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On the Diophantine Equation $(a^n-1)(b^n-1)=x^2$

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Abstract: In 2002, F. Luca and P. G. Walsh studied the diophantine equations of the form $(a^n-1)(b^n-1)=x^2$, for all (a,b) in the range $2\le b<a\le 100$ with sixty-nine exceptions. In this paper, we study two of the exceptions. In fact, we consider the equations of the form $(a^n-1)(b^n-1)=x^2$, with $(a,b)=(33,3),(33,9)$.

Keywords: Diophantine Equation; Legendre Symbol

关于不定方程$(a^n-1)(b^n-1)=x^2$解的研究#

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摘　要：2002 年，F. Luca 和 P. G. Walsh 研究了不定方程$(a^n-1)(b^n-1)=x^2$在$2\le b<a\le 100$范围内解的情况(除 69 种例外)。在本文中，我们研究了其中的两种例外。也就是，我们考虑的是不定方程$(a^n-1)(b^n-1)=x^2$在$(a,b)=(33,3),(33,9)$时解的情况。

关键词：不定方程；Legendre 符号

1. 引言

2000 年，Szalay[1]证明了不定方程$(2^n-1)(3^n-1)=x^2$在$n,x\in N$范围内无解，$(2^n-1)(5^n-1)=x^2$在$n,x\in N$范围内仅有唯一解$n=1,x=2$和$(2^n-1)\left(\left(2^k\right)^n-1\right)=x^2$在$k,n,x\in N$范围内仅有唯一解$k=2,n=3,x=21$。2000 年，Hajdu 和 Szalay[2]证明了不定方程$(2^n-1)(6^n-1)=x^2$在$n,x\in N$范围内无解和$(a^n-1)(a^{kn}-1)=x^2,a>1,k>1,kn>2$的正整数解仅有$(a,n,k,x)=(2,3,2,21),(3,1,5,22),(7,1,4,120)$。

2002 年，F. Luca 和 P. G. Walsh[3]证明了不定方程$(a^k-1)(b^k-1)=x^n$仅有有限多组正整数解$(k,n,x)(n>1)$。此外，他们考虑了在$2\le b<a\le 100$范围内，除 69 种情况以外其他所有数组(a,b)满足不定方程$(a^k-1)(b^k-1)=x^n$的解的情况$(k,n,x),k>1$，得到下列结论：

定理 A([3]，定理 3.1) 令$2\le b<a\le 100$都是正整数，假设(a,b)不是下列情况之一：

1. $\{(22,2);(22,4)\}$；

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2. $\left\{(a,b);(a-1)(b-1)\right.$是平方数，$a\equiv b(\mathrm{mod}\,2)$且$\left.(a,b)\neq(9,3),(64,8)\right\}$；

3. $\left\{(a,b);(a-1)(b-1)\right.$是平方数，$a+b\equiv 1(\mathrm{mod}\,2)$，且$\left.ab\equiv 0(\mathrm{mod}\,4)\right\}$。

若$\left(a^n-1\right)\left(b^n-1\right)=x^2$，则$n=2$。当$(a,b)=(4,2)$时，方程在$n=3$时有解。

对于其他相关问题，参见[4-7]。

在本文中，我们考虑两种例外：$(a,b)=(13,4),(28,13)$，并且得到以下结论：

定理 1 如果不定方程$\left(9^n-1\right)\left(33^n-1\right)=x^2$在$n,x\in N$范围内有解，那么$n\equiv 1(\mathrm{mod}\,13440)$。

定理 2 如果不定方程$\left(3^n-1\right)\left(33^n-1\right)=x^2$在$n,x\in N$范围内有解，那么$n\equiv 1(\mathrm{mod}\,3600)$。

设a和m是互素的两个正整数，我们称满足$a^x\equiv 1(\mathrm{mod}\,m)$的最小正整数$n$为$a$对模$m$的阶，记作$ord_m(a)$。设$p$是奇素数，$a$是整数，用$(a/p)$定义 Legendre 符号。

2. 定理的证明

2.1. 定理 1 的证明

容易验证当$n\le 4$时不定方程$\left(9^n-1\right)\left(33^n-1\right)=x^2$仅有唯一解：$n=1,x=16$。

假设$(n,x)(n\ge 5)$是方程$\left(9^n-1\right)\left(33^n-1\right)=x^2$的一组解，现在我们分成下列 18 种情况来讨论。

情况 1. $n\equiv 0(\mathrm{mod}\,4)$。此时$n$可以唯一地表示成$n=4\cdot 5^k l$，其中$5\nmid l,k\ge 0$。通过对$k$运用数学归纳法，我们有

$$9^{4\cdot 5^k l}\equiv 1+2^5\times 41\cdot 5^{k+1}l\left(\mathrm{mod}\,5^{k+2}\right), \tag{1}$$

$$33^{4\cdot 5^k l}\equiv 1+2^7\times 17\times 109\cdot 5^{k+1}l\left(\mathrm{mod}\,5^{k+2}\right). \tag{2}$$

由假设以及(1)和(2)式，我们得到$x^2/5^{2k+2}\equiv 2^{12}\times 17\times 41\times 109 l^2(\mathrm{mod}\,5)$，这和$\left(2^{12}\times 17\times 41\times 109 l^2/5\right)=(3/5)=-1$矛盾。

情况 2. $n\equiv 3(\mathrm{mod}\,4)$。由$ord_5(9)=2,ord_5(33)=4$，我们有$x^2\equiv(4-1)\left(3^3-1\right)\equiv 3(\mathrm{mod}\,5)$，这与$(3/5)=-1$矛盾。

情况 3. $n\equiv 5(\mathrm{mod}\,12)$。由$ord_7(9)=3,ord_7(33)=6$，我们有$x^2\equiv\left(9^5-1\right)\left(33^5-1\right)\equiv\left(2^2-1\right)\left(5^5-1\right)\equiv 6(\mathrm{mod}\,7)$，这与$(6/7)=-1$矛盾。

情况 4. $n\equiv 2,10(\mathrm{mod}\,12)$。由$ord_{13}(9)=3,ord_{13}(33)=12$，我们有$x^2\equiv\left(9^2-1\right)\left(33^2-1\right),\left(9^{10}-1\right)\left(33^{10}-1\right)\equiv 5,11(\mathrm{mod}\,13)$，这些都和$(5/13)=(11/13)=-1$矛盾。

情况 5. $n\equiv 13,21(\mathrm{mod}\,24)$。即$n\equiv 5(\mathrm{mod}\,8),n1(\mathrm{mod}\,2)$。由$ord_{17}(9)=8,ord_{17}(33)=2$，我们有$x^2\equiv\left(9^5-1\right)(33-1)\equiv 3(\mathrm{mod}\,17)$，这与$(3/17)=-1$矛盾。

情况 6. $n\equiv 18,33,42,49,57,73,78,97,102(\mathrm{md}120)$。即$n\equiv 3,12,4,13,7(\mathrm{mod}\,15)$，$n\equiv 3,2,4,3,2(\mathrm{mod}\,5)$。由$ord_{31}(9)=15,ord_{31}(33)=5$，我们有$x^2\equiv\left(9^3-1\right)\left(33^3-1\right),\left(9^{12}-1\right)\left(33^2-1\right),\left(9^4-1\right)\left(33^4-1\right),\left(9^{13}-1\right)\left(33^3-1\right),\left(9^7-1\right)\left(33^2-1\right)\equiv 12,3,6,26,27(\mathrm{mod}\,31)$，这些都和$(12/31)=(3/31)=(6/31)=(26/31)=(27/31)=-1$矛盾。

情况 7. $n\equiv 6,9,54,66,114(\mathrm{mod}\,120)$。即$n\equiv 2,1(\mathrm{mod}\,4),n\equiv 6,14,9(\mathrm{mod}\,20)$。由$ord_{41}(9)=4,ord_{41}(33)=20$，我们有$x^2\equiv\left(9^2-1\right)\left(33^6-1\right),\left(9^2-1\right)\left(33^{14}-1\right),(9-1)\left(33^9-1\right)\equiv 22,35,34(\mathrm{mod}\,41)$，这些都和$(22/41)=(35/41)=(34/41)=-1$矛盾。

情况 8. $n\equiv 81,105(\mathrm{mod}\,120)$。即$n\equiv 9(\mathrm{mod}\,24),n\equiv 1(\mathrm{mod}\,8)$，我们有$x^2\equiv\left(9^9-1\right)(33-1)\equiv 76(\mathrm{mod}\,97)$，这与$(76/97)=-1$矛盾。

情况 9. $n\equiv 25,90,121,145,150(\mathrm{mod}\,240)$。即$n\equiv 25,30,1,25(\mathrm{mod}\,60)$，$n\equiv 25,10,70,41,65(\mathrm{mod}\,80)$。由$ord_{241}(9)=60,ord_{241}(33)=80$，我们有$x^2\equiv\left(9^{25}-1\right)\left(33^{25}-1\right),\left(9^{30}-1\right)\left(33^{10}-1\right),\left(9^{30}-1\right)\left(33^{70}-1\right),(9-1)\left(33^{41}-1\right),$

$\left(9^{25}-1\right)\left(33^{65}-1\right)\equiv 140,62,227,210,224\left(\bmod 241\right)$，这些都
$\left(140/241\right)=\left(62/241\right)=\left(227/241\right)=\left(210/241\right)=\left(224/241\right)=-1$ 矛盾。

情况 10. $n\equiv 210,241,270,481,510,961,1170,1201,1410,1470\left(\bmod 1680\right)$。即 $n\equiv 42,17,46,33,6,9,50,25,10,14\left(\bmod 56\right)$， $n\equiv 98,17,46,33,62,65,50,81,66,14\left(\bmod 112\right)$。由 $ord_{113}\left(9\right)=56, ord_{113}\left(33\right)=112$，我们有 $x^2\equiv \left(9^{42}-1\right)\left(33^{98}-1\right),\left(9^{17}-1\right)\left(33^{17}-1\right),\left(9^{46}-1\right)\left(33^{46}-1\right),\left(9^{33}-1\right)\left(33^{33}-1\right),\left(9^{6}-1\right)\left(33^{62}-1\right),\left(9^{9}-1\right)\left(33^{65}-1\right),\left(9^{50}-1\right)\left(33^{50}-1\right),\left(9^{25}-1\right)\left(33^{81}-1\right),\left(9^{10}-1\right)\left(33^{66}-1\right),\left(9^{14}-1\right)\left(33^{14}-1\right)\equiv 73,65,17,59,103,110,101,46,43,42\left(\bmod 113\right)$，这些都和 $\left(73/113\right)=\left(65/113\right)=\left(17/113\right)=\left(59/113\right)=\left(103/113\right)=\left(110/113\right)=\left(101/113\right)=\left(46/113\right)=\left(43/113\right)=\left(42/113\right)=-1$ 矛盾。

情况 11. $n\equiv 1710,5070,750,4110,2670,6030,1230,4590,2130,5490,690,4050,2610,5970,1650,5010,1441,2401,3121,3361,4081,5041\left(\bmod 6720\right)$。即 $n\equiv 2,6,1\left(\bmod 8\right), n\equiv 18,50,14,46,33,49\left(\bmod 64\right)$。由 $ord_{193}\left(9\right)=8, ord_{193}\left(33\right)=64$，我们有 $x^2\equiv \left(9^{2}-1\right)\left(33^{18}-1\right),\left(9^{2}-1\right)\left(33^{50}-1\right),\left(9^{6}-1\right)\left(33^{14}-1\right),\left(9^{6}-1\right)\left(33^{46}-1\right),\left(9-1\right)\left(33^{33}-1\right),\left(9-1\right)\left(33^{49}-1\right)\equiv 52,174,120,44,114,146\left(\bmod 193\right)$，这些都和
$\left(52/193\right)=\left(174/193\right)=\left(120/193\right)=\left(44/193\right)=\left(114/193\right)=\left(146/193\right)=-1$ 矛盾。

情况 12. $n\equiv 30,3390,2430,5790,2910,6270,450,3810,930,4290,3330,6690,721,5761\left(\bmod 6720\right)$。即 $n\equiv 30,78,54,30,6,54,49\left(\bmod 84\right)$， $n\equiv 30,78,222,114,258,306,49\left(\bmod 336\right)$。由 $ord_{337}\left(9\right)=84, ord_{337}\left(33\right)=336$，我们有 $x^2\equiv \left(9^{30}-1\right)\left(33^{30}-1\right),\left(9^{78}-1\right)\left(33^{78}-1\right),\left(9^{54}-1\right)\left(33^{222}-1\right),\left(9^{30}-1\right)\left(33^{114}-1\right),\left(9^{6}-1\right)\left(33^{258}-1\right),\left(9^{54}-1\right)\left(33^{306}-1\right),\left(9^{49}-1\right)\left(33^{49}-1\right)\equiv 218,284,183,284,124,5,264\left(\bmod 337\right)$，这些都和
$\left(218/337\right)=\left(284/337\right)=\left(183/337\right)=\left(124/337\right)=\left(5/337\right)=\left(264/337\right)=-1$ 矛盾。

情况 13. $n\equiv 4530,2370,1681,4801,6481\left(\bmod 6720\right)$。即 $n\equiv 94,130,113,97,209\left(\bmod 224\right)$， $n\equiv 381,130,337,321,209\left(\bmod 448\right)$。由 $ord_{449}\left(9\right)=224, ord_{449}\left(33\right)=448$，我们有 $x^2\equiv \left(9^{94}-1\right)\left(33^{318}-1\right),\left(9^{130}-1\right)\left(33^{130}-1\right),\left(9^{113}-1\right)\left(33^{337}-1\right),\left(9^{97}-1\right)\left(33^{321}-1\right),\left(9^{209}-1\right)\left(33^{209}-1\right)\equiv 68,355,119,363,326\left(\bmod 449\right)$，这些都和
$\left(68/449\right)=\left(335/449\right)=\left(119/449\right)=\left(363/449\right)=-1$ 矛盾。

情况 14. $n\equiv 7710,5730,6721\left(\bmod 13440\right)$。即 $n\equiv 30,290,1\left(\bmod 320\right), n\equiv 30,610,321\left(\bmod 640\right)$。由 $ord_{641}\left(9\right)=320, ord_{641}\left(33\right)=640$，我们有 $x^2\equiv \left(9^{30}-1\right)\left(33^{30}-1\right),\left(9^{290}-1\right)\left(33^{610}-1\right),\left(9-1\right)\left(33^{321}-1\right)\equiv 614,418,369\left(\bmod 641\right)$，这些都和 $\left(614/641\right)=\left(418/641\right)=\left(369/641\right)=-1$ 矛盾。

情况 15. $n\equiv 990,25890\left(\bmod 26880\right)$。即 $n\equiv 6,18\left(\bmod 24\right), n\equiv 222,34\left(\bmod 256\right)$。由 $ord_{769}\left(9\right)=24, ord_{769}\left(33\right)=256$，我们有 $x^2\equiv \left(9^{6}-1\right)\left(33^{222}-1\right),\left(9^{18}-1\right)\left(33^{34}-1\right)\equiv 412,567\left(\bmod 769\right)$，这些都和 $\left(412/769\right)=\left(567/769\right)=-1$ 矛盾。

情况 16. $n\equiv 14430,41310,39330,66210\left(\bmod 80640\right)$。即 $n\equiv 66,18\left(\bmod 84\right)$， $n\equiv 318,990,18,690\left(\bmod 1008\right)$。由 $ord_{1009}\left(9\right)=84, ord_{1009}\left(33\right)=1008$，我们有 $x^2\equiv \left(9^{66}-1\right)\left(33^{318}-1\right),\left(9^{66}-1\right)\left(33^{990}-1\right),\left(9^{18}-1\right)\left(33^{18}-1\right),\left(9^{18}-1\right)\left(33^{690}-1\right)\equiv 653,792,138,513\left(\bmod 1009\right)$，这些都和 $\left(653/1009\right)=\left(792/1009\right)=\left(513/1009\right)=-1$ 矛盾。

情况 17. $n\equiv 12450,93090,254370,335010,68190,148830,310110,390750\left(\bmod 403200\right)$。即 $n\equiv 60,45\left(\bmod 105\right)$， $n\equiv 900,690,270,60,990,780,360,150\left(\bmod 1050\right)$。由 $ord_{1051}\left(9\right)=105, ord_{1051}\left(33\right)=1050$，我们有 $x^2\equiv \left(9^{60}-1\right)\left(33^{900}-1\right),\left(9^{60}-1\right)\left(33^{690}-1\right),\left(9^{60}-1\right)\left(33^{270}-1\right),\left(9^{60}-1\right)\left(33^{60}-1\right),\left(9^{45}-1\right)\left(33^{990}-1\right),\left(9^{45}-1\right)\left(33^{780}-1\right),\left(9^{45}-1\right)\left(33^{360}-1\right),\left(9^{45}-1\right)\left(33^{150}-1\right)\equiv 166,637,519,78,728,368,331,166\left(\bmod 1051\right)$，这些都和
$\left(637/1051\right)=\left(519/1051\right)=\left(78/1051\right)=\left(728/1051\right)=\left(368/1051\right)=\left(331/1051\right)=-1$ 矛盾。

情况 18. $n\equiv 173730,229470\left(\bmod 403200\right)$。即 $n\equiv 30,120\left(\bmod 150\right), n\equiv 930,270\left(\bmod 1200\right)$。由 $ord_{1021}\left(9\right)=150, ord_{1201}\left(33\right)=1200$，我们有 $x^2\equiv \left(9^{30}-1\right)\left(33^{930}-1\right),\left(9^{120}-1\right)\left(33^{270}-1\right)\equiv 1010,542\left(\bmod 1201\right)$，这些都和 $\left(1010/1201\right)=\left(542/769\right)=-1$。

定理证毕。

猜想 不定方程 $\left(9^n-1\right)\left(33^n-1\right)=x^2$ 在 $n,x\in N$ 范围内仅有唯一解： $n=1,x=16$ 。

2.2. 定理 2 的证明

容易验证当 $n\le 4$ 时不定方程 $\left(3^n-1\right)\left(33^n-1\right)=x^2$ 仅有唯一解： $n=1,x=8$ 。

假设 $(n,x)(n\ge 5)$ 是方程 $\left(3^n-1\right)\left(33^n-1\right)=x^2$ 的一组解，现在我们分成下列 15 种情况来讨论。

情况 1. $n\equiv 0\left(\bmod 6\right)$。此时 n 可以唯一地表示成 $n=6\cdot 7^k l$，其中 $7\nmid l,k\ge 0$。通过对 k 运用数学归纳法，我们有

$$3^{6\cdot 7^k l}\equiv 1+2^3\times 13\cdot 7^{k+1}l\left(\bmod 7^{k+2}\right), \tag{1}$$

$$33^{6\cdot 7^k l}\equiv 1+2^6\times 17\times 151\times 1123\cdot 7^{k+1}l\left(\bmod 7^{k+2}\right). \tag{2}$$

由假设以及(1)和(2)式，我们得到 $x^2/7^{2k+2}\equiv 2^9\times 13\times 17\times 151\times 1123l^2\left(\bmod 7\right)$，这和 $\left(2^9\times 13\times 17\times 151\times 1123l^2/7\right)=\left(5/7\right)=-1$ 矛盾。

情况 2. $n\equiv 2,4\left(\bmod 6\right)$。由 $ord_7\left(3\right)=ord_7\left(33\right)=6$，我们有 $x^2\equiv\left(3^2-1\right)\left(33^3-1\right),\left(3^4-1\right)\left(33^4-1\right)\equiv 3,3\left(\bmod 7\right)$，这与 $\left(3/7\right)=-1$ 矛盾。

情况 3. $n\equiv 5,11\left(\bmod 12\right)$。即 $n\equiv 2\left(\bmod 3\right)$。由 $ord_{13}\left(3\right)=3,ord_{13}\left(33\right)=12$，我们有 $x^2\equiv\left(3^2-1\right)\left(33^5-1\right),\left(3^2-1\right)\left(33^{11}-1\right)\equiv 2,8\left(\bmod 13\right)$，这些都和 $\left(2/13\right)=\left(8/13\right)=-1$ 矛盾。

情况 4. $n\equiv 7,13,15,27,31,39,43,45\left(\bmod 48\right)$。即 $n\equiv 7,11,13,15\left(\bmod 16\right),n\equiv 1\left(\bmod 2\right)$。由 $ord_{17}\left(3\right)=16,ord_{17}\left(33\right)=2$，我们有 $x^2\equiv\left(3^7-1\right)\left(33-1\right),\left(3^{11}-1\right)\left(33-1\right),\left(3^{13}-1\right)\left(33-1\right),\left(3^{15}-1\right)\left(33-1\right)\equiv 14,5,12,7\left(\bmod 17\right)$，这些都和 $\left(14/17\right)=\left(5/17\right)=\left(12/17\right)=\left(7/17\right)=-1$ 矛盾。

情况 5. $n\equiv 19,33,37\left(\bmod 48\right)$。即 $n\equiv 3,1,5\left(\bmod 8\right)$。由 $ord_{97}\left(3\right)=48,ord_{97}\left(33\right)=8$，我们有 $x^2\equiv\left(3^{19}-1\right)\left(33^3-1\right),\left(3^{33}-1\right)\left(33-1\right),\left(3^{37}-1\right)\left(33^5-1\right)\equiv 56,30,21\left(\bmod 97\right)$，这些都和 $\left(56/97\right)=\left(30/97\right)=\left(21/97\right)=-1$ 矛盾。

情况 6. $n\equiv 3,9,69,73,97,99,153,193,213,217\left(\bmod 240\right)$。即 $n\equiv 3,9,13,7\left(\bmod 30\right)$，$n\equiv 3,4,3,2\left(\bmod 5\right)$。由 $ord_{31}\left(3\right)=30,ord_{31}\left(33\right)=5$，我们有 $x^2\equiv\left(3^3-1\right)\left(33^3-1\right),\left(3^9-1\right)\left(33^4-1\right),\left(3^{13}-1\right)\left(33^3-1\right),\left(3^7-1\right)\left(33^2-1\right)\equiv 27,17,6,17\left(\bmod 31\right)$，这些都和 $\left(27/31\right)=\left(17/31\right)=\left(6/31\right)=-1$ 矛盾。

情况 7. $n\equiv 21,121,169,201\left(\bmod 240\right)$。即 $n\equiv 21,1,49,81\left(\bmod 120\right),n\equiv 21,41,9,41\left(\bmod 80\right)$。由 $ord_{241}\left(3\right)=120,ord_{241}\left(33\right)=80$，我们有 $x^2\equiv\left(3^{21}-1\right)\left(33^{21}-1\right),\left(3-1\right)\left(33^{41}-1\right),\left(3^{81}-1\right)\left(33^{41}-1\right)\equiv 222,173,139,220\left(\bmod 241\right)$，这些都和 $\left(222/241\right)=\left(173/241\right)=\left(139/241\right)=\left(220/241\right)=-1$ 矛盾。

情况 8. $n\equiv 49,195\left(\bmod 240\right)$。即 $n\equiv 1,3\left(\bmod 8\right),n\equiv 9,15\left(\bmod 20\right)$。由 $ord_{41}\left(3\right)=8,ord_{41}\left(33\right)=20$，我们有 $x^2\equiv\left(3-1\right)\left(33^9-1\right),\left(3^3-1\right)\left(33^{15}-1\right)\equiv 3,29\left(\bmod 41\right)$，这些都和 $\left(3/41\right)=\left(29/41\right)=-1$ 矛盾。

情况 9. $n\equiv 25,51,57,105,117,145,165\left(\bmod 240\right)$。即 $n\equiv 5,1,7\left(\bmod 10\right),n\equiv 5,11,17\left(\bmod 20\right)$。由 $ord_{61}\left(3\right)=10,ord_{61}\left(33\right)=20$，我们有 $x^2\equiv\left(3^5-1\right)\left(33^5-1\right),\left(3-1\right)\left(33^{11}-1\right),\left(3^7-1\right)\left(33^{17}-1\right)\equiv 24,54,24\left(\bmod 61\right)$，这些都和 $\left(24/61\right)=\left(54/61\right)=-1$ 矛盾。

情况 10. $n\equiv 241,481\left(\bmod 720\right)$。即 $n\equiv 7,13\left(\bmod 18\right),n\equiv 7,4\left(\bmod 9\right)$。由 $ord_{37}\left(3\right)=18,ord_{37}\left(33\right)=9$，我们有 $x^2\equiv\left(3^7-1\right)\left(33^7-1\right),\left(3^{13}-1\right)\left(33^4-1\right)\equiv 18,32\left(\bmod 37\right)$，这些都和 $\left(18/37\right)=\left(32/37\right)=-1$ 矛盾。

情况 11. $n\equiv 387\left(\bmod 720\right)$。即 $n\equiv 27\left(\bmod 45\right),n\equiv 27\left(\bmod 90\right)$。由 $ord_{181}\left(3\right)=45,ord_{181}\left(33\right)=90$，我们有 $x^2\equiv\left(3^{27}-1\right)\left(33^{27}-1\right)\equiv 74\left(\bmod 181\right)$，这与 $\left(74/181\right)=-1$ 矛盾。

情况 12. $n\equiv 627,867,1441,2067,2161,2881,3027\left(\bmod 3600\right)$。即 $n\equiv 27,67,41,61,81\left(\bmod 100\right),n\equiv 27,17,41,11,31\left(\bmod 50\right)$。由 $ord_{101}\left(3\right)=100,ord_{101}\left(33\right)=50$，我们有 $x^2\equiv\left(3^{27}-1\right)\left(33^{27}-1\right),\left(3^{67}-1\right)\left(33^{17}-1\right),\left(3^{41}-1\right)$

$\left(33^{41}-1\right),\left(3^{61}-1\right)\left(33^{11}-1\right),\left(3^{81}-1\right)\left(33^{31}-1\right)\equiv 51,94,27,18,10\left(\bmod 101\right)$，这些都和$\left(51/101\right)=\left(94/101\right)=\left(27/101\right)=\left(18/101\right)=\left(10/101\right)=-1$矛盾。

情况 13. $n\equiv 1587,2307,2787,3507\left(\bmod 3600\right)$。即$n\equiv 37,7\left(\bmod 50\right),n\equiv 3\left(\bmod 6\right)$。由$ord_{151}\left(3\right)=50,ord_{151}\left(33\right)=6$，我们有$x^2\equiv\left(3^{37}-1\right)\left(33^3-1\right),\left(3^7-1\right)\left(33^3-1\right)\equiv 146,7\left(\bmod 151\right)$，这些都和$\left(146/151\right)=\left(7/151\right)=-1$矛盾。

情况 14. $n\equiv 147,1347\left(\bmod 3600\right)$。即$n\equiv 147\left(\bmod 400\right),n\equiv 67\left(\bmod 80\right)$。由$ord_{401}\left(3\right)=400,ord_{401}\left(33\right)=80$，我们有$x^2\equiv\left(3^{147}-1\right)\left(33^{67}-1\right)\equiv 139\left(\bmod 401\right)$，这与$\left(139/401\right)=-1$矛盾。

情况 15. $n\equiv 721\left(\bmod 3600\right)$。即$n\equiv 46\left(\bmod 75\right),n\equiv 121\left(\bmod 600\right)$。由$ord_{601}\left(3\right)=24,ord_{601}\left(33\right)=600$，我们有$x^2\equiv\left(3^{46}-1\right)\left(33^{121}-1\right)\equiv 224\left(\bmod 601\right)$，这与$\left(224/601\right)=-1$矛盾。

定理证毕。

猜想 不定方程$\left(3^n-1\right)\left(33^n-1\right)=x^2$在$n,x\in N$范围内仅有唯一解：$n=1,x=8$。

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