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Caputo分数阶微分方程解的唯一性

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摘 要

本文主要研究一类具有Riemann-Stieltjes边值条件的Caputo分数阶微分方程。利用Green函数的性质，Banach收缩原理，证明了方程解的唯一性。

关键词

Caputo分数阶微分方程，唯一性，解

Uniqueness of Positive Solutions for the Caputo Fractional Differential Equation

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Abstract

In this paper, we consider a class of the Caputo fractional differential equation with Riemann-Stieltjes integral boundary conditions. Making use of the properties of the

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Green function, the Banach contraction principle, uniqueness result of the equation is proved.

Keywords

Caputo Fractional Differential Equation, Uniqueness, Solution

1. 引言

本文研究具有Riemann-Stieltjes边值条件的Caputo分数阶微分方程(BVP):

$$\begin{cases} {}^{c}D^{\alpha}x(t) + a(t)f(t,x(t)) = 0, \ 0 < t < 1, \\ x(0) = x''(0) = 0, \ x(1) = \int_0^1 b(s)x(s)dA(s), \end{cases} \tag{1.1}$$

其中$2 < \alpha < 3$, D^{α}_{0+}是Caputo 微分. $\int_0^1 b(s)x(s)dA(s)$表示具有广义测度的Riemann-Stieltjes积分, $A : [0,1] \to (-\infty,+\infty)$是有界变差函数, $a : (0,1) \to [0,+\infty)$, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$连续.

在数学的背景下, 得益于近年来非线性分析理论的快速发展, 涉及不同边界条件的分数阶微分方程越来越引起学者的兴趣 [1–7]. 分数阶微分方程边值问题的研究, 大多考虑的是分数阶导数定义下的多点, 积分边值问题, 对于Riemann-Stieltjes积分定义下的积分边值问题的研究, 相对较少. 因此, 本文研究的方程是十分有意义的.

2. 预备知识

定义2.1 [8,9] (Riemann-Liouville) α 阶积分定义为

$$I^{\alpha}_{0+}x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds,$$

其中$n-1 \le \alpha < n$, n 为整数.

定义2.2 [8,9] (Riemann-Liouville) α 阶导数定义为

$$D_{0+}^{\alpha}x(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_{0}^{t}(t-s)^{n-\alpha-1}x(s)ds,$$

其中$n-1\leq\alpha<n$, n 为整数.

引理2.1 [8,9] 若$\alpha>0$, $x\in L(0,1)$, $D_{0+}^{\alpha}x\in L(0,1)$, 则

$$I_{0+}^{\alpha}D_{0+}^{\alpha}x(t)=x(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n},$$

其中$c_i\in(-\infty,+\infty)$, $i=1,2,...,n$, $n-1<\alpha\leq n$.

引理2.2 假设$h\in C(0,1)\cap L(0,1)$ 则

$$\begin{cases} {}^{c}D^{\alpha}x(t)+h(t)=0,\ 0<t<1, \\ x(0)=x''(0)=0,\ x(1)=\displaystyle\int_{0}^{1}b(s)x(s)dA(s) \end{cases}\tag{2.1}$$

有唯一的表达式

$$x(t)=\int_{0}^{1}G(t,s)h(s)ds,\tag{2.2}$$

其中

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}\dfrac{t}{(1-\chi)}\left((1-s)^{\alpha-1}-\displaystyle\int_{s}^{1}b(t)(t-s)^{\alpha-1}dA(t)\right)-(t-s)^{\alpha-1}, & 0\leq s\leq t\leq 1,\\ \dfrac{t}{(1-\chi)}\left((1-s)^{\alpha-1}-\displaystyle\int_{s}^{1}b(t)(t-s)^{\alpha-1}dA(t)\right), & 0\leq t\leq s\leq 1,\end{cases}$$

$$\chi=\int_{0}^{1}tb(t)dA(t).$$

证明 由引理2.1 和条件

$$x(0)=x''(0)=0,$$

BVP(2.1) 等价于下面的积分方程

$$x(t)=-\int_{0}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+ct,\tag{2.3}$$

结合条件

$$x(1)=\int_{0}^{1}b(s)x(s)dA(s),$$

可以得到

$$c = x(1) + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds.$$

所以

$$c = \frac{1}{\Gamma(\alpha)(1-\chi)} \left(\int_0^1 (1-s)^{\alpha-1} h(s)ds - \int_0^1 \int_0^t b(t)(t-s)^{\alpha-1} h(s)dsdA(t) \right), \quad (2.4)$$

将(2.4)式代入(2.3)式, 我们有

$$\begin{aligned} x(t) = & \frac{t}{\Gamma(\alpha_1)(1-\chi_1)} \left(\int_0^1 (1-s)^{\alpha_1-1} h_1(s)ds \right. \\ & \left. -\mu_1 \int_0^1 \int_s^1 b_1(t)(t-s)^{\alpha_1-1} h_1(s)dA_1(t)ds \right) - \frac{1}{\Gamma(\alpha_1)} \int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} h_1(s)ds. \end{aligned}$$

即, (2.2)式成立. □

由Green 函数$G(t,s)$的表达式, 可以证明下面的引理2.3成立.

引理2.3

(1)

$$G(t,s) \geq 0, G(t,s) 3[0,1] \times [0,1] Y.$$

(2)

$$G(t,s) \leq \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\chi)} \doteq \Phi(s), \quad t, \ s \in [0,1].$$

假设$X = C[0,1]$, 定义范数

$$\|x\| = \max_{t\in[0,1]} |x(t)|.$$

则X是Banach 空间. 对任意的$(x) \in X$, 定义$T : X \to X$

$$T(x)(t) = \int_0^1 G(t,s)a(s)f(s,x(s))ds, \quad 0 \leq t \leq 1.$$

则x是BVP(1.1)的解当且仅当x是T的不动点.

3. 主要结果

本文, 我们假设下面的条件$(\mathbf{H}_1)$ 成立.

$(\mathbf{H}_0)$ $\int_0^1 \Phi(s)a(s) < +\infty$.

$(\mathbf{H}_1)$ 存在$\zeta \geq 0$,有

$$|f(t,u_1) - f(t,u_2)| \leq \zeta|u_1 - u_2|,\ t \in [0,1],\ u_1, u_2 \in [0,+\infty).$$

由Ascoli-Arzela 定理, 可得下面的引理3.1成立.

引理3.1 假设$(\mathbf{H}_0)$ 成立, 则$T : X \to X$是全连续算子.

定理3.1 假设$(\mathbf{H}_0)(\mathbf{H}_1)$ 成立, $\Delta\zeta < 1$, where

$$\Delta = \int_0^1 \Phi(s)a(s)ds. \tag{3.1}$$

则BVP(1.1)有唯一解.

证明 假设

$$\sup|f(t,0)| = \varpi < +\infty,$$

由$(\mathbf{H}_1)$可得

$$|f(t,u)| \leq \varpi + \zeta|u|.$$

令

$$r = \frac{\Delta\varpi}{1-\Delta\zeta},\ P_r = \{x \in X : \|x\| < r\}.$$

接下来, 我们证明$TP_r \subset P_r$. 对任意的$x \in P_r$,

$$\begin{aligned} |Tx(t)| &\leq \max_{t\in[0,1]} \left|\int_0^1 G(t,s)a(s)f(s,x(s))ds\right| \\ &\leq \int_0^1 \Phi(s)a(s)f(s,x(s))ds \\ &\leq \int_0^1 \Phi(s)a(s)(\varpi + \zeta|x|)ds \\ &\leq \Delta(\varpi + \zeta\|x\|), \end{aligned}$$

因此

$$\|Tx\| \leq \Delta(\varpi + \zeta\|x\|).$$

$$\begin{aligned} \|T(x,y)\| &= \|T_1(x,y)\| + \|T_2(x,y)\| \\ &\leq \Delta_1(\varpi_1 + \zeta_1\|x\| + \eta_1\|y\|) + \Delta_2(\varpi_2 + \zeta_2\|x\| + \eta_2\|y\|) \\ &\leq r. \end{aligned}$$

对任意的$x_1, x_2 \in X$, $t \in [0,1]$, 由引理2.3和$(\mathbf{H}_1)$,

$$
\begin{aligned}
&|Tx_2(t)-Tx_1(t)| \\
&\le \int_0^1 G(t,s)a(s)\left|f(s,x_2(s))-f(s,x_1(s))\right|ds \\
&\le \int_0^1 \Phi(s)a(s)\left|f(s,x_2(s))-f(s,x_1(s))\right|ds \\
&\le \Delta\zeta\|x_2-x_1\|.
\end{aligned}
$$

所以

$$\|Tx_2-Tx_1\| \le \Delta\zeta\|x_2-x_1\|.$$

由于$\Delta\zeta<1$, T 有唯一的不动点,即BVP(1.1)有唯一解. □

4. 结论

本文研究了Caputo微分方程的解, 利用Banach压缩原理, 给出了解的存在唯一性(定理3.1). 由于f是抽象函数, 在实际中, 存在大量满足本文条件的函数, 说明了定理的有效性.

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