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Generalization and Application of Impulse Integral Inequality with Power

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Abstract

In this paper, we give the upper bounds estimation of unknown function with power of impulse integral inequality. The result is used to estimate the upper bound of impulsive differential systems.

Keywords

Impulse Integral Inequalities, Estimation of Unknown Function, Impulsive Differential Systems

含Power的脉冲积分不等式的推广与应用

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摘 要

本文研究了含有幂次的脉冲积分不等式，给出了未知函数的上界估计。利用此结果估计了脉冲微分系统的上界。

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关键词

脉冲积分不等式，未知函数估计，脉冲微分系统

1. 引言

Gronwall-Bellman 型积分不等式是研究微分方程和积分方程的重要工具，通过对积分不等式中未知函数的估计，可以研究某些微分方程解的存在性、有界性、唯一性和稳定性等定性性质(例如，文献[1]-[17])。通过对脉冲积分不等式中未知函数进行估计，可以研究某些脉冲微分方程和解的一些性质。2013 年严勇[16]研究了含有时滞的脉冲积分不等式

$$u(t) \le \alpha(t) + \int_{t_0}^{t} f(t,s)u(\tau(s))\mathrm{d}s + \int_{t_0}^{t} f(t,s)\left(\int_{t_0}^{s} g(s,\theta)u(\tau(\theta))\mathrm{d}\theta\right)\mathrm{d}s + q(t)\sum_{t_0<t_i<t}\beta_i u^m(t_i-0), \forall t \ge t_0.$$

2015 年米玉珍，钟吉玉[13]研究了含有未知函数的复合函数的积分不等式

$$u(t) \le \alpha(t) + \int_{t_0}^{t} f(t,s)\int_{t_0}^{s} g(s,t)w(u(\tau))\mathrm{d}\tau\mathrm{d}s + q(t)\sum_{t_0<t_i<t}\beta_i \mu^m(t_i-0), \forall t \ge t_0.$$

其中，$w(u)$ 是定义在 $[0,\infty)$ 上的单调不减连续函数且当 $u>0$ 时，$w(u)>0$。

本文在上述研究成果的基础上，研究了一类含有 power 的脉冲积分不等式

$$u(t) \le \alpha(t) + \int_{t_0}^{t} f_1(t,s)\left[u^2(s) + \int_{t_0}^{s} f_2(t,\tau)u(\tau)\mathrm{d}\tau\right]^p \mathrm{d}s + \sum_{t_0<t_i<t}\beta_i u^m(t_i-0). \tag{1.1}$$

其中，$u(t)$ 是 $[t_0,\infty)$ 上只有第一类不连续点 $\left\{t_i : t_0 < t_1 < t_2 \cdots, \lim_{i\to\infty} t_i = \infty\right\}$ 的非负逐段连续函数，$p \in (0,\infty)$，$m \in (1,\infty)$ 以及 $\beta_i \in [0,\infty)$ 都是给定的常数。

2. 主要结论

假设：

(H_1) $\alpha(t)$ 是定义在 $[t_0,\infty)$ 上的连续函数，且 $\alpha(t) \ge 1$；

(H_2) $f_i(t,s), i=1,2$ 是定义在 $[t_0,\infty)\times[t_0,\infty)$ 上的非负连续函数;

(H_3) $\beta_i \ge 0$ 是常数。

定理 2.1： 具有第一类不连续点 $\left\{t_i : t_0 < t_1 < t_2 \cdots, \lim_{i\to\infty} t_i = \infty\right\}$ 的非负逐段连续函数 $u(t), t \ge t_0 \ge 0$ 满足积分不等式(1.1)，则函数 $u(t)$ 有下面的估计式：

$$u(t) \le E_{i+1}(t) + \int_{t_i}^{t} \tilde{f}_1(t,s)A_{I+1}(s)\mathrm{d}s, \forall t \in [t_i, t_{i+1}],$$

其中

$$\tilde{f}_1(t,s) = \max_{t_0 \le \tau \le t} f_i(\tau,s), i=1,2,$$

$$E_1 = e_1(t) + \max_{t_0 \le \tau \le t} |a(\tau)|, t \in [t_0, t_1)$$

$$E_{i+1}(t) = e_1(t) + \sum_{k=1}^{i+1}\int_{t_k}^{t_k+1} \tilde{f}_1(T,s)\left[u^2(s) + \int_{t_i}^{s} \tilde{f}_2(T,\ \tau)u(\tau)\mathrm{d}\tau\right]^p \mathrm{d}s + \sum_{k=1}^{i}\beta_k u^m(t_k-0)$$
$$\forall t \in [t_k, t_{k+1}), i = 1,2,\cdots,$$

$$A_{i+1}(t)=\frac{E_{i+1}^{2P}\exp\left(2p\left(E_{i+1}(s)-E_{i+1}(t_i)\right)+p\int_{t_i}^{s}\tilde{f}_2(T,s)\right)\mathrm{d}s}{1-E_{i+1}^{2P}(t_i)\int_{t_i}^{t}\tilde{f}_1(T,s)\left(\exp\left(2pE_{i+1}(s)-E_{i+1}(t_i)\right)+p\int_{t_i}^{s}\tilde{f}_2(T,\tau)\mathrm{d}\tau\right)\mathrm{d}s}$$

$$i=1,2,\cdots,$$

$$1>E_{i+1}^{2P}(t_i)\int_{t_i}^{t}\tilde{f}_1(T,s)(\exp\left(2pE_{i+1}(s)-E_{i+1}(t_i)+p\int_{t_i}^{s}\tilde{f}_2(T,\tau)\mathrm{d}\tau\right)\mathrm{d}s,\ \ \forall t\in[t_i,t_{i+1}).$$

证明： 首先，我们考虑情况 $t\in[t_0,t_1)$，任取 $t\in[t_0,T)$，可得

$$u(t)\le e_1(t)+\int_{t_0}^{t}\tilde{f}_1(T,s)\left[u^2(s)+\int_{t_0}^{s}\tilde{f}_2(T,\tau)u(\tau)\mathrm{d}\tau\right]^p\mathrm{d}s,$$

令

$$v(t)=e_1(t)+\int_{t_0}^{t}\tilde{f}_1(T,s)\left[u^2(s)+\int_{t_0}^{s}\tilde{f}_2(T,\tau)u(\tau)\mathrm{d}\tau\right]^p\mathrm{d}s, \tag{2.1}$$

则 $v(t)$ 为非负不减的连续函数，且

$$u(t)\le v(t),v(t_0)=e_1(t_0), \tag{2.2}$$

对式(2.1)求导，我们可得

$$\begin{aligned}v'(t)&=e_1'(t)+\tilde{f}_1(T,t)\left[u^2(s)+\int_{t_0}^{t}\tilde{f}_2(T,s)u(s)\mathrm{d}s\right]^p\\&\le e_1'(t)+\tilde{f}_1(T,t)\left[v^2(s)+\int_{t_0}^{t}\tilde{f}_2(T,s)v(s)\mathrm{d}s\right]^p\end{aligned} \tag{2.3}$$

令

$$z(t)=v^2(t)+\int_{t_0}^{t}\tilde{f}_2(T,s)v(s)\mathrm{d}s$$

则 $z(t_0)=e_1^2(t_0)$，由 H_1 中 $\alpha(t)\ge 1$，得 $v(t)\le z(t)$，对 $z(t)$ 求导，由(2.3)可得

$$\begin{aligned}z'(t)&=2v(t)v'(t)+\tilde{f}_2(T,t)v(t)\\&\le 2z(t)(e_1'(t)+\tilde{f}_1(T,t)z^p(t)+\tilde{f}_2(T,t)z(t),\end{aligned} \tag{2.4}$$

由(2.4)得

$$z^{-[p+1]}z'(t)-(2e_1'(t)+\tilde{f}_2(T,t)z^{-p}\le 2\tilde{f}_1(T,t), \tag{2.5}$$

令 $\delta(t)=z^{-p}(t)$，则 $\delta'(t)=-pz^{-[p+1]}z'(t)$，(2.5)可变为

$$\delta'(t)+p\left(2e_1'(t)+\tilde{f}_2(T,t)\right)\delta(t)\ge -2p\tilde{f}_1(T,t), \tag{2.6}$$

(2.6)两边同时乘以 $\exp\left(\int_{t_0}^{t}p2e_1'(s)+\tilde{f}_2(T,t)\right)\mathrm{d}s$，可得

$$\left(\delta(t)\exp\left(\int_{t_0}^{t}p2e_1'(s)+\tilde{f}_2(T,t)\right)\mathrm{d}s\right)'\ge -2p\tilde{f}_1(T,t)\left(\exp\left(\int_{t_0}^{t}p2e_1'(s)+\tilde{f}_2(T,t)\right)\right)\mathrm{d}s \tag{2.7}$$

从 t_0 到 t 积分(2.7)的两边，我们得到

$$\delta(t)\ge\frac{1-e_1^{2p}(t_0)\int_{t_0}^{t}\tilde{f}_2(T,s)\left(\exp\left(2p\left(e_1(s)-e_1(t_0)\right)+p\int_{t_0}^{s}\tilde{f}_2(T,\tau)\right)\mathrm{d}\tau\right)\mathrm{d}s}{\exp\left(2p\left(e_1(s)-e_1(t_0)\right)+p\int_{t_0}^{t}\tilde{f}_2(T,s)\right)\mathrm{d}s} \tag{2.8}$$

由公式 $\delta(t)=z^{-p}(t)$ 和(2.8)得

$$z^{p}(t)\le\frac{e_1^{2p}\exp\left(2p\left(e_1(s)-e_1(t_0)\right)+p\int_{t_0}^{t}\tilde f_2(T,s)\right)\mathrm{d}s}{1-e_1^{2p}(t_0)\int_{t_0}^{t}\tilde f_1(T,s)\left(\exp\left(2p\left(e_1(s)-e_1(t_0)\right)+p\int_{t_0}^{s}\tilde f_2(T,\tau)\right)\mathrm{d}\tau\right)\mathrm{d}s}\tag{2.9}$$

令 $A_1(t)$ 等于(2.9)的右边，由(2.3)和(2.9)得

$$v'(t)\le e_1'(t)+\tilde f_1(T,t)z^{p}\le e_1'(t)+\tilde f_1(T,t)A_1(t)\tag{2.10}$$

(2.10)两边从 t_0 到 t 积分得

$$v(t)\le e_1(t)+\int_{t_0}^{t}\tilde f_1(T,s)A_1(s)\mathrm{d}s\tag{2.11}$$

由(2.2)和(2.11)可得

$$u(t)\le e_1(t)+\int_{t_0}^{t}\tilde f_1(T,s)A_1(s)\mathrm{d}s\tag{2.12}$$

由 T 的任意性，可得当 $t\in[t_0,t_1)$ 时有

$$u(t)\le e_1(t)+\int_{t_0}^{t}\tilde f_1(T,s)A_1(s)\mathrm{d}s\tag{2.13}$$

当 $t\in[t_0,t_1)$ 时我们证明了估计式。

当 $t\in[t_1,t_2)$ 时，任意取定 $T_1\in[t_1,t_2)$，对于任意的 $t\in[t_1,T_1]$，不等式(1.1)变为

$$\begin{aligned}u(t)\le{}& e_1(t)+\int_{t_0}^{t}\tilde f_1(T,s)\left[u^2(s)+\int_{t_0}^{s}\tilde f_2(T,\tau)u(\tau)\mathrm{d}\tau\right]^{p}\mathrm{d}s+\beta_1\mu^{m}(t_1-0)\\&+\int_{t_1}^{t}\tilde f(T,s)\left[u^2(s)+\int_{t_1}^{s}\tilde f_2(T,\tau)u(\tau)\mathrm{d}\tau\right]^{p}\mathrm{d}s,\end{aligned}\tag{2.14}$$

令 $r(t)$ 表示(2.14)的右边，且令 $E_2=e_1(t)+\int_{t_0}^{t_1}\tilde f_1(T,s)\left[u^2(s)+\int_{t_0}^{s}\tilde f_2(T,\tau)u(\tau)\mathrm{d}\tau\right]^{p}\mathrm{d}s+\beta_1\mu^{m}(t_1-0)$，则 $r(t)$ 是单调不减函数，且有

$$\begin{aligned}&u(t)\le r(t)\\&u(t_1)\le r(t)=E_2(t_1)\end{aligned}\tag{2.15}$$

$r(t)$ 两边关于 t 求导得

$$\begin{aligned}r'(t)&\le E_2'(t)+\tilde f_1(T,t)\left[u^2(t)+\int_{t_1}^{t}\tilde f_2(T,s)u(s)\mathrm{d}s\right]^{p}\\&\le E_2'(t)+\tilde f_1(T,t)\left[r^2(t)+\int_{t_1}^{t}\tilde f_2(T,s)r(s)\mathrm{d}s\right]^{p}\\&\le E_2'(t)+\tilde f_1(T,t)z_1^{p}(t)\end{aligned}\tag{2.16}$$

其中 $z_1(t)=r^2(t)+\int_{t_0}^{t}\tilde f_2(T,s)r(s)\mathrm{d}s$，(2.16)化为了(2.3)的形式，用相同的过程可得估计式为

$$u(t)\le E_2(t)+\int_{t_1}^{t}\tilde f_1(t,s)A_2(s)\mathrm{d}s,t\in[t_1,t_2)\tag{2.17}$$

我们证明了当 $t\in[t_1,t_2)$ 时估计式成立。

同理，对任意自然数 k，当 $t\in[t_k,t_{k+1})$ 时，我们可以得到未知函数的估计式

$$u(t)\le E_{k+1}(t)+\int_{t_k}^{t}\tilde f_1(t,s)A_{k+1}(s)\mathrm{d}s,t\in[t_k,t_{k+1}),\tag{2.18}$$

综上定理被证明。

3. 在脉冲微分方程中的应用

本节我们用得到的结果给出脉冲微分系统解的上界估计。考虑脉冲微分系统

$$\frac{\mathrm{d}(x(t))}{\mathrm{d}t}=F(t,x),\quad t\neq t_i, t\in[t_0,\infty), \tag{3.19}$$

$$\Delta(x)\big|_{t=t_i}=\beta_i x(t_i-0), \tag{3.20}$$

$$x(t_0)=c\text{，}$$

其中：$0\le t_0<t_1<t_2<\cdots,\lim\limits_{i\to\infty}t_i=\infty, c>1$ 是常数，$F(t,x)$ 关于 t，x 在 $[t_0,\infty)\times(-\infty,+\infty)$ 上连续。

假设(3.19)中 $F(t,x)$ 满足

$$|F(t,x)|\le f_1(t)\left(\left[\left|x^2(t)\right|+\int_{t_0}^{t_1}f_2(t,s)|x|(s)\mathrm{d}s\right]^3\right) \tag{3.21}$$

其中 $f_1(t), f_2(t)$ 是 $[t_0,\infty)$ 上连续的非负函数。

推论 1：在条件(3.21)成立的情况下，系统(3.19)，(3.20)所有的解 $x(t)$ 满足估计式：

$$x(t)\le\int_{t_i}^{t}f_1(t,s)A_{i+1}(s)\mathrm{d}s, \forall t\in[t_i,t_{i+1}), \tag{3.22}$$

其中

$$E_1=e_1(t)=c, , t\in[t_0,t_1),$$

$$E_{i+1}(t)=c+\sum_{k=1}^{i+1}\int_{t_k}^{k+1}f_1(s)\left[x^2(s)+\int_{t_i}^{s}f_2(\tau)x(\tau)\mathrm{d}\tau\right]^3\mathrm{d}s+\sum_{k=1}^{i}\beta_k x(t_k-0),$$
$$i=0,1,2,\cdots,\forall t\in[t_i,t_{i+1}), i=0,1,2,\cdots,$$

$$A_{i+1}(t)=\frac{E_{i+1}^6\exp\left(3\int_{t_i}^{t}f_2(T,s)\right)\mathrm{d}s}{1-E_{i+1}^6(t_i)\int_{t_i}^{t}f_1(s)\left(\exp\left(3\int_{t_i}^{s}f_2(\tau)\mathrm{d}\tau\right)\right)\mathrm{d}s}, i=0,1,2,\cdots,$$

$$1>E_{i+1}^6(t_i)\int_{t_i}^{t}f_1(s)\exp\left(3\int_{t_i}^{s}f_2(\tau)\mathrm{d}\tau\right)\mathrm{d}s, \forall t\in[t_i,t_{i+1}).$$

证明：脉冲微分方程(3.19)与(3.20)等价于积分方程

$$x(t)=c+\int_{t_0}^{t}f(s,x(s))\mathrm{d}s+\sum_{t_0<t_i<t}\beta_i x(t_i-0), t\in[t_0,\infty). \tag{3.23}$$

利用条件(3.21)，由(3.23)，可得

$$|x(t)|\le c+\int_{t_0}^{t}f_1(s)\left(\left|x^2(s)\right|+\int_{t_0}^{s}f_2(s)\left(|x(s)|\right)^3\right)\mathrm{d}s+\sum_{t_0<t_i<t}\beta_i x(t_i-0), \tag{3.24}$$

令 $u(t)=|x(t)|$，由(3.24)，我们可得不等式

$$u(t)\le c+\int_{t_0}^{t}f_1(t,s)\left[u^2(s)+\int_{t_0}^{s}f_2(t,\tau)u(\tau)\mathrm{d}\tau\right]^3\mathrm{d}s+\sum_{t_0<t_i<t}\beta_i u(t_i-0) \tag{3.25}$$

我们看出(3.25)是(1.1)的特殊形式。且(3.25)中的函数满足定理 2.1 的条件，由定理 2.1，我们可以推出 $x(t)$ 的估计式(3.22)。

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