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一类分数阶薛定谔–泊松系统非平凡解的存在性

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摘 要

本文研究一类具有变号权的分数阶薛定谔–泊松系统

$$\begin{cases} -(\Delta)^s u + u + k(x)\phi u = a(x)|u|^{p-1}u, & x \in \mathbb{R}^3, \\ -(\Delta)^t \phi = k(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

非平凡解的存在性，其中 $\frac{3s+4t}{s+t} < p < \frac{3+2s}{3-2s}$，$s, t \in (0,1)$ 且 $4s+2t > 3$，$a(x) \in C(\mathbb{R}^3)$ 变号且 $\lim_{|x|\to\infty} a(x) = a^{\infty} < 0$，$k(x) \in C(\mathbb{R}^3) \cap L^{\frac{6}{4s+2t-3}}(\mathbb{R}^3)$. 应用山路引理，本文得到该系统至少存在一个非平凡解.

关键词

分数阶薛定谔–泊松系统，变号权，非平凡解

Existence of Nontrivial Solution for a Class of Fractional Schrödinger-Poisson System

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Abstract

In this paper, we are concerned with the existence of nontrivial solution for a class of fractional Schrödinger-Poisson system:

$$\begin{cases} -(\Delta)^s u + u + k(x)\phi u = a(x)|u|^{p-1}u, & x \in \mathbb{R}^3, \\ -(\Delta)^t \phi = k(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $\frac{3s+4t}{s+t} < p < \frac{3+2s}{3-2s}$**,** $s, t \in (0,1)$ **and** $4s+2t > 3$**,** $a(x) \in C(\mathbb{R}^3)$ **is a sign-changing function with** $\lim_{|x|\to\infty} a(x) = a^\infty < 0$**,** $k(x) \in C(\mathbb{R}^3) \cap L^{\frac{6}{4s+2t-3}}(\mathbb{R}^3)$**. By using mountain pass theorem, we obtain that this system has at least one nontrivial solution.**

Keywords

Fractional Schrödinger-Poisson System, Sign-Changing Weight, Nontrivial Solution

1. 引言

近年来, 许多文献考虑如下分数阶薛定谔 - 泊松系统

$$\begin{cases} (-\Delta)^s u + V(x)u + K(x)\phi u = f(x,u), & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1}$$

此处 $s,t\in(0,1)$, 分数阶 Laplacian 算子 $(-\Delta)^s$ 定义为

$$(-\Delta)^s v(x)=C_{N,s}P.V.\int_{\mathbb{R}^N}\frac{v(x)-v(y)}{|x-y|^{N+2s}}\mathrm{d}y,\ v\in\mathbb{S}(\mathbb{R}^N),$$

此处 $P.V.$ 表示柯西主值, $C_{N,s}$ 标准化常数, $\mathbb{S}(\mathbb{R}^N)$ 是由快速衰减函数组成的施瓦茨函数空间. 注意到分数阶 Laplacian 算子 $(-\Delta)^s$ 是在文献 [1,2]首次引入. 更多有关 $(-\Delta)^s$ 信息, 请参考 [3]及其中的参考文献. 当前, 有关系统 (1) 的讨论绝大多数是有关解或变号解的存在性, 多解性的结果, 如 [4–15]. 其中, 文献 [5,6,12,14]考虑了系统 (1) 或类似问题变号解的存在性; 文献 [4,10]研究了系统 (1) 在临界条件下高能量解的存在性; 文献 [7,13]讨论了系统 (1) 解的存在性及集中性; 文献 [8,9,11,15]考察了系统 (1) 基态解的存在性或多重性. 然而, 通过梳理相关文献发现: 针对具有变号权的分数阶薛定谔- 泊松系统解的存在性问题的研究却非常少. 另一方面, 我们注意到, 余晓辉在 [16]中考虑了以下薛定谔-泊松系统

$$\begin{cases}-\Delta u+u+\phi u=a(x)|u|^{p-1}u,\ x\in\mathbb{R}^3,\\ -\Delta\phi=k(x)u^2,\ x\in\mathbb{R}^3,\end{cases}\tag{2}$$

其中 $3\le p<5$, $a(x)\in C(\mathbb{R}^3)$ 变号且 $\lim\limits_{|x|\to\infty}a(x)=a_\infty<0$, $k(x)\in C(\mathbb{R}^3)\cap L^2(\mathbb{R}^3)$. 利用山路引理 [17], 作者获得了薛定谔-泊松系统 (2) 至少有一个非平凡解的存在性结果. 受以上文献的启发, 本文考虑如下分数阶薛定谔-泊松系统

$$\begin{cases}-(\Delta)^s u+u+k(x)\phi u=a(x)|u|^{p-1}u,\ \ x\in\mathbb{R}^3\\ -(\Delta)^t\phi=k(x)u^2,\ \ x\in\mathbb{R}^3,\end{cases}\tag{3}$$

非平凡解的存在性, 其中 $\frac{3s+4t}{s+t}<p<\frac{3+2s}{3-2s}$, $s,t\in(0,1)$ 且 $4s+2t>3$, $a(x)$ 和 $k(x)$ 满足:

(A1) $a(x)\in C(\mathbb{R}^3,\mathbb{R})$ 为一类变号函数且满足 $a^\infty=\lim_{|x|\to\infty}a(x)<0$.

(A2) $k(x)\in C(\mathbb{R}^3,\mathbb{R})$, $k(x)\ge 0$ 且 $k(x)\in L^{\frac{6}{4s+2t-3}}(\mathbb{R}^3)$.

本文的主要结果为如下定理:

定理1.1. 假设(A1), (A2)成立, 则分数阶薛定谔-泊松系统 (3) 至少存在一个非平凡解.

2 定理1.1的证明

对于固定的 $u\in H^s(\mathbb{R}^3)$, 定义 $D^{t,2}(\mathbb{R}^3)$ 上的线性算子

$$L_u(v)=\int_{\mathbb{R}^3}k(x)u^2vdx,$$

那么就有

$$\begin{aligned}|L_u(v)| &\le \int_{\mathbb{R}^3} k(x)u^2|v|dx \\ &\le (\int_{\mathbb{R}^3} k(x)^{\frac{6}{4s+2t-3}}dx)^{\frac{4s+2t-3}{6}}(\int_{\mathbb{R}^3} u^{\frac{6}{3-2s}})^{\frac{3-2s}{3}}(\int_{\mathbb{R}^3} v^{2_t^*})^{\frac{1}{2_t^*}} \\ &\le C\|k(x)\|_{L^{\frac{6}{4s+2t-3}}(\mathbb{R}^3)}\|u\|^2_{H^s(\mathbb{R}^3)}\|v\|_{D^{t,2}(\mathbb{R}^3)}.\end{aligned}$$

因此由 Riesz 表示定理可知, 存在唯一的 $\phi_u \in D^{t,2}(\mathbb{R}^3)$, 使得

$$\langle \phi_u, v\rangle_{D^{t,2}(\mathbb{R}^3)} = L_u(v),\ \ \forall v \in D^{t,2}(\mathbb{R}^3),$$

即

$$\int_{\mathbb{R}^3}(-\Delta)^{\frac{t}{2}}\phi_u(-\Delta)^{\frac{t}{2}}vdx = \int_{\mathbb{R}^3} k(x)u^2vdx,\ \ \forall v \in D^{t,2}(\mathbb{R}^3),$$

也就是说 ϕ_u 是(3)中第二个方程的弱解. 将 ϕ_u 带入第一个方程就得到

$$-(\Delta)^s u + u + k(x)\phi_u u = a(x)|u|^{p-1}u. \tag{4}$$

因此求解方程(3)等价于求解方程(4). 而方程(4)的解对应能量泛函

$$\begin{aligned}F(u) = \frac{1}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2 + u^2dx + \frac{1}{4}\int_{\mathbb{R}^3}k(x)\phi_u u^2dx \\ - \frac{1}{p+1}\int_{\mathbb{R}^3}a(x)|u|^{p+1}dx,\ \ u \in H^s(\mathbb{R}^3)\end{aligned}$$

的临界点.

定义 Φ: $H^s(\mathbb{R}^3) \to D^{t,2}(\mathbb{R}^3)$ 为 $\Phi(u) = \phi_u$, 则有下面引理.

引理2.1

(*i*) Φ 连续.

(*ii*) Φ 将有界集映到有界集.

(*iii*) 若 $\{u_n\}$ 在 $H^s(\mathbb{R}^3)$ 中有界且 $u_n \rightharpoonup u$, 那么有

$$\int_{\mathbb{R}^3} k(x)\phi_{u_n}u_n^2dx \to \int_{\mathbb{R}^3} k(x)\phi_u u^2dx.$$

证: (i) 对所有 $u \in H^s(\mathbb{R}^3)$, 有

$$|L_u| = \|\phi_u\|_{D^{t,2}(\mathbb{R}^3)} = \|\Phi(u)\|_{D^{t,2}(\mathbb{R}^3)}$$

成立. 所以, 为了证明 Φ 连续, 只需证: $u \mapsto L_u$ 是连续的.

$$\begin{aligned}|L_{u_n}(v)-L_u(v)| &\le \int_{\mathbb{R}^3} k(x)|v||u_n^2-u^2|dx \\ &\le C\|k\|_{L^{\frac{6}{4s+2t-3}}(\mathbb{R}^3)}\|v\|_{D^{t,2}(\mathbb{R}^3)}\|u_n^2-u^2\|_{L^{\frac{6}{3-2s}}(\mathbb{R}^3)}\end{aligned}$$

因为 v 是任意的, 当 $n\to\infty$, 有 $|L_{u_n}(v)-L_u(v)|\to 0$.

(ii) 因为 $\|\phi_u\|_{D^{t,2}(\mathbb{R}^3)}=|L_u|\le C\|k\|_{L^{\frac{6}{4t+2s-3}}(\mathbb{R}^3)}\|u\|^2_{H^s(\mathbb{R}^3)}$. 所以引理中(ii)成立.

(iii) 在文献[16]中已经证明了 $\phi_{u_n}\rightharpoonup\phi_u$, 下面证明

$$\int_{\mathbb{R}^3} k(x)\phi_{u_n}u_n^2dx \to \int_{\mathbb{R}^3} k(x)\phi_u u^2dx.$$

首先我们注意到由于 $\phi_{u_n}\rightharpoonup\phi_u$, 所以我们有

$$\int_{\mathbb{R}^3} k(x)(\phi_{u_n}-\phi_u)u^2dx\to 0. \tag{5}$$

下面证明

$$\int_{\mathbb{R}^3} k(x)\phi_{u_n}|u_n^2-u^2|dx\to 0. \tag{6}$$

因为 $k(x)\in L^{\frac{6}{4s+2t-3}}(\mathbb{R}^3)$ 且连续, 因此对 $\forall\varepsilon>0$, $\exists\rho=\rho(\varepsilon)>0$, 使得当 $|x|\ge\rho$ 时 $k(x)\le\varepsilon$, 于是我们得到

$$\begin{aligned}\int_{|x|\ge\rho} k(x)\phi_{u_n}|u_n^2-u^2|dx &\le \varepsilon[\int_{|x|\ge\rho}\phi_{u_n}u_n^2dx+\int_{|x|\ge\rho}\phi_{u_n}u^2dx] \\ &\le \varepsilon C\|\phi_{u_n}\|_{D^{t,2}(\mathbb{R}^3)}(\|u_n\|^2+\|u\|^2) \\ &\le C\varepsilon.\end{aligned}$$

另一方面, 假设 $s,t\in(0,1)$, 观察到如果 $4s+2t>3$, 则有 $2\le\frac{12}{3+2t}<\frac{6}{3-2s}$, 因此 $H^s(\mathbb{R}^3)\hookrightarrow L^{\frac{12}{3+2t}}(\mathbb{R}^3)$. 故有:

$$\begin{aligned}&\int_{|x|\le\rho} k(x)\phi_{u_n}|u_n^2-u^2|dx \\ &\le \|k\|_{L^\infty(\mathbb{R}^3)}[\int_{|x|\le\rho}\phi_{u_n}|u_n^2-u^2|dx] \\ &\le \|k\|_{L^\infty(\mathbb{R}^3)}C\|\phi_{u_n}\|_{D^{t,2}(\mathbb{R}^3)}[\int_{|x|\le\rho}|u_n^2-u^2|^{\frac{12}{3+2t}}dx]^{\frac{3+2t}{6}} \\ &\to 0.\end{aligned}$$

这样就证明了(6), 综合(5)和(6)我们得到 $\int_{\mathbb{R}^3}k(x)\phi_{u_n}u_n^2dx\to\int_{\mathbb{R}^3}k(x)\phi_u u^2dx$. 引理证毕.

引理2.2. 泛函 F 满足(PS)条件.

证: 设 $u_n \subset H^s(\mathbb{R}^3)$ 满足 $|F(u_n)| \leq c$ 和 $F'(u_n) \to 0$, 那么有

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 + u_n^2 dx + \int_{\mathbb{R}^3} k(x)\phi_{u_n} u_n^2 dx - \int_{\mathbb{R}^3} a(x)|u_n|^{p+1} dx = o(1)\|u_n\| \tag{7}$$

$$\frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 + u_n^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} k(x)\phi_{u_n} u_n^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^3} a(x)|u_n|^{p+1} dx \leq c.$$

利用上面两式进一步得到

$$[\frac{1}{2} - \frac{1}{p+1}]\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 + u_n^2 dx + [\frac{1}{4} - \frac{1}{p+1}]\int_{\mathbb{R}^3} k(x)\phi_{u_n} u_n^2 \leq c + o(1)\|u_n\|.$$

因为 $p > \frac{3s+4t}{s+t}$, 所以有

$$[\frac{1}{2} - \frac{1}{p+1}]\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 + u_n^2 dx \leq c + o(1)\|u_n\|.$$

由上式知 u_n 有界. 因此我们可以假定 $u_n \rightharpoonup u$, 下面证明 $u_n \to u$. 只需证明 $\|u_n\| \to \|u\|$. 由方程(7)我们有

$$\begin{aligned}&\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 + u_n^2 dx + \int_{\mathbb{R}^3} k(x)\phi_{u_n} u_n^2 + \int_{\mathbb{R}^3} a^-(x)|u_n|^{p+1} dx \\ &= \int_{\mathbb{R}^3} a^+(x)|u_n|^{p+1} dx + o(1).\end{aligned}$$

由于 $\lim_{|x|\to\infty} a(x) = a^\infty < 0$, 因此 $a^+(x)$ 有紧支集, 利用索伯列夫嵌入定理知道

$$\int_{\mathbb{R}^3} a^+(x)|u_n|^{p+1} dx = \int_{\mathbb{R}^3} a^+(x)|u|^{p+1} dx + o(1).$$

所以得到

$$\begin{aligned}&\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 + u_n^2 dx + \int_{\mathbb{R}^3} k(x)\phi_{u_n} u_n^2 dx + \int_{\mathbb{R}^3} a^-(x)|u_n|^{p+1} dx \\ &= \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 + u^2 dx + \int_{\mathbb{R}^3} k(x)\phi_{u} u^2 dx + \int_{\mathbb{R}^3} a^-(x)|u|^{p+1} dx + o(1).\end{aligned} \tag{8}$$

我们断言

$$\int_{\mathbb{R}^3} a^-(x)|u|^{p+1} dx \leq \liminf_{n\to\infty}\int_{\mathbb{R}^3} a^-(x)|u_n|^{p+1} dx.$$

事实上由

$$
\begin{aligned}
&\int_{\mathbb{R}^3} a^-(x)||u_n|^{p+1}-|u|^{p+1}-|u_n-u|^{p+1}|dx\\
&\le \|a(x)\|_{L^\infty(\mathbb{R}^3)}\int_{\mathbb{R}^3}||u_n|^{p+1}-|u|^{p+1}-|u_n-u|^{p+1}|dx\\
&=o(1)
\end{aligned}
$$

知

$$
\int_{\mathbb{R}^3} a^-(x)|u_n|^{p+1}dx=\int_{\mathbb{R}^3} a^-(x)|u|^{p+1}dx+\int_{\mathbb{R}^3} a^-(x)|u_n-u|^{p+1}dx+o(1),
$$

也就是断言成立.

如果 $u_n \nrightarrow u$, 那么 $\|u_n\|<\|u\|$, 再由 $\int_{\mathbb{R}^3} k(x)\phi_{u_n}u_n^2dx\to\int_{\mathbb{R}^3} k(x)\phi_u u^2dx$ 和 $\int_{\mathbb{R}^3} a^-(x)|u|^{p+1}dx\le \liminf_{n\to\infty}\int_{\mathbb{R}^3} a^-(x)|u_n|^{p+1}dx$. 知

$$
\begin{aligned}
&\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2+u_n^2dx+\int_{\mathbb{R}^3}k(x)\phi_{u_n}u_n^2dx+\int_{\mathbb{R}^3}a^-(x)|u_n|^{p+1}dx\\
>&\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2+u^2dx+\int_{\mathbb{R}^3}k(x)\phi_u u^2dx+\int_{\mathbb{R}^3}a^-(x)|u|^{p+1}dx+o(1),
\end{aligned}
$$

这与 (8) 式矛盾. 因此 $u_n\to u$, 证毕.

定理1的证明:首先证明存在 $\alpha,\rho>0$ 使得 $F_{\partial B_\rho}>\alpha>0$. 事实上由索伯列夫嵌入定理有

$$
F(u)\ge\frac{1}{2}\|u\|^2-C\|a\|_{L^\infty}\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^3)},
$$

由上式可知存在 $\rho>0$, 使得 $F_{\partial B_\rho}>\alpha>0$.

我们再证明存在 $\eta\in H^s(\mathbb{R}^3)$, $\|\eta\|>\rho$, 使得 $F(\eta)<0$, 事实上, 选取函数 $\varphi_\theta=\theta^{s+t}\varphi(\theta x)\in H^s(\mathbb{R}^3)$, $\theta\in\mathbb{R}^+$, $\varphi_\theta\ne0$, 使得 $supp\varphi_\theta\subset suppa^+$, 那么就有

$$
F(\theta^{s+t}\varphi(\theta x))\le\frac{\theta^{4s+2t-3}}{2}\|\varphi\|^2+\|k\|_\infty\frac{\theta^{4s+2t-3}}{4}\int_{\mathbb{R}^3}\phi_\varphi\varphi^2dx-\frac{\theta^{(p+1)(s+t)-3}}{p+1}\int_{\mathbb{R}^3}a^+|\varphi|^{p+1}dx.
$$

因为 $p>\frac{3s+4t}{s+t}$, 所以 $(p+1)(s+t)-3>4s+2t-3$. 当 $\theta\to+\infty$ 时, $F(\theta^{s+t}\varphi(\theta x))\to-\infty$. 因此, 对某个 θ_0 充分大时, 令 $\eta=\varphi_{\theta_0}=\theta_0^{s+t}\varphi(\theta_0x)$, 所以有 $F(\eta)<0$. 定义

$$
\begin{aligned}
&\Gamma=\{\gamma\in C([0,1],H^s(\mathbb{R}^3)):\gamma(0)=0,\gamma(1)=\eta\},\\
&c=\inf_{\gamma\in\Gamma}\sup_{\theta\in[0,1]}F(\gamma(\theta)),
\end{aligned}
$$

那么由山路引理知 c 是 F 的一个非平凡的临界点, 即分数阶薛定谔-泊松系统 (3) 至少存在一个非平凡解.

3 结论

本文通过利用山路引理, 得到了分数阶薛定谔–泊松系统至少存在一个非平凡解. 其主要思路是找到一个有界的 (PS) 序列, 使该序列满足 (PS) 条件, 由此可以证明非平凡解的存在性.

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