):\partial_t u^1+\partial_x\big(f'(u^0)u^1\big)=\partial_x^2 g(u^0), \tag{8}$$

$$O(\varepsilon^2):\partial_t u^2+\partial_x\big(f'(u^0)u^2\big)=\partial_x^2\big(B(u^0)u^1\big)-\frac{1}{2}\partial_x\big(f''(u^0)(u^1)^2\big). \tag{9}$$

其次, 在激波层 $\{x=s_1(t)\}$ 附近, 类似于外部展开可以得到近似解

$$u_1^{IN}(x,t)\sim u_s^0(\xi,t)+\varepsilon u_s^1(\xi,t)+\varepsilon^2 u_s^2(\xi,t)+\cdots \tag{10}$$

这里 ξ 是通过下列伸缩变换得到

$$\xi=\frac{x-s_1(t)}{\varepsilon}+\delta_1(t,\varepsilon), \tag{11}$$

其中 δ_1 是激波位置的扰动

$$\delta_1(t,\varepsilon)=\delta_1^0(t)+\varepsilon\delta_1^1(t)+\varepsilon^2\delta_1^2(t)+\cdots. \tag{12}$$

将 u_1^{IN} 代入方程组 (1) 得到

$$O(\frac{1}{\varepsilon}):\partial_\xi f(u_s^0)-\dot{s}_1\partial_\xi u_s^0-\partial_\xi^2 g(u_s^0)=0, \tag{13}$$

$$O(1):\partial_\xi^2\big(B(u_s^0)u_s^1\big)-\partial_\xi\big(f'(u_s^0)u_s^1\big)+\dot{s}_1\partial_\xi u_s^1=\dot{\delta}_1^0\partial_\xi u_s^0+\partial_t u_s^0, \tag{14}$$

$$\begin{aligned}O(\varepsilon):\partial_\xi^2\big(B(u_s^0)u_s^2\big)-\partial_\xi\big(f'(u_s^0)u_s^2\big)+\dot{s}_1\partial_\xi u_s^2&=\dot{\delta}_1^1\partial_\xi u_s^0+\dot{\delta}_1^0\partial_\xi u_s^1+\partial_t u_s^1\\&+\frac{1}{2}\partial_\xi\big(f''(u_s^0)(u_s^1,u_s^1)\big)-\frac{1}{2}\partial_\xi^2\big(B'(u_s^0)(u_s^1,u_s^1)\big).\end{aligned} \tag{15}$$

其中 $\dot{s}_1=\frac{ds}{dt}$, $\dot{\delta}=\frac{d\delta}{dt}$. 在激波层外的匹配区域中, 我们需要利用泰勒展开使内外部的近似解达成一致, 得到如下的匹配条件, 即当 $\xi\to\pm\infty$ 有

$$u_s^0(\xi,t)=u^0(s_1(t)\pm 0,t)+o(1), \tag{16}$$

$$u_s^1(\xi,t)=u^1(s_1(t)\pm 0,t)+(\xi-\delta_1^0)\partial_x u^0(s_1(t)\pm 0,t)+o(1), \tag{17}$$

$$\begin{aligned}u_s^2(\xi,t)=&u^2(s_1(t)\pm 0,t)+(\xi-\delta_1^0)\partial_x u^1(s_1(t)\pm 0,t)-\delta_1^1\partial_x u^0(s_1(t)\pm 0,t)\\&+\frac{1}{2}(\xi-\delta_1^0)^2\partial_x^2 u^0(s_1(t)\pm 0,t)+o(1),\end{aligned} \tag{18}$$

与在激波 $\{x=s_1(t)\}$ 附近的展开方法相同, 我们得到在激波层 $\{x=s_2(t)\}$ 附近的近似解

$$u_2^{IN}(x,t)\sim\hat{u}_s^0(\zeta,t)+\varepsilon\hat{u}_s^1(\zeta,t)+\varepsilon^2\hat{u}_s^2(\zeta,t)+\varepsilon^3\hat{u}_s^3(\zeta,t). \tag{19}$$

$$\zeta=\frac{x-s_2(t)}{\varepsilon}+\delta_2(t,\varepsilon), \tag{20}$$

$$\delta_2(t,\varepsilon)=\delta_2^0(t)+\varepsilon\delta_2^1(t)+\varepsilon^2\delta_2^2(t)+\cdots. \tag{21}$$

$$
\begin{aligned}
O(\frac{1}{\varepsilon}) :&\partial_\zeta f(\hat{u}_s^0) - \dot{s}_2\partial_\zeta \hat{u}_s^0 - \partial_\zeta^2\big(g(\hat{u}_s^0)\big) = 0, && (22)\\
O(1) :&\partial_\zeta^2\big(B(\hat{u}_s^0)\hat{u}_s^1\big) - \partial_\zeta\big(f'(\hat{u}_s^0)u_s^1\big) + \dot{s}_2\partial_\zeta\hat{u}_s^1 = \dot{\delta}_2^0\partial_\zeta\hat{u}_s^0 + \partial_t\hat{u}_s^0, && (23)\\
O(\varepsilon) :&\ \partial_\zeta^2\big(B(\hat{u}_s^0)\hat{u}_s^2\big) - \partial_\zeta\big(f'(\hat{u}_s^0)\hat{u}_s^2\big) + \dot{s}_2\partial_\zeta\hat{u}_s^2 = \dot{\delta}_2^1\partial_\zeta\hat{u}_s^0 + \dot{\delta}_2^0\partial_\zeta\hat{u}_s^1 + \partial_t\hat{u}_s^1\\
&+ \frac{1}{2}\partial_\zeta\big(f''(\hat{u}_s^0)(\hat{u}_s^1,\hat{u}_s^1)\big) - \frac{1}{2}\partial_\zeta^2\big(B^{'}(\hat{u}_s^0)(\hat{u}_s^1,\hat{u}_s^1)\big), && (24)
\end{aligned}
$$

其中 $\dot{s}_2 = \frac{ds}{dt}$, $\dot{\delta} = \frac{d\delta}{dt}$. 有如下匹配条件

$$
\begin{aligned}
\hat{u}_s^0(\zeta,t) =& u^0(s_2(t)\pm 0,t) + o(1), && (25)\\
\hat{u}_s^1(\zeta,t) =& u^1(s_2(t)\pm 0,t) + (\zeta-\delta_2^0)\partial_x u^0(s_2(t)\pm 0,t) + o(1), && (26)
\end{aligned}
$$

$$
\begin{aligned}
\hat{u}_s^2(\zeta,t) =& u^2(s_2(t)\pm 0,t) + (\zeta-\delta_2^0)\partial_x u^1(s_2(t)\pm 0,t) - \delta_2^1\partial_x u^0(s_2(t)\pm 0,t)\\
&+ \frac{1}{2}(\zeta-\delta_2^0)^2\partial_x^2 u^2(s_2(t)\pm 0,t) + o(1). && (27)
\end{aligned}
$$

2.2. 粘性激波剖面的构造

在 $x = s_1(t)$ 附近考虑以下常微分方程组

$$
\begin{aligned}
&f(\phi)_\xi - \dot{s}_1\phi_\xi = f(\phi)_{\xi\xi}, && (28)\\
&\phi(\xi,t) \to u_l = u^0(s_1(t)-0,t), \quad \xi\to-\infty, && (29)\\
&\phi(\xi,t) \to u_r = u^0(s_1(t)+0,t), \quad \xi\to+\infty, && (30)
\end{aligned}
$$

其中, (29) - (30) 为边界条件, 满足

$$
\dot{s}_1(u_l - u_r) = f(u_l) - f(u_r). \quad (31)
$$

则有 (28) - (30) 与下列方程组等价

$$
\begin{aligned}
&(\phi-u_r)_{\xi\xi} = B^{-1}(\phi)(f'(\phi)-\dot{s}_1 I)(\phi-u_r)_\xi - B^{-1}(\phi)dB(\phi)\big((\phi-u_r)_\xi,(\phi-u_r)_\xi\big), && (32)\\
&\phi(\xi,t) - u_l \to u, \quad \xi\to-\infty, && (33)\\
&\phi(\xi,t) - u_r \to 0, \quad \xi\to+\infty, && (34)
\end{aligned}
$$

其中 $u = u_l - u_l$. 设 $\mathcal{C} = \{u\in\mathbb{R}^N |\ (32)\ -\ (34)$ 有解 $\Phi(u,\xi) = \phi - u_r, \forall\xi\in\mathbb{R}^1\}$, 则关于 $\mathcal{C}$ 有以下命题成立

命题 2.1. 在假设 $(A1)-(A3)$ 成立的前提下, 存在一个 0 的小邻域 $\mathcal{V}$ 其中 $0\in\mathbb{R}^N$, $\mathcal{V}\in\mathcal{U}$, 使得 (i) - (iii) 成立, 其中

(i) $\mathcal{C}$ 是$\mathcal{V}$ 中经过0 的光滑流形, $0 \in \mathcal{V}$;
(ii) $T_0\mathcal{C} = \tilde{N}(0)$, $\tilde{N}(0)$ 是 $B^{-1}(\phi)(A - \dot{s}I)$ 负特征值对应特征向量生成的不变子空间;
(iii) $\mathcal{O}_{u_r}$ 为 u_r 的小邻域,
那么对于 $\Phi(u,\xi)$, 当 $\phi \in \mathcal{O}_{u_r}$ 时, 存在常数 c, 当 $\alpha \geq 0$ 时, $\mid \partial_u\partial_\xi\Phi(u,\xi) \mid \leq c\exp(-\alpha\xi)$ 成立.
在证明命题 2.1 前先介绍一个引理.

引理 2.1. $B^{-1}(\phi)(A(0) - \dot{s}_1 I)$ 的负特征值的个数与$A(0) - \dot{s}_1 I$ 相同, 即$B^{-1}(\phi)(A(0) - \dot{s}_1 I)$

命题 2.2. 存在 u^1, $u^1_{s_1}$ 使得下列方程组成立

$$\partial_t u^1 + \partial_x\big(f'(u^0)u^1\big) = g_{xx}(u^0), \tag{35}$$

$$\partial_\xi\big(f'(u^0_{s_1})u^1_{s_1}\big) - \dot{s}_1 \cdot \partial_\xi u^1_{s_1} = \partial^2_\xi\big(B(u^0_{s_1})u^1_{s_1}\big) - \dot{\delta}^0_1 \cdot \partial_\xi u^0_{s_1} - \partial_t u^0_{s_1}, \tag{36}$$

$$u^1_{s_1}(\xi \to \pm\infty) = \partial_x u^0(s_1(t) \pm 0, t)(\xi_1 - \delta^0_1) + u^1(s_1(t) \pm 0, t) + o(1). \tag{37}$$

命题 2.3. 由假设 $(A1) - (A3)$, 有 $T_0\mathcal{C} \cap E^+A(0) = 0$.
由假设 $(A1)$ 可知, $A(u)$ 在 $\Omega \cap \mathcal{C}$ 内可逆, 再结合命题 2.3 可以得到$T_{u^0(s(t)+0),t}\,\mathcal{C} \cap E^+A(u^0(s(t) + 0), t)) = \{0\}$. 由此, 下面关于 u^1 的初边值问题满足一致 Lopastinski 条件

$$\partial_t u^1 + \partial_x\big(f'(u^0)u^1\big) = g_{xx}(u^0), \tag{38}$$

$$u^1(s_1(t) + 0, t) \in \mathcal{C}_1, \tag{39}$$

$$u^1(0, x) = 0. \tag{40}$$

为了得到 (38) - (40) 好的适定性, 我们介绍下面引理 [29]

引理 2.2. 考虑初边值问题

$$\begin{aligned} &\partial_t u + A(u)\partial_x u = 0, \quad x > 0\,,\ 0 < t < T, \\ &B(t, u(t, x = 0)) = 0,\ 0 < t < T, \\ &u(t = 0, x) = u_0(x), \quad x > 0. \end{aligned}$$

假设上述方程是严格双曲的, $u_0(x) \in C^1_b(\mathbb{R}^1_+)$, $B \in C^1([0,T] \times \mathcal{U})$, 并且初边值满足 0 阶相容性条件: $B(0, u_0(0) = 0$, 设 $a := u_0(0)$, 假设线性方程组

$$\partial_t v + A(a)\partial_x v = 0, \quad x > 0\,,\ 0 < t < T,$$

$$\begin{aligned} &d_u B(0, a)v(t, 0) = h(t),\ t > 0, \\ &v(t = 0, x) = v_0(x), \quad x > 0. \end{aligned}$$

满足一致 Lopastinski 条件, 则存在 $T^* > 0$ 使拟线性初边值问题有唯一的分段 C^1 解, 如果初边值条件也满足 1 阶相容性条件, 即$B_t(0, a) = d_u B(0, a)A(a)u_0^{'}(0)$, 则解是 C^1 的. 假设 (38) - (40) 满足任意阶相容性条件, 再由引理 2.2, 存在唯一的 $u^1(t, x) \in H^\infty([0, T] \times \mathbb{R}^1_+)$ 使 (38) - (40) 有解. 到此

为止, 我们确定了 u^1, u_s^1, 再用相同的方法可以得到 $u^2, u_s^2, \hat{u}_s^1, \hat{u}_s^2$ 的存在性.

2.3. 近似解

现在我们利用截断函数来构造近似解. 我们先将远离激波和激波附近的解依次定义为

$$
\begin{aligned}
&O(x,t)=u^0(x,t)+\varepsilon u^1(x,t)+\varepsilon^2u^2(x,t), x\neq s_i(t), i=1,2, &(41)\\
&I_1(x,t)=\phi(\frac{x-s_1(t)}{\varepsilon}+\delta_1^0+\varepsilon\delta_1^1+\varepsilon^2\delta_1^2,t)+\varepsilon u_s^1(\frac{x-s_1(t)}{\varepsilon}+\delta_1^0+\varepsilon\delta_1^1+\varepsilon^2\delta_1^2,t)\\
&\qquad+\varepsilon^2u_s^2(\frac{x-s_1(t)}{\varepsilon}+\delta_1^0+\varepsilon\delta_1^1+\varepsilon^2\delta_1^2,t), &(42)\\
&I_2(x,t)=\hat{\phi}(\frac{x-s_2(t)}{\varepsilon}+\delta_2^0+\varepsilon\delta_2^1+\varepsilon^2\delta_2^2,t)+\varepsilon \hat{u}_s^1(\frac{x-s_2(t)}{\varepsilon}+\delta_2^0+\varepsilon\delta_2^1+\varepsilon^2\delta_2^2,t)\\
&\qquad+\varepsilon^2\hat{u}_s^2(\frac{x-s_2(t)}{\varepsilon}+\delta_2^0+\varepsilon\delta_2^1+\varepsilon^2\delta_2^2,t). &(43)
\end{aligned}
$$

其中 u_i, u_s^i, $\hat{u}_s^i$ 的构造在上一节中已经给出. 设截断函数 $m_i(y)\in C_0^\infty(\mathbb{R})$, $i=1,2$ 并满足 $0\le m_i(y)\le 1$:

$$
m_i(y)=\begin{cases}1, & |y|\le 1,\\ h_i(y), & 1<|y|<2,\\ 0, & |y|\ge 2,\end{cases}
$$

其中 $h_i(y)$ 是光滑函数, $y=\frac{x-s_i(t)}{\varepsilon^\gamma}$, 满足 $0<h_i(y)<1$. 由此, 定义方程 (1) 的近似解为

$$
v^\varepsilon(x,t)=m_1I_1+m_2I_2+(1-m_1-m_2)O+d(x,t)=\bar{v}^\varepsilon(x,t)+d(x,t), \tag{44}
$$

其中, $d(x,t)$ 为高阶修正项, $\gamma\in(\frac{4}{5},1)$. 由激波层内外部近似解的构造, $v^\varepsilon(x,t)$ 满足方程

$$
\partial_tv^\varepsilon+\partial_xf(v^\varepsilon)-\varepsilon[g(v^\varepsilon)]_{xx}=\sum_{i=1}^{5}q_i(x,t), \tag{45}
$$

其中, $q_i(x,t)$ 是光滑函数定义如下

$$
\begin{aligned}
q_1(x,t)=&(1-m_1-m_2)\{[f(O)-f(u^0)-\varepsilon f'(u^0)u^1-\varepsilon^2f'(u^0)u^2-\frac{\varepsilon^2}{2}f''(u^0)(u^1,u^1)\\
&-\varepsilon^3[B(u^0)u_x^2+B'(u^0)u^2(\varepsilon u_x^1+\varepsilon^2u_x^2)+\frac{1}{2}B''(u^0)(u^1+\varepsilon u^2)^2(u_x^0+\varepsilon u_x^1+\varepsilon^2u_x^2)\},\\
q_2(x,t)=&m_1\{(f(I_1)-f(u_s^0)-\varepsilon f'(u_s^0)u_s^1-\varepsilon^2f'(u_s^0)u_s^2-\frac{\varepsilon^2}{2}f''(u_s^0)(u_s^1,u_s^1)\\
&+\varepsilon^3\dot{\delta}_1^2\partial_xu_s^0+\varepsilon^3\dot{\delta}_1^1\partial_xu_s^1+\varepsilon^4\dot{\delta}_1^2\partial_xu_s^1+\varepsilon^3\dot{\delta}_1^0\partial_xu_s^2+\varepsilon^4\dot{\delta}_1^1\partial_xu_s^2+\varepsilon^5\dot{\delta}_1^2\partial_xu_s^2\\
&+\frac{\varepsilon^3}{2}\partial_x[B'(u_s^0)\partial_x(u_s^0,u_s^0)]+\varepsilon\partial_x[B''(u_s^0)\partial_xu_s^0]-\varepsilon\partial_x^2[B(u_s^0)\partial_xu_s^0]\\
&-\varepsilon^3\partial_x^2[B'(u_s^0)\partial_xu_s^1(\partial_xu_s^1+\varepsilon\partial_xu_s^2]\},
\end{aligned}
$$

$$q_3(x,t) = m_1\{(f(I_2) - f(\hat{u}_s^0) - \varepsilon f'(\hat{u}_s^0)\hat{u}_s^1 - \varepsilon^2 f'(\hat{u}_s^0)\hat{u}_s^2 - \frac{\varepsilon^2}{2} f''(\hat{u}_s^0)(\hat{u}_s^1, \hat{u}_s^1)$$
$$+ \varepsilon^3\dot{\delta}_2^2\partial_x\hat{u}_s^0 + \varepsilon^3\dot{\delta}_2^1\partial_x\hat{u}_s^1 + \varepsilon^4\dot{\delta}_2^2\partial_x\hat{u}_s^1 + \varepsilon^3\dot{\delta}_2^0\partial_x\hat{u}_s^2 + \varepsilon^4\dot{\delta}_2^1\partial_x\hat{u}_s^2 + \varepsilon^5\dot{\delta}_2^2\partial_x\hat{u}_s^2$$
$$+ \frac{\varepsilon^3}{2}\partial_x[B'(\hat{u}_s^0)\partial_x(\hat{u}_s^0, \hat{u}_s^0)] + \varepsilon\partial_x[B''(\hat{u}_s^0)\partial_x\hat{u}_s^0] - \varepsilon\partial_x^2[B(\hat{u}_s^0)\partial_x\hat{u}_s^0]$$
$$- \varepsilon^3\partial_x^2[B'(\hat{u}_s^0)\partial_x\hat{u}_s^1(\partial_x\hat{u}_s^1 + \varepsilon\partial_x\hat{u}_s^2]\},$$

$$q_4(x,t) = \partial_t m_1(I_1 - O) + \partial_t m_2(I_2 - O) + f(m_1I_1 + m_2I_2 + (1 - m_1 - m_2)O)_x$$
$$- \{m_1 f(I_1) + m_2 f(I_2) + (1 - m_1 - m_2)f(O)\}_x + \varepsilon(B(O)O_x)_x$$
$$+ \partial_x m_1(f(I_1) - f(O)) + \partial_x m_2(f(I_2) - f(O))$$
$$+ \varepsilon\{m_1(B(I_1) - B(O))(I_1 - O)_x + m_2(B(I_2) - B(O))(I_2 - O)_x\}_x$$
$$- \varepsilon\partial_x m_1(B(I_1) - B(O))(I_1 - O)_x - \varepsilon\partial_x m_2(B(I_2) - B(O))(I_2 - O)_x$$
$$+ \varepsilon\{m_1 B(O)(I_1 - O)_x + m_2 B(O)(I_2 - O)_x\}_x$$
$$- \varepsilon\partial_x m_1 B(O)(I_1 - O)_x - \varepsilon\partial_x m_2 B(O)(I_2 - O)_x$$
$$+ \varepsilon\{m_1(B(I_1) - B(O))O_x + m_2(B(I_2) - B(O))O_x\}_x$$
$$- \varepsilon\partial_x m_1[B(I_1) - B(O)]O_x - \varepsilon\partial_x m_2[B(I_2) - B(O)]O_x$$
$$- \varepsilon\{g(m_1I_1 + m_2I_2 + (1 - m_1 - m_2)O)\}_{xx},$$

$$q_5(x,t) = d_t - \varepsilon(g(v^\varepsilon) - g(v^\varepsilon - d))_{xx} + (f(v^\varepsilon) - f(v^\varepsilon - d))_x.$$

(i) supp $q_1 \subseteq \{(x,t) : |x - s_i(t)| \geq \varepsilon^\gamma, 0 \leq t \leq T, i = 1, 2\}$,

$$\partial_x^l q_1(x,t) = O(1)\varepsilon^{3-l\gamma},\ (\int_0^T \|\partial_x^l q_1(\cdot,t)\|^2 dt)^{\frac{1}{2}} \leq O(1)\varepsilon^{3-(l-1/2)\gamma}, l = 0, 1, 2; \tag{46}$$

(ii) supp $q_2 \subseteq \{(x,t) : |x - s_1(t)| \leq 2\varepsilon^\gamma, 0 \leq t \leq T\}$,

$$\partial_x^l q_2(x,t) = O(1)\varepsilon^{(2-l)\gamma}, l = 0, 1, 2; \tag{47}$$

(iii) supp $q_3 \subseteq \{(x,t) : |x - s_2(t)| \leq 2\varepsilon^\gamma, 0 \leq t \leq T\}$,

$$\partial_x^l q_3(x,t) = O(1)\varepsilon^{(2-l)\gamma}, l = 0, 1, 2; \tag{48}$$

(iv) supp$q_4 \subseteq \{(x,t) : \varepsilon^\gamma \leq |x - s_i(t)| \leq 2\varepsilon^\gamma, 0 \leq t \leq T\}$,

$$\partial_x^l q_4(x,t) = O(1)\varepsilon^{(3-l)\gamma}, l = 0, 1, 2. \tag{49}$$

记 $R^{\varepsilon}=\sum_{i=1}^{4}q_i(x,t)$, 则 $R^{\varepsilon}=O(1)\varepsilon^{2\gamma}$. 现在令 $d(x,t)$ 是下列扩散问题

$$\begin{cases} d_t=\varepsilon(B(\bar{v}^{\varepsilon})d_x)_x-R^{\varepsilon}, \\ d(x,t)=0. \end{cases} \tag{50}$$

的解, 则 v^{ε} 满足如下形式的守恒方程, 即

$$\partial_t v^{\varepsilon}+\partial_x f(v^{\varepsilon})-\varepsilon\partial_x\big(B(v^{\varepsilon})\partial_x v^{\varepsilon}\big)=\varepsilon\big(B(\bar{v}^{\varepsilon})d_x\big)_x-\varepsilon\big(g(v^{\varepsilon})-g(\bar{v}^{\varepsilon})\big)_{xx}+\big(f(v^{\varepsilon})-f(\bar{v}^{\varepsilon})\big)_x. \tag{51}$$

下面估计线性扩散波 $d(x,t)$. 通过 $d(x,t)$ 和 $q_i(x,t)$ 的结构, 我们可以得到以下引理

引理 2.3. 设 $d(x,t)$ 是方程 (50) 的解, 则 $\forall t\in[0,T]$ 有下列估计

$$\sup_{0\le t\le T}\int_{\mathbb{R}}|d(x,t)|^2\,dx+\varepsilon\int_0^T\int_{\mathbb{R}}|\partial_x d(x,t)|^2\,dxdt\le C\varepsilon^{5\gamma}, \tag{52}$$

$$\sup_{0\le t\le T}\int_{\mathbb{R}}|\partial_x d(x,t)|^2\,dx+\varepsilon\int_0^T\int_{\mathbb{R}}\left|\partial_x^2 d(x,t)\right|^2\,dxdt\le C\varepsilon^{5\gamma-2}, \tag{53}$$

$$\sup_{0\le t\le T}\int_{\mathbb{R}}\left|\partial_x^2 d(x,t)\right|^2\,dx+\varepsilon\int_0^T\int_{\mathbb{R}}\left|\partial_x^3 d(x,t)\right|^2\,dxdt\le C\varepsilon^{5\gamma-4}, \tag{54}$$

$$\sup_{0\le t\le T}\parallel d(x,t)\parallel_{L^{\infty}(\mathbb{R})}\le C\varepsilon^{\frac{5\gamma-1}{2}}, \tag{55}$$

$$\sup_{0\le t\le T}\parallel \partial_x d(x,t)\parallel_{L^{\infty}(\mathbb{R})}\le C\varepsilon^{\frac{5\gamma-3}{2}}, \tag{56}$$

引理 2.4. 对于 (42) 定义的 $v^{\varepsilon}(x,t)$, 有

$$v^{\varepsilon}(x,t)=\begin{cases} u^0(x,t)+O(1)\varepsilon, & |x-s_i(t)|\ge\varepsilon^{\gamma}, i=1,2 \\ u_s^0(\xi,t)+O(1)\varepsilon^{\gamma}, & |x-s_1(t)|\le 2\varepsilon^{\gamma} \\ \hat{u}_s^0(\zeta,t)+O(1)\varepsilon^{\gamma}, & |x-s_2(t)|\le 2\varepsilon^{\gamma}. \end{cases} \tag{57}$$

3. 稳定性分析

3.1. 误差方程

假设 $u^{\varepsilon}(x,t)$ 是方程 (1.1) 的真实解, 令

$$u^{\varepsilon}(x,t)=v^{\varepsilon}(x,t)+\bar{w}(x,t),\quad x\in\mathbb{R}^1,\quad t\in[0,T]. \tag{58}$$

$$\begin{aligned} &\bar{w}_t+\partial_x\big(f'(v^{\varepsilon})\bar{w}\big)+Q_{1x} \\ =&\varepsilon\partial_x^2\big(B(v^{\varepsilon})\bar{w}\big)+\varepsilon Q_{2xx}-\varepsilon\big(B(\bar{v}^{\varepsilon})d_x\big)_x+\varepsilon[B(v^{\varepsilon})d-Q_3]_{xx}+[-f'(v^{\varepsilon})d+Q_4]_x, \end{aligned} \tag{59}$$

$$\bar{w}(x,0) \equiv 0.$$

其中

$$Q_1 = f(u^\varepsilon) - f(v^\varepsilon) - f'(v^\varepsilon)\bar{w}, \tag{60}$$

$$Q_2 = g(u^\varepsilon) - g(v^\varepsilon) - g'(v^\varepsilon)\bar{w}, \tag{61}$$

$$Q_3 = g(\bar{v}^\varepsilon) - g(v^\varepsilon) + g'(v^\varepsilon)d, \tag{62}$$

$$Q_4 = f(\bar{v}^\varepsilon) - f(v^\varepsilon) + f'(v^\varepsilon)d. \tag{63}$$

令$\bar{w}(x,t) = \tilde{w}_x(x,t)$. 对 (58) 积分可得

$$\tilde{w}_t + f'(v^\varepsilon)\tilde{w} + Q_1 = \varepsilon\big(B(v^\varepsilon)\tilde{w}_x\big)_x + \varepsilon Q_{2x} - \varepsilon B(\bar{v}^\varepsilon)d_x + \varepsilon[B(v^\varepsilon)d - Q_3]_x + [-f'(v^\varepsilon)d + Q_4], \tag{64}$$

$$\tilde{w}(x,0) \equiv 0.$$

进行尺度变换

$$\tilde{w}(x,t) = \varepsilon w(y,\tau), \quad y = \frac{x - s(t)}{\varepsilon}, \quad \tau = \frac{t}{\varepsilon}. \tag{65}$$

$$\begin{aligned} &\tilde{w}_\tau - \dot{s}w_y + f'(v^\varepsilon)w_y + Q_1 \\ &= \big(B(v^\varepsilon)w_y\big)_y + Q_{2y} - B(\bar{v}^\varepsilon)d_y + [B(v^\varepsilon)d - Q_3]_y + [-f'(v^\varepsilon)d + Q_4], \end{aligned} \tag{66}$$

$$\tilde{w}(y,0) \equiv 0.$$

命题 3.1. (先验估计) 假设柯西问题 (66) 有唯一解 $w \in C^1([0,\tau_0]; H^2(\mathbb{R}))$, 满足

$$\sup_{0\le t\le T} \|w(\cdot,\tau)\|_{H^2} \le c\varepsilon^\delta, \tag{67}$$

那么有

$$\sup_{0\le t\le T} \|w(\cdot,\tau)\|_{H^2}^2 + \int_0^{\tau_0} \|w_y(\cdot,\tau)\|_{H^2}^2 d\tau \le c\varepsilon^{5\gamma-3}, \tag{68}$$

其中, $\delta > 0$. (3.9) 中 $f'(v^\varepsilon)$ 为任意矩阵, 为保证能量估计顺利进行, 要将其对角化, 先定义

$$\theta(y,\tau) = L(v^\varepsilon)w(y,\tau), \tag{69}$$

$$M(y,\tau) = (\partial_y L(v^\varepsilon)) \cdot R(v^\varepsilon), \tag{70}$$

$$N(y,\tau) = (\partial_\tau L(v^\varepsilon)) \cdot R(v^\varepsilon). \tag{71}$$

据此, (66) 可以转化为

$$\theta_\tau - N\theta + (\Lambda - \dot{s})\theta_y + LQ_1$$
$$= L[B(v^\varepsilon)(R\theta)_y]_y + LQ_{2y} - LB(\bar{v}^\varepsilon)d_y + L(B(v^\varepsilon)d)_y - LQ_{3y} - \Lambda Ld + LQ_4 \tag{72}$$
$$\theta(y,0) \equiv 0.$$

3.2. 基本能量估计

引理 3.1. 假设 $w \in C^1([0,\tau_0]; H^2(\mathbb{R}))$ 为 (66) 的解, 满足命题 3.1 的假设条件, 那么

$$\|w(\cdot,\tau)\|^2 + \int_0^\tau \|w_y(\cdot,\tau)\|^2 d\sigma \le c\varepsilon^{5\gamma-3}.$$

证明. 将 (72) 式左乘 θ 并在等式两边积分得

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}\parallel\theta\parallel^2 =& \int \theta N\theta dy + \int \theta(\Lambda-\dot{s})\theta_y dy - \int \theta(\Lambda-\dot{s})M\theta dy\\ &+ \int \theta L[B(v^\varepsilon)(R\theta)_y]_y dy + \int \theta L(Q_4 - Q_1)dy + \int \theta L(Q_{2y} - Q_{3y})dy\\ &- \int \theta LB(\bar{v}^\varepsilon)d_y dy + \int \theta LB((v^\varepsilon)d)_y dy - \int \theta\Lambda Lddy. \end{aligned}\tag{73}$$

由假设 (i) - (iii) 可以得到存在常数 $c_0 > 0$ 使得

$$c_0 \parallel \theta_y \parallel^2 \le \int \theta_y L(v^\varepsilon)B(v^\varepsilon)R(v^\varepsilon)\theta_y dy.$$

于是利用分部积分 (73) 式可转变为

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}\parallel\theta\parallel^2 + c_0\parallel\theta_y\parallel^2 \le& \int \theta N\theta dy - \int \theta(\Lambda-\dot{s})\theta_y dy + \int \theta(\Lambda-\dot{s})M\theta dy\\ &- \int \theta_y L(v^\varepsilon)B(v^\varepsilon)R_y\theta dy - \int \theta\partial_y L(v^\varepsilon)B(v^\varepsilon)R_y\theta dy\\ &- \int \theta\partial_y L(v^\varepsilon)B(v^\varepsilon)R\theta_y dy + \int \theta L((v^\varepsilon)(Q_4 - Q_1)dy\\ &+ \int \theta L(v^\varepsilon)(Q_{2y} - Q_{3y})dy - \int \theta LB(\bar{v}^\varepsilon))d_y dy\\ &+ \int \theta L(v^\varepsilon)[B(v^\varepsilon)d]_y dy - \int \theta\Lambda L(v^\varepsilon)ddy. \end{aligned}\tag{74}$$

下面对 (74) 式右端每一项进行分开估计, 由引理 2.3 可以推得

$$\partial_\tau v^\varepsilon = O(1)\varepsilon. \tag{75}$$

$$\int \theta N\theta dy = \int \theta\partial_\tau L(v^\varepsilon)R(v^\varepsilon)\theta dy \qquad \le c\varepsilon\int |\theta\nabla L(v^\varepsilon)R(v^\varepsilon)\theta|dy \le c\varepsilon \parallel \theta \parallel^2 .$$

现在估计第二项,分区域进行估计, 因为两个激波不相互作用, 所以我们将整个区域划分为以下几个部分

$$E_O := \{x \le s_1(t) - 2\varepsilon^\gamma\} \cup \{s_1(t) + 2\varepsilon^\gamma \le x \le s_2(t) - 2\varepsilon^\gamma\} \cup \{x \ge s_2(t) + 2\varepsilon^\gamma\}, \tag{76 a}$$

$$E_{I_1} := \{|x - s_1(t)| \le 2\varepsilon^\gamma\}, \tag{76 b}$$

$$E_{I_2} := \{|x - s_2(t)| \le 2\varepsilon^\gamma\}, \tag{76 c}$$

$$E_{M_1} := \{\varepsilon^\gamma \le |x - s_1(t)| \le 2\varepsilon^\gamma\}, \tag{76 d}$$

$$E_{M_2} := \{\varepsilon^\gamma \le |x - s_2(t)| \le 2\varepsilon^\gamma\}. \tag{76 e}$$

由引理 2.3 可知

$$\partial_y v^\varepsilon = m_1\phi_y + m_2\hat{\phi}_y + O(1)\varepsilon. \tag{77}$$

再由激波剖面的性质, 对 $-\int\theta(\Lambda - \dot{s})\theta_y dy$ 分区域估计

$$\begin{aligned}
&-\int \theta(\Lambda - \dot{s})\theta_y dy \\
\le& -\frac{1}{2}\int m_1|\partial_y\lambda_p(\phi)|\theta_p^2 dy + c\varepsilon^\gamma\int m_1|\partial_y\lambda_p(\phi)|\theta_p^2 dy + (c + c\varepsilon^\gamma)\int\sum_{k\ne p} m_1|\partial_y\lambda_p(\phi)|\theta_k^2 dy \\
&-\frac{1}{2}\int m_2|\partial_y\lambda_p(\hat{\phi})|\theta_p^2 dy + c\varepsilon^\gamma\int m_2|\partial_y\lambda_p(\hat{\phi})|\theta_p^2 dy + (c + c\varepsilon^\gamma)\int\sum_{k\ne p} m_2|\partial_y\lambda_p(\hat{\phi})|\theta_k^2 dy \\
&+ c\varepsilon \parallel \theta \parallel^2 .
\end{aligned}$$

由 (70) 和激波剖面的性质, 可以得到

$$\begin{aligned}
\int \theta(\Lambda - \dot{s})M\theta dy \le& (c\varepsilon^\gamma + c\mu_1)\int m_1|\partial_y\lambda_p(\phi)|\theta_p^2 dy + (c + c\varepsilon^\gamma)\int\sum_{k\ne p} m_1|\partial_y\lambda_p(\phi)|\theta_k^2 dy \\
&+ (c\varepsilon^\gamma + c\mu_2)\int m_2|\partial_y\lambda_p(\hat{\phi})|\theta_p^2 dy + (c + c\varepsilon^\gamma)\int\sum_{k\ne p} m_2|\partial_y\lambda_p(\hat{\phi})|\theta_k^2 dy \\
&+ c\varepsilon \parallel \theta \parallel^2 .
\end{aligned}$$

$$\begin{aligned}
-\int \theta_y L(v^\varepsilon)B(v^\varepsilon)\partial_y R(v^\varepsilon)\theta dy =& -\int \theta_y L(v^\varepsilon)B(v^\varepsilon)\nabla R(v^\varepsilon)[m_1\phi_y + m_2\hat{\phi}_y + O(1)\varepsilon]\theta dy \\
\le& c\varepsilon^\delta\int m_1|\partial_y\lambda_p(\phi)|\theta^2 dy + c\varepsilon^\delta\int m_2|\partial_y\lambda_p(\hat{\phi})|\theta^2 dy + c\varepsilon \parallel \theta \parallel^2 .
\end{aligned}$$

$$-\int \theta\partial_y L(v^\varepsilon)B(v^\varepsilon)\partial_y R(v^\varepsilon)\theta dy \quad \le c\varepsilon^\delta\int m_1|\partial_y\lambda_p(\phi)|\theta^2 dy + c\varepsilon^\delta\int m_2|\partial_y\lambda_p(\hat{\phi})|\theta^2 dy + c\varepsilon \parallel \theta \parallel^2 .$$

由 (60) 和 (63) 有

$$|Q_4 - Q_1| \leq c|d|^2 + c|w_y|^2 \leq c|d|^2 + c|\partial_y R(v^\varepsilon)\theta|^2 + c|R(v^\varepsilon)\theta_y|^2.$$

$$\begin{aligned}
&\int \theta L(v^\varepsilon)(Q_4 - Q_1)dy \\
\leq&(c + c\varepsilon^\gamma)\int |\theta|d|^2|dy + (c + c\varepsilon^\gamma)\int |\theta|\partial_y R(v^\varepsilon)\theta|^2|dy + (c + c\varepsilon^\gamma)\int |\theta|R(v^\varepsilon)\theta_y|^2|dy \\
=&\sum_{i=1}^{3} J_i.
\end{aligned}$$

$$J_1 = (c + c\varepsilon^\gamma)\int |\theta|d|^2|dy \leq (c + c\varepsilon^\gamma)[\varepsilon \int |\theta|^2 dy + \varepsilon^{-1}\int |d|^4 dy \leq c\varepsilon \parallel \theta \parallel^2 + c\varepsilon^{10\gamma-4}.$$

$$\begin{aligned}
J_2 \leq&(c\mu_1 + c\varepsilon^\gamma) \parallel \theta \parallel_{L^\infty} \int m_1|\partial_y \lambda_p(\phi)|\theta^2 dy + (c\varepsilon^\gamma + c\mu_2) \parallel \theta \parallel_{L^\infty} \int m_2|\partial_y \lambda_p(\hat{\phi})|\theta^2 dy + c\varepsilon \parallel \theta \parallel^2, \\
J_3 =&(c + c\varepsilon^\gamma)\int |\theta|R(v^\varepsilon)\theta_y|^2|dy \leq (c + c\varepsilon^\gamma) \parallel \theta \parallel_{L^\infty} \parallel \theta_y \parallel^2 .
\end{aligned}$$

综上可以得到

$$\begin{aligned}
\int \theta L(v^\varepsilon)(Q_4 - Q_1)dy \leq c\varepsilon^{10\gamma-3} + c\varepsilon \parallel \theta \parallel^2 + &(c + c\varepsilon^\gamma) \parallel \theta \parallel_{L^\infty} \parallel \theta_y \parallel^2 \\
&+ (c\mu_1 + c\varepsilon^\gamma) \parallel \theta \parallel_{L^\infty} \int m_1|\partial_y \lambda_p(\phi)|\theta^2 dy \\
&+ (c\mu_2 + c\varepsilon^\gamma) \parallel \theta \parallel_{L^\infty} \int m_2|\partial_y \lambda_p(\hat{\phi})|\theta^2 dy.
\end{aligned}$$

$$\begin{aligned}
\int \theta L(v^\varepsilon)(Q_{2y} - Q_{3y})dy \leq c &\int |\theta_y L(v^\varepsilon)(|d|^2 + |\partial_y R(v^\varepsilon)\theta|^2 + |R(v^\varepsilon)\theta_y|^2)|dy \\
&+ c\int |\theta \partial_y L(v^\varepsilon)(|d|^2 + |\partial_y R(v^\varepsilon)\theta|^2 + |R(v^\varepsilon)\theta_y|^2)|dy \\
&= \sum_{i=4}^{5} J_i
\end{aligned}$$

$$\begin{aligned}
J_4 \leq&\frac{c_0}{6} \parallel \theta_y \parallel^2 + c\varepsilon^{10\gamma-3} + c\varepsilon^4 \parallel \theta \parallel^2 + c\varepsilon^{2\delta_0} \parallel \theta_y \parallel^2 \\
&+(c\mu_1 + c\varepsilon^\gamma)\int m_1|\partial_y \lambda_p(\phi)|\theta^2 dy + (c\mu_2 + c\varepsilon^\gamma)\int m_2|\partial_y \lambda_p(\hat{\phi})|\theta^2 dy. \\
J_5 \leq& c\varepsilon^{10\gamma-3} + c\varepsilon \parallel \theta \parallel^2 + (c + c\varepsilon^\gamma) \parallel \theta \parallel_{L_\infty} \parallel \theta_y \parallel^2 \\
&+(c\mu_1 + c\varepsilon^\gamma) \parallel \theta \parallel_{L_\infty} \int m_1|\partial_y \lambda_p(\phi)|\theta^2 dy + (c\mu_2 + c\varepsilon^\gamma) \parallel \theta \parallel_{L_\infty} \int m_2|\partial_y \lambda_p(\hat{\phi})|\theta^2 dy.
\end{aligned}$$

所以综上有

$$\int \theta L(v^\varepsilon)(Q_{2y}-Q_{3y})dy \le c\varepsilon^{10\gamma-3}+c\varepsilon \parallel \theta \parallel^2 +[(c+c\varepsilon^\gamma)\parallel \theta \parallel_{L^\infty} +c\varepsilon^{2\delta_0}+\frac{c_0}{6}]\parallel \theta_y \parallel^2$$
$$+(c\mu_1+c\varepsilon^\gamma)(\parallel \theta \parallel_{L^\infty}+1)\int m_1|\partial_y\lambda_p(\phi)|\theta^2 dy$$
$$+(c\mu_2+c\varepsilon^\gamma)(\parallel \theta \parallel_{L^\infty}+1)\int m_2|\partial_y\lambda_p(\hat{\phi})|\theta^2 dy.$$
$$-\int \theta L(v^\varepsilon)B(\bar{v}^\varepsilon)d_y dy \le c\varepsilon \parallel \theta \parallel^2 + c\varepsilon^{5\gamma-2}.$$

先对下一项作用分部积分, 再把 (77) 代入

$$\int \theta L(v^\varepsilon)[B(v^\varepsilon)d]_y dy \le (c\mu_1+c\varepsilon^\gamma)\int m_1|\partial_y\lambda_p(\phi)|\theta^2 dy+(c\mu_2+c\varepsilon^\gamma)\int m_2|\partial_y\lambda_p(\hat{\phi})|\theta^2 dy$$
$$+c\varepsilon^2 \parallel \theta \parallel^2 + c\varepsilon \parallel \theta_y \parallel^2 + c\varepsilon^{5\gamma-2}.$$
$$-\int \theta \Lambda L d\, dy \le \varepsilon \parallel \theta \parallel^2 + c\varepsilon^{5\gamma-2}.$$

将所有的估计代入 (74), 整理后有

$$\frac{1}{2}\frac{d}{d\tau}\parallel \theta \parallel^2 + c_0 \parallel \theta_y \parallel^2 + \frac{1}{2}\int m_1|\partial_y\lambda_p(\phi)|\theta_p^2 dy+\frac{1}{2}\int m_2|\partial_y\lambda_p(\hat{\phi})|\theta_p^2 dy$$
$$\le (c\mu_1+c\varepsilon^\gamma)(\parallel \theta \parallel_{L^\infty}+1)\int m_1|\partial_y\lambda_p(\phi)|\theta_p^2 dy$$
$$+(c\mu_2+c\varepsilon^\gamma)(\parallel \theta \parallel_{L^\infty}+1)\int m_2|\partial_y\lambda_p(\hat{\phi})|\theta_p^2 dy$$
$$+(c+c\varepsilon^\gamma)(\parallel \theta \parallel_{L^\infty}+1)\int \sum_{k\neq p} m_1|\partial_y\lambda_p(\phi)|\theta_k^2 dy$$
$$+(c+c\varepsilon^\gamma)(\parallel \theta \parallel_{L^\infty}+1)\int \sum_{k\neq p} m_2|\partial_y\lambda_p(\hat{\phi})|\theta_k^2 dy$$
$$+c\varepsilon \parallel \theta \parallel^2 +[(c+c\varepsilon^\gamma)\parallel \theta \parallel_{L^\infty}+\frac{c_0}{6}]\parallel \theta_y \parallel^2 + c\varepsilon^{5\gamma-2}.$$

我们可以选择 ε, $\mu_i(i=1,2)$, $\parallel \theta \parallel_{L^\infty}$ 足够小使得

$$(c\mu_i+c\varepsilon^\gamma)(\parallel \theta \parallel_{L^\infty}+1)\le \frac{1}{4},\quad i=1,2,$$
$$(c+c\varepsilon^\gamma)\parallel \theta \parallel_{L^\infty}+c\varepsilon^{2\delta_0}+\frac{c_0}{6}\le \frac{c_0}{2}.$$

$$\frac{d}{d\tau}\parallel \theta \parallel^2 + c_0 \parallel \theta_y \parallel^2 + \int m_1|\partial_y\lambda_p(\phi)|\theta_p^2 dy+\int m_2|\partial_y\lambda_p(\hat{\phi})|\theta_p^2 dy$$
$$\le (c+c\varepsilon^\gamma)\int \sum_{k\neq p} m_1|\partial_y\lambda_p(\phi)|\theta_k^2 dy+(c+c\varepsilon^\gamma)\int \sum_{k\neq p} m_2|\partial_y\lambda_p(\hat{\phi})|\theta_k^2 dy \quad + c\varepsilon \parallel \theta \parallel^2 + c\varepsilon^{5\gamma-2}. \quad (78)$$

下面我们需要进一步估计关于 θ_k 的项使其被主波控制, 利用 (72) 式我们可以得到

$$\int m_1|\partial_y\lambda_p(\phi)|\theta_k^2dy$$

$$=2\int_{E_{I_1}}(\frac{\lambda_p(\phi)-\dot{s}_1}{\lambda_k(\phi)-\dot{s}_1})\theta_k[(\lambda_k(\phi)-\lambda_k(v^\varepsilon))\theta_{ky}-\theta_{k\tau}+\{(\Lambda-\dot{s})M\theta+N\theta+L(Q_4-Q_1)$$

$$+L[B(v^\varepsilon)(R\theta)_y]_y+L(Q_{2y}-Q_{3y})-LB(\bar{v}^\varepsilon)d_y+L(B(v^\varepsilon)d)_y-\Lambda Ld\}_k]dy=\sum_1 K_i.$$

下面开始分别估计 $K_i,\ i=1,2...10.$

$$K_1=\int_{E_{I_1}}\partial_y(\frac{\lambda_p(\phi)-\dot{s}_1}{\lambda_k(\phi)-\dot{s}_1})(\lambda_k(v^\varepsilon)-\lambda_k(\phi))\theta_k^2dy+\int_{E_{I_1}}(\frac{\lambda_p(\phi)-\dot{s}_1}{\lambda_k(\phi)-\dot{s}_1})\partial_y(\lambda_k(v^\varepsilon)-\lambda_k(\phi))\theta_k^2dy$$
$$=K_1^1+K_1^2.$$

由激波剖面的性质有

$$|\partial_y(\frac{\lambda_p(\phi)-\dot{s}_1}{\lambda_k(\phi)-\dot{s}_1})|\le c|\phi_y|\le c|\partial_y\lambda_p(\phi)|.$$

$$K_1^1\le c\varepsilon^\gamma\int m_1|\partial_y\lambda_p(\phi)|\theta_k^2dy+c\varepsilon\parallel\theta_k\parallel^2.$$
$$K_1^2=\le c\mu_1\varepsilon^\gamma\int m_1|\partial_y\lambda_p(\phi)|\theta_k^2dy+c\varepsilon\parallel\theta_k\parallel^2.$$

进而

$$K_1\le c\varepsilon^\gamma\int m_1|\partial_y\lambda_p(\phi)|\theta_k^2dy+c\varepsilon\parallel\theta_k\parallel^2.$$

$$K_2=-2\int_{E_{I_1}}(\frac{\lambda_p(\phi)-\dot{s}_1}{\lambda_k(\phi)-\dot{s}_1})\theta_k\theta_{k\tau}dy\le c\varepsilon^\gamma\int m_1|\partial_y\lambda_p(\phi)|\theta_k^2dy+c\varepsilon\parallel\theta_k\parallel^2.$$

$$K_3\le c\mu_1\int m_1|\partial_y\lambda_p(\phi)|\theta_k^2dy+c\varepsilon\parallel\theta_k\parallel^2.$$

$$K_4=2\int_{E_{I_1}}(\frac{\lambda_p(\phi)-\dot{s}_1}{\lambda_k(\phi)-\dot{s}_1})\theta_k[N\theta]_kdy\le c\varepsilon\parallel\theta_k\parallel^2.$$

$$K_5\le(c+c\varepsilon^\gamma)\mu_1\parallel\theta_k\parallel_{L^\infty}\int m_1|\partial_y\lambda_p(\phi)|\theta^2dy+c\varepsilon\parallel\theta\parallel^2$$

$$+(c+c\varepsilon^\gamma)\mu_1\parallel\theta\parallel_{L^\infty}\parallel\theta_y\parallel^2+c\varepsilon^{10\gamma-3}.$$

接下来, 估计粘性项, 利用激波剖面的性质在第一个激波附近进行估计可得

$$K_6 \leq (c+c\varepsilon^{\gamma})\mu_1 \int m_1|\partial_y \lambda_p(\phi)|\theta_k^2 dy + c\varepsilon \parallel \theta_k \parallel^2 + c\mu_1 \parallel \theta_{ky} \parallel^2 .$$

接下来, 利用分部积分处理 K_7

$$K_7 \leq c\varepsilon^{5\gamma-1} + (c+c\varepsilon^{\gamma})\mu_1 \int m_1|\partial_y \lambda_p(\phi)|\theta_k^2 dy + c\varepsilon \parallel \theta_k \parallel^2 + (c+c\varepsilon^{\gamma})(\parallel \theta_k \parallel_{L^\infty} + \mu_1) \parallel \theta_{ky} \parallel^2 .$$

运用柯西不等式, 再利用引理 2.3 估计最后三项

$$K_8 = -2\int_{E_{I_1}} (\frac{\lambda_p(\phi)-\dot{s}_1}{\lambda_k(\phi)-\dot{s}_1})\theta_k[L(v^{\varepsilon})B(\bar{v}^{\varepsilon})d_y]_k dy \leq c\varepsilon^{5\gamma-2} + c\varepsilon \parallel \theta \parallel^2,$$

$$K_9 \leq c\varepsilon^{5\gamma-1} + c\varepsilon \parallel \theta_y \parallel^2 + (c+c\varepsilon^{\gamma})\mu_1 \int m_1|\partial_y \lambda_p(\phi)|\theta_k^2 dy + c\varepsilon^2 \parallel \theta_k \parallel^2,$$

$$K_{10} \leq c\varepsilon \parallel \theta_k \parallel^2 + c\varepsilon^{5\gamma-1}.$$

将以上估计合并并整理得

$$\begin{aligned}
&\sum_{k\neq p}\int m_1|\partial_y \lambda_p(\phi)|\theta_k^2 dy \\
&\leq -\frac{d}{d\tau}[\sum_{k\neq p}\int (\frac{\lambda_p(\phi)-\dot{s}_1}{\lambda_k(\phi)-\dot{s}_1})\theta_k^2 dy] + (c+c\varepsilon^{\gamma})(\mu_1 + \parallel \theta \parallel_{L^\infty})\int m_1 \sum_{k\neq p}|\partial_y \lambda_p(\phi)|\theta_k^2 dy \\
&+(c+c\varepsilon^{\gamma})(\mu_1 + \parallel \theta \parallel_{L^\infty})\int m_1|\partial_y \lambda_p(\phi)|\theta_p^2 dy + c\varepsilon \parallel \theta \parallel^2 \\
&+(c+c\varepsilon^{\gamma})\mu_1 \parallel \theta_y \parallel^2 + c\varepsilon^{5\gamma-2}.
\end{aligned} \tag{79}$$

在第二个激波附近我们可以得到与上式类似的结果

$$\begin{aligned}
&\sum_{k\neq p}\int m_2|\partial_y \lambda_p(\hat{\phi})|\theta_k^2 dy \\
&\leq -\frac{d}{d\tau}[\sum_{k\neq p}\int (\frac{\lambda_p(\hat{\phi})-\dot{s}_2}{\lambda_k(\hat{\phi})-\dot{s}_2})\theta_k^2 dy] + (c+c\varepsilon^{\gamma})(\mu_2 + \parallel \theta \parallel_{L^\infty})\int m_2 \sum_{k\neq p}|\partial_y \lambda_p(\hat{\phi})|\theta_k^2 dy \\
&+(c+c\varepsilon^{\gamma})(\mu_2 + \parallel \theta \parallel_{L^\infty})\int m_2|\partial_y \lambda_p(\hat{\phi})|\theta_p^2 dy + c\varepsilon \parallel \theta \parallel^2 + (c+c\varepsilon^{\gamma})\mu_2 \parallel \theta_y \parallel^2 + c\varepsilon^{5\gamma-2}.
\end{aligned} \tag{80}$$

取 $\mu_1, \mu_2, \parallel \theta \parallel_{L^\infty}, \varepsilon$ 足够小, 将 (79), (80) 带入 (78), 在 $[0,\tau]$ 上积分, 由 $\theta(y,\tau=0)=0$, 有

$$\begin{aligned}
&\parallel \theta \parallel^2 + \frac{c_0}{2}\int_0^{\tau} \parallel \theta_y \parallel^2 d\sigma + \frac{1}{2}\int_0^{\tau}\int m_1|\partial_y \lambda_p(\phi)|\theta_p^2 dy d\sigma + \frac{1}{2}\int_0^{\tau}\int m_2|\partial_y \lambda_p(\hat{\phi})|\theta_p^2 dy d\sigma \\
&\leq c\mu_1 \sum_{k\neq p} \parallel \theta_k \parallel^2 + c\mu_2 \sum_{k\neq p} \parallel \theta_k \parallel^2 + c\varepsilon \int_0^{\tau} \parallel \theta \parallel^2 d\sigma + c\tau\varepsilon^{5\gamma-2}.
\end{aligned}$$

再根据 Gronwall 不等式和 $w_y = R\theta_y - RM\theta$

$$\int_0^\tau \| w_y(\cdot,\tau) \|^2 \, d\sigma \le c\varepsilon^{5\gamma-3}.$$

3.3. 高阶估计

引理 3.2. 假设 $w \in C^1([0,\tau_0]; H^2(\mathbb{R}))$ 满足命题 3.1 的假设条件, 那么有

$$\|\partial_y w(\cdot,\tau)\|_{H^1}^2 + \int_0^\tau \|\partial_y w(\cdot,\tau)\|_{H^2}^2 d\sigma \le c\varepsilon^{5\gamma-3}.$$

证明. 对 (66) 式两边同时作用 ∂_y, 左乘 $\partial_y w$, 并在 $\mathbb{R}$ 上积分得

$$\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau} \| \partial_y w \|^2 + c_1 \int | \partial_y^2 w |^2 \, dy \\
\le& \int \partial_y^2 w \cdot f'(v^\varepsilon) w_y dy + \int \partial_y^2 w \cdot Q_1 dy - \int \partial_y^2 w \cdot \partial_y B(v^\varepsilon) w_y dy \\
&- \int \partial_y^2 w \cdot Q_2 dy + \int \partial_y^2 w \cdot B(\bar{v}^\varepsilon) d_y dy \\
&+ \int \partial_y^2 w \cdot \partial_y [B(v^\varepsilon) d - Q_3] dy - \int \partial_y^2 w \cdot [-f'(v^\varepsilon) d + Q_4] dy.
\end{aligned}$$

由 (60) , (61) (62) , 引理 2.3 , 通过简单计算得到

$$\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau} \| \partial_y w \|^2 + c_1 \int | \partial_y^2 w |^2 \, dy \\
\le& \frac{c_1}{2} \| \partial_y^2 w \|^2 + c\varepsilon^{2\delta} \| w_y \|^2 + c\varepsilon^{2\delta} \| \partial_y^2 w \|^2 + c\varepsilon^{5\gamma-1}.
\end{aligned}$$

令 ε 足够小, 再由 Gronwall 不等式可推出 (81) 式

$$\| \partial_y w \|^2 + c_1 \int_0^\tau \| \partial_y^2 w \|^2 \, dy \le c\varepsilon^{5\gamma-3}. \tag{81}$$

重复之前的步骤, 对 (66) 式两边关于 y 求二阶导, 再左乘 $\partial_y^2 w$, 并在 $\mathbb{R}$ 上积分可得

$$\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau} \| \partial_y^2 w \|^2 + c_1 \| \partial_y^3 w \|^2 \, dy \\
\le& \int \partial_y^3 w \cdot [f'(v^\varepsilon) w_y]_y dy + \int \partial_y^3 w \cdot Q_{1y} dy - \int \partial_y^3 w \cdot [\partial_y B(v^\varepsilon) w_y]_y dy \\
&- \int \partial_y^3 w \cdot Q_{2yy} dy + \int \partial_y^3 w \cdot [B(\bar{v}^\varepsilon) d_y]_y dy \\
&+ \int \partial_y^3 w \cdot \partial_y^2 [B(v^\varepsilon) d - Q_3] dy - \int \partial_y^3 w \cdot [-f'(v^\varepsilon) d + Q_4]_y dy.
\end{aligned}$$

对右边每一项进行估计得到

$$\frac{1}{2}\frac{d}{d\tau} \parallel \partial_y^2 w \parallel^2 + c_1 \int \mid \partial_y^3 w \mid^2 dy$$
$$\leq \frac{c_1}{2} \parallel \partial_y^2 w \parallel^2 + c\varepsilon^{2\delta} \parallel w_y \parallel^2 + c\varepsilon^{2\delta} \parallel \partial_y^2 w \parallel^2 + c\varepsilon^{2\delta} \parallel \partial_y^3 w \parallel^2 + c\varepsilon^{5\gamma-1}.$$

令 ε 足够小, 再由 Gronwall 不等式可得

$$\parallel \partial_y^2 w \parallel^2 + c_1 \int_0^\tau \parallel \partial_y^3 w \parallel^2 d\sigma \leq c\varepsilon^{5\gamma-3}. \tag{82}$$

将 (81) 和 (82) 相加即可得到引理 3.2. 下面我们证明命题 3.1 中假设是成立的, 由 Sobolev 不等式有

$$\sup_{0\leq\tau\leq T/\varepsilon} \|w_y(\cdot,\tau)\|_{L^\infty(\mathbb{R})} \leq c \sup_{0\leq\tau\leq T/\varepsilon} \|w_y(\cdot,\tau)\|_{L^2(\mathbb{R})}^{1/2} \cdot \sup_{0\leq\tau\leq T/\varepsilon} \|w_{yy}(\cdot,\tau)\|_{L^2(\mathbb{R})}^{1/2}$$
$$\leq c\varepsilon^{5\gamma-3}.$$

因为 $\gamma \in (\frac{4}{5}, 1)$, 得证.最后我们证明命题 1.

证明. 由命题 3.1 有

$$\sup_{0\leq t\leq T} \|u^\varepsilon(\cdot,t) - v^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R})}^2 = \sup_{0\leq t\leq T} \|\tilde{w}_x(\cdot,t)\|_{L^2(\mathbb{R})}^2$$
$$= \varepsilon \sup_{0\leq\tau\leq T/\varepsilon} \|w_y(\cdot,\tau)\|_{L^2(\mathbb{R})}^2$$
$$\leq c\varepsilon^{5\gamma-2}.$$

同时, 通过命题 2.4 中近似解的构造我们可以知道

$$\sup_{0\leq t\leq T} \|v^\varepsilon(\cdot,t) - u^0(\cdot,t)\|_{L^2(\mathbb{R})}^2 \leq c\varepsilon^\gamma.$$

从而得到 (4) 成立

$$\sup_{0\leq t\leq T} \parallel u^\varepsilon(\cdot,t) - u^0(\cdot,t) \parallel_{L^2(\mathbb{R})}^2$$
$$\leq \sup_{0\leq t\leq T} \parallel u^\varepsilon(\cdot,t) - v^\varepsilon(\cdot,t) \parallel_{L^2(\mathbb{R})}^2 + \sup_{0\leq t\leq T} \parallel v^\varepsilon(\cdot,t) - u^0(\cdot,t) \parallel_{L^2(\mathbb{R})}^2$$
$$\leq c\varepsilon^{5\gamma-3} + c\varepsilon^\gamma$$
$$\leq c\varepsilon^\gamma.$$

最后, 我们证明 (5)

$$\|u^\varepsilon - v^\varepsilon(\cdot,\tau)\|_{L^\infty(\mathbb{R})} \leq \|w_y(\cdot,\tau)\|_{L^\infty(\mathbb{R})}$$
$$\leq c \sup_{0\leq\tau\leq T/\varepsilon} \|w_y(\cdot,\tau)\|_{L^2(\mathbb{R})}^{1/2} \cdot \sup_{0\leq\tau\leq T/\varepsilon} \|w_{yy}(\cdot,\tau)\|_{L^2(\mathbb{R})}^{1/2}$$
$$\leq c\varepsilon.$$

因此可得出结论, 定理 1 得证.

参考文献

[1] Hoff, D. and Liu, T.P. (1989) The Inviscid Limit for the Navier-Stokes Equations of Compressible Isentropic Flow with Shock Data. *Indiana University Mathematics Journal*, **36**, 861-915.

[2] Goodman, J. and Xin, Z.P. (1992) Viscous Limits for Piecewise Smooth Solutions to Systems of Conservation Laws. *Archive for Rational Mechanics and Analysis*, **121**, 235-265. https://doi.org/10.1007/BF00410614

[3] Maslov, A.A. and Mironov, S.G. (1999) Viscous Shock Layer on a Plate in Hypersonic Flow. *European Journal of Mechanics*, **18**, 213-226. https://doi.org/10.1016/S0997-7546(99)80023-5

[4] Huang, F.M. and Wang, Y. (2010) Hydrodynamic Limit of the Boltzmann Equation with Contact Discontinuities. *Communications in Mathematical Physics*, **295**, 293-326. https://doi.org/10.1007/s00220-009-0966-2

[5] Eyink, G.L. and Drivas, T.D. (2015) Spontaneous Stochasticity and Anomalous Dissipation for Burgers Equation. *Journal of Statistical Physics*, **158**, 386-432. https://doi.org/10.1007/s10955-014-1135-3

[6] Wang, J. (2010) Boundary Layers for Compressible Navier-Stokes Equations with Outflow Boundary Condition. *Journal of Differential Equations*, **248**, 1143-1174. https://doi.org/10.1016/j.jde.2009.12.001

[7] Yu, S.H. (1999) Zero-Dissipation Limit of Solutions with Shocks for Systems of Hyperbolic Conservation Laws. *Archive for Rational Mechanics and Analysis*, **146**, 275-370. https://doi.org/10.1007/s002050050143

[8] Ma, S.X. (2009) Viscous Limits to Piecewise Smooth Solutions for the Navier-Stokes Equations of One-Dimensional Compressible Viscous Heat-Conducting Fluids. *Methods and Applications of Analysis*, **16**, 1-32. https://doi.org/10.4310/MAA.2009.v16.n1.a1

[9] Zheng, T.T. and Zhao, J.N. (2012) On the Stability of Contact Discontinuity for Cauchy Problem of Compress Navier-Stokes Equations with General Initial Data. *Science China Mathematics*, **55**, 2005-2026. https://doi.org/10.1007/s11425-012-4441-8

[10] Ou, Y.B. (2013) The Vanishing Viscosity Method for the Sensitivity Analysis of an Optimal Control Problem of Conservation Laws in the Presence of Shocks. *Nonlinear Analysis Real World Applications*, **14**, 1947-1974. https://doi.org/10.1016/j.nonrwa.2013.02.001

[11] Xin, Z.P. (1993) Zero Dissipation Limit to Rarefaction Waves for the One-Dimensional Navier-Stokes Equations of Compressible Isentropic Gases. *Communications on Pure and Applied Mathematics*, **46**, 621-665. https://doi.org/10.1002/cpa.3160460502

[12] Wang, Y. (2008) Zero Dissipation Limit of the Compressible Heat-Conducting Navier-Stokes Equations in the Presence of the Shock. *Acta Mathematica Scientia*, **28**, 727-748. https://doi.org/10.1016/S0252-9602(08)60074-0

[13] Wang, H.Y. (2005) Zero Dissipation Limit to Rarefaction Waves for the p-System. *Acta Mathematica Sinica, English Series*, **21**, 1229-1210. https://doi.org/10.1007/s10114-004-0515-z

[14] Huang, F.M. and Li, J. (2010) Asymptotic Stability of Combination of Viscous Contact Wave with Rarefaction Waves for One-Dimensional Compressible Navier-Stokes System. *Archive for Rational Mechanics and Analysis*, **197**, 89-116. https://doi.org/10.1007/s00205-009-0267-0

[15] Huang, F.M., Li, M.G. and Wang, Y. (2012) Zero Dissipation Limit to Rarefaction Wave with Vacuum for One-Dimensional Compressible Navier-Stokes Equations. *SIAM Journal on Mathematical Analysis*, **44**, 1742-1759. https://doi.org/10.1137/100814305

[16] Huang, F.M. and Wang, Y. (2013) The Limit of the Boltzmann Equation to the Euler Equations for Riemann Problems. *SIAM Journal on Mathematical Analysis*, **45**, 1741-1811. https://doi.org/10.1137/120898541

[17] Huang, F.M. and Wang, Y. (2012) Vanishing Viscosity Limit of Compressible Navier-Stokes Equations for Solutions to a Riemann Problem. *Archive for Rational Mechanics and Analysis*, **203**, 379-413. https://doi.org/10.1007/s00205-011-0450-y

[18] Huang, F.M. and Wang, Y. (2015) Vanishing Viscosity of Isentropic Navier-Stokes Equations for Interacting Shocks. *Science China Mathematics*, **58**, 653-672. https://doi.org/10.1007/s11425-014-4962-4

[19] Li, H.L. (2018) Stability of the Superposition of a Viscous Contact Wave with Two Rarefaction Waves to the Bipolar Vlasov-Poisson-Boltzmann System. *SIAM Journal on Mathematical Analysis*, **50**. https://doi.org/10.1137/17M1150888

[20] Ma, S.X. (2010) Zero Dissipation Limit to Strong Contact Discontinuity for the 1-D Compressible Navier-Stokes Equations. *Journal of Differential Equations*, **248**, 95-110. https://doi.org/10.1016/j.jde.2009.08.016

[21] Liu, T.P. and Xin, Z.P. (1992) Stability of Viscous Shock Waves Associated with a System of Nonstrictly Hyperbolic Conservations Laws. *Communication on Pure and Applied Mathematics*, **45**, 361-388. https://doi.org/10.1002/cpa.3160450402

[22] Zhang, Y.H. (2013) Zero Dissipation Limit to a Riemann Solution Consisting of Two Shock Waves for the 1D Compressible Isentropic Navier-Stokes Equations. *Science China*, **56**, 2205-2232. https://doi.org/10.1007/s11425-013-4690-1

[23] Zeng, H. (2009) Stability of a Superposition of Shock Waves with Contact Discontinuities for Systems of Viscous Conservation Laws. *Journal of Differential Equations*, **246**, 2081-2102. https://doi.org/10.1016/j.jde.2008.07.034

[24] Hong, H. and Wang, T. (2017) Zero Dissipion Limit to a Riemann Solution for the Compressinle Navier-Stokes System of General Gas. *Acta Mathematica Scientia*, **37**, 1177-1208. https://doi.org/10.1016/S0252-9602(17)30067-X

[25] Serre, D. (1998) Global Solutions ($-\infty < t < +\infty$) of Parabolic Systems of Conservation Laws. *Annales de l'institut Fourier (Grenoble)*, **48**, 1069-1091. https://doi.org/10.5802/aif.1649

[26] Shi, X.D. (2016) Vanishing Viscosity for Non-Isentropic Gas Dynamics with Interacting Shocks. *Acta Mathematica Scientia*, **36**, 1699-1720. https://doi.org/10.1016/S0252-9602(16)30100-X

[27] Grenier, E. and Gues, O. (1998) Boundary Layers for Viscous Perturbations of Noncharacteristic Quasilinear Hyperbolic Problems. *Journal of Differential Equations*, **143**, 110-146. https://doi.org/10.1006/jdeq.1997.3364

[28] Wang, J. (2009) Boundary Layers for Parabolic Perturbations of Quasi-Linear Hyperbolic Problems. *Mathematical Methods in the Applied Sciences*, **32**, 2416-2438. https://doi.org/10.1002/mma.1144

[29] Li, T.T. and Yu, W.C. (1985) Boundary Value Problems for Quasilinear Hyperbolic Systems. Duke University Mathematics Series, Vol. V. Duke University, Mathematics Department, Durham, NC.