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The Initial Boundary Value Problems for a Class of *n*-Dimensional Nonlinear Evolution Systems of Higher Order

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Abstract

In this paper, we study a class of *n*-dimensional nonlinear wave system of high order by using the Galerkin method, Sobolev embedding theorem and prior estimate of solution, and prove the existence and uniqueness of global strong solution to the initial boundary value problem.

Keywords

Nonlinear Wave System, Prior Estimate, Galerkin Method, Sobolev Embedding Theorem, The Global Strong Solution

一类高阶*n*维非线性发展方程组的初边值问题

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摘　要

本文研究了一类高阶*n*维非线性波动方程组，通过解的先验估计，利用Galerkin方法和Sobolev嵌入定理，

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证明了初边值问题整体强解的存在性和唯一性。

关键词

非线性波动方程组，先验估计，Galerkin方法，Sobolev嵌入定理，整体强解

1. 引言

在本文中， $\boldsymbol{u} \cdot \boldsymbol{v} = \sum_{i=1}^{N} u_i v_i$ ， $(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega} \boldsymbol{u} \cdot \boldsymbol{v} \mathrm{d}x$ ， $|\boldsymbol{u}|^2_{L^2(\Omega)} = (\boldsymbol{u}, \boldsymbol{u})$ ，

$$[\boldsymbol{u}, \boldsymbol{v}] = \int_0^t (\boldsymbol{u}, \boldsymbol{v}) \mathrm{d}t ， \|\boldsymbol{u}\|^2_{L^2(\Omega)} = [\boldsymbol{u}, \boldsymbol{u}] ， |\boldsymbol{u}| = \left(\sum_{i=1}^{N} u_i^2 \right)^{\frac{1}{2}} ，$$

$$\boldsymbol{u} = \boldsymbol{u}(x, t) = (u_1, u_2, \cdots, u_N)^{\mathrm{T}} ， \boldsymbol{v} = \boldsymbol{v}(x, t) = (v_1, v_2, \cdots, v_N)^{\mathrm{T}} ，$$

$$\boldsymbol{f}(\boldsymbol{u}_t) = (f_1(\boldsymbol{u}_t), f_2(\boldsymbol{u}_t), \cdots, f_N(\boldsymbol{u}_t))^{\mathrm{T}} ， \boldsymbol{\varphi}(x) = (\varphi_1(x), \varphi_2(x), \cdots, \varphi_N(x))^{\mathrm{T}} ，$$

$$\boldsymbol{\psi}(x) = (\psi_1(x), \psi_2(x), \cdots, \psi_N(x))^{\mathrm{T}}$$

其中 $\boldsymbol{\varphi}(x) \in H^{2M}(\Omega) \bigcap H_0^M(\Omega)$ ， $\boldsymbol{\psi}(x) \in H^{2M}(\Omega) \bigcap H_0^M(\Omega)$ ，"T"表示转置。

有三种因素影响弹性杆内波的传播：非线性、弥散以及耗散。文[1]研究了在上述三种因素影响下的细长弹性杆中纵向应变波的传播问题。

文[2]在此基础上研究了一类四阶非线性耗散色散波动方程的初边值问题，非线性耗散色散波动方程组关于空间变量导数为2阶的情形在文[3]中有所研究，本文主要在此基础上研究了高阶的情形，关于高阶 n 维非线性耗散色散波动方程组的研究在已有文献中还未见到。

本文主要研究了下述在粘弹性力学中具有实际背景的一类高阶 n 维非线性波动方程组：

$$\begin{cases} \boldsymbol{u}_{tt} + (-1)^M \Delta^M \boldsymbol{u} + (-1)^M \Delta^M \boldsymbol{u}_t + (-1)^M \Delta^M \boldsymbol{u}_{tt} = \boldsymbol{f}(\boldsymbol{u}_t), & (x, t) \in \Omega \times [0, T] , & (1.1) \\ \boldsymbol{u}|_{t=0} = \varphi(x), \quad \boldsymbol{u}_t|_{t=0} = \psi(x) , & (1.2) \\ D^\gamma \boldsymbol{u}|_{\partial\Omega \times [0, T]} = 0, \quad 0 \le |\gamma| \le M-1 & (1.3) \end{cases}$$

其中 Ω 为 R^n 中具有光滑边界的有界区域， $T > 0$ ， M 为正整数， $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ 为多重指标， $\gamma_i (i = 1, 2, \cdots, n)$ 为非负整数，且 $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ 。 $\boldsymbol{\varphi}(x) \in H^{2M}(\Omega) \bigcap H_0^M(\Omega), \boldsymbol{\psi}(x) \in H^{2M}(\Omega) \bigcap H_0^M(\Omega)$ ，假设 f 满足：

$$\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}, \ \boldsymbol{f} \in C^1 \tag{1.4}$$

且 Jacobian 矩阵 $\dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}$ 半有界，即 $\exists k_0 > 0, \forall \xi \in R^n$ 满足：

$$\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}} \xi, \xi \right) \le k_0 (\xi, \xi), \tag{1.5}$$

$$|\boldsymbol{f}(\boldsymbol{u})| \le a_1 + b_1 |\boldsymbol{u}|^{\frac{p}{2}} (a_1, b_1 \text{为正常数}), \tag{1.6}$$

当 $2M < n$ 时， $2 \le p < \dfrac{2n}{n-2M}$ ；当 $2M \ge n$ 时， $2 \le p < +\infty$ 。

2. 主要结论

定义：$\boldsymbol{u} = \boldsymbol{u}(x,t)$ $\left((x,t) \in \Omega \times [0,T], \Omega \subset R^n\right)$ 称为问题(1.1)~(1.3)在 $\Omega \times [0,T]$ 上的整体强解，若

$$\boldsymbol{u} \in L^\infty\left(0,T;H^{2M}(\Omega) \cap H_0^M(\Omega)\right),$$

$$\boldsymbol{u}_t \in L^\infty\left(0,T;H^{2M}(\Omega) \cap H_0^M(\Omega)\right),$$

$$\boldsymbol{u}_{tt} \in L^\infty\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H_0^M(\Omega)\right),$$

对一切 $\boldsymbol{d}(x,t) = \left(d_1(x,t), d_2(x,t), \cdots, d_N(x,t)\right) \in C\left([0,T];L^2(\Omega)\right)$ 成立

$$\int_0^T \left(\boldsymbol{u}_{tt} + (-1)^M \Delta^M \boldsymbol{u} + (-1)^M \Delta^M \boldsymbol{u}_t + (-1)^M \Delta^M \boldsymbol{u}_{tt} - \boldsymbol{f}(\boldsymbol{u}_t), \boldsymbol{d}(x,t)\right)\mathrm{d}t = 0$$

且

$$\boldsymbol{u}\big|_{t=0} = \boldsymbol{\varphi}(x), \boldsymbol{u}_t\big|_{t=0} = \boldsymbol{\psi}(x)$$

定理 1：若 $\boldsymbol{\varphi}(x) \in H^{2M}(\Omega) \cap H_0^M(\Omega), \boldsymbol{\psi}(x) \in H^{2M}(\Omega) \cap H_0^M(\Omega)$，条件(1.5)~(1.6)成立，则问题(1.1)~(1.3)存在上述意义下的整体强解 $\boldsymbol{u} = \boldsymbol{u}(x,t)$。

定理 2：若 $\boldsymbol{\varphi}(x) \in H^{2M}(\Omega) \cap H_0^M(\Omega), \boldsymbol{\psi}(x) \in H^{2M}(\Omega) \cap H_0^M(\Omega)$，条件(1.5)~(1.6)成立，则非线性问题(1.1)~(1.3)的整体强解唯一。

3. 主要结论的证明

首先介绍定理 1 的证明思想，即第一步构造问题(1.1)~(1.3)的近似解，然后对近似解作出先验估计，在作出先验估计基础上，由列紧性原理可证得问题(1.1)~(1.3)的整体强解。下面进行定理 1 的证明：

设 $\{w_j(x) | j = 1,2,\cdots\}$ 为问题 $\begin{cases} (-1)^M \Delta^M w_j = \lambda_j w_j \\ D^\gamma w_j(x)\big|_{\partial\Omega} = 0 \quad 0 \leq |\gamma| \leq M-1 \end{cases}$ 的特征函数系，则由文[4]中的引理 1.1 有结论：$\{w_j(x) | j = 1,2,\cdots\}$ 分别构成 $L^2(\Omega)$，$H_0^M(\Omega), H^{2M} \cap H_0^M(\Omega)$ 的正交基底。

构造问题(1.1)~(1.3)的近似解

$$u_{mi} = u_{mi}(x,t) = \sum_{j=1}^m a_{mij}(t) w_j(x), \tag{2.1}$$

并满足初值条件

$$u_{mi}(x,0) = \varphi_{mi}(x) = \sum_{j=1}^m a_{mij} w_j(x) \ \left(a_{mij}\text{将在下面选取}\right), \tag{2.2}$$

$$u_{mit}(x,0) = \psi_{mi}(x) = \sum_{j=1}^m b_{mij} w_j(x) \ \left(b_{mij}\text{将在下面选取}\right). \tag{2.3}$$

由 Galerkin 方法，该近似解应满足如下非线性常微分方程组的初值问题

$$(*) \begin{cases} \left(u_{mitt}, w_k\right) + \left((-1)^M \Delta^M u_{mi}, w_k\right) + \left((-1)^M \Delta^M u_{mit}, w_k\right) + \left((-1)^M \Delta^M u_{mitt}, w_k\right) = \left(f_i(\boldsymbol{u}_{mt}), w_k\right), 0 \leq t \leq T, (2.4) \\ u_{mi}(x,0) = \varphi_{mi}(x), \\ u_{mit}(x,0) = \psi_{mi}(x), \\ i = 1,2,\cdots,N; k = 1,2,\cdots m. \end{cases}$$

由于 $\boldsymbol{\varphi}(x) = \left(\varphi_1(x), \varphi_2(x), \cdots, \varphi_N(x)\right)^{\mathrm{T}} \in H^{2M}(\Omega) \cap H_0^M(\Omega)$，

$\boldsymbol{\psi}(x) = \left(\psi_1(x), \psi_2(x), \cdots, \psi_N(x)\right)^{\mathrm{T}} \in H^{2M}(\Omega) \bigcap H_0^M(\Omega)$，$\left\{w_j(x) \big| j = 1, 2, \cdots\right\}$ 构成

$H^{2M}(\Omega) \bigcap H_0^M(\Omega)$ 的一组正交基，可选取适当的 a_{mij}，b_{mij} $(i = 1, 2, \cdots, N; j = 1, 2, 3, \cdots)$ 使得当 $m \to +\infty$ 时，在空间 $H^{2M}(\Omega) \bigcap H_0^M(\Omega)$ 中成立强收敛，$\varphi_{mi}(x) \to \varphi_i(x)$，$\psi_{mi}(x) \to \psi_i(x)$，$(i = 1, 2, \cdots, N)$。因假设 $\boldsymbol{f} \in C^1$，由非线性常微分方程组理论中 Peano 存在性定理知问题 $(*)$ 存在局部解 $u_{mi}(x, t)(i = 1, 2, 3, \cdots, N)$，为得到问题 $(*)$ 在 $[0, T]$ 上的整体解，需证明问题 $(*)$ 中方程的右端 $\left(f_i(\boldsymbol{u}_{mt}), w_k\right)$ 的绝对值能用一与 $\boldsymbol{u}_m(x, t)$ 无关的正常数控制住，为此作出近似解 $u_{mi}(x, t)(i = 1, 2, 3, \cdots, N)$ 的先验估计如下：

引理 3.1：设 $\boldsymbol{\varphi}(x) \in H_0^M(\Omega), \boldsymbol{\psi}(x) \in H_0^M(\Omega)$，选取 a_{mij}, b_{mij}，使当 $m \to +\infty$ 时，$\boldsymbol{u}_m(x, 0) \to \boldsymbol{\varphi}(x)$ 及 $\boldsymbol{u}_{mt}(x, 0) \to \boldsymbol{\psi}(x)$ 在 $H_0^M(\Omega)$ 中强收敛，若条件(1.5)，(1.6)式成立，则有估计：

$$\left|\boldsymbol{u}_m\right|_{L^2(\Omega)}^2 + \left|\boldsymbol{u}_{mt}\right|_{L^2(\Omega)}^2 + \left|\nabla^M \boldsymbol{u}_m\right|_{L^2(\Omega)}^2 + \left|\nabla^M \boldsymbol{u}_{mt}\right|_{L^2(\Omega)}^2 + \left\|\nabla^M \boldsymbol{u}_{mt}\right\|_{L^2(\Omega)}^2 \leq C_1 \quad \left(C_1 \text{为与} \boldsymbol{u}_m \text{无关的正常数}\right).$$

引理 3.2：设 $\boldsymbol{\varphi}(x) \in H^{2M}(\Omega) \bigcap H_0^M(\Omega), \boldsymbol{\psi}(x) \in H^{2M}(\Omega) \bigcap H_0^M(\Omega)$，选取 a_{mij}, b_{mij}，使当 $m \to +\infty$ 时，$\boldsymbol{u}_m(x, 0) \to \boldsymbol{\varphi}(x)$ 及 $\boldsymbol{u}_{mt}(x, 0) \to \boldsymbol{\psi}(x)$ 在 $H^{2M}(\Omega) \bigcap H_0^M(\Omega)$ 中强收敛，若条件(1.5)，(1.6)式成立，则有估计：

$$\left|\boldsymbol{u}_{mtt}\right|_{L^2(\Omega)}^2 + \left|\nabla^M \boldsymbol{u}_{mtt}\right|_{L^2(\Omega)}^2 + \left|\nabla^M \boldsymbol{u}_{mt}\right|_{L^2(\Omega)}^2 + \left\|\nabla^M \boldsymbol{u}_{mt}\right\|_{L^2(\Omega)}^2 \leq C_2 \quad \left(C_2 \text{为与} \boldsymbol{u}_m \text{无关的正常数}\right).$$

引理 3.3：设 $\boldsymbol{\varphi}(x) \in H^{2M}(\Omega) \bigcap H_0^M(\Omega), \boldsymbol{\psi}(x) \in H^{2M}(\Omega) \bigcap H_0^M(\Omega)$，选取 a_{mij}, b_{mij} 使当 $m \to +\infty$ 时，$\boldsymbol{u}_m(x, 0) \to \boldsymbol{\varphi}(x)$ 及 $\boldsymbol{u}_{mt}(x, 0) \to \boldsymbol{\psi}(x)$ 在 $H^{2M}(\Omega) \bigcap H_0^M(\Omega)$ 中强收敛，若条件(1.5)，(1.6)式成立，则有估计：

$$\left|\nabla^M \boldsymbol{u}_{mt}\right|_{L^2(\Omega)}^2 + \left|\Delta^M \boldsymbol{u}_m\right|_{L^2(\Omega)}^2 + \left|\Delta^M \boldsymbol{u}_{mt}\right|_{L^2(\Omega)}^2 + \left\|\Delta^M \boldsymbol{u}_{mt}\right\|_{L^2(\Omega)}^2 \leq C_3 \quad \left(C_3 \text{为与} \boldsymbol{u}_m \text{无关的正常数}\right).$$

引理 3.4：在引理 3.3 条件下成立不等式

$$\left|\Delta^M \boldsymbol{u}_{mtt}\right|_{L^2(\Omega)}^2 \leq C_4 \quad \left(C_4 \text{为与} \boldsymbol{u}_m \text{无关的正常数}\right).$$

引理 3.1 的证明：

方程(2.4)两边同乘 $a'_{mik}(t)$

$$\left(u_{mitt}, a'_{mik}(t) w_k\right) + \left((-1)^M \Delta^M u_{mi}, a'_{mik}(t) w_k\right) + \left((-1)^M \Delta^M u_{mit}, a'_{mik}(t) w_k\right) + \left((-1)^M \Delta^M u_{mitt}, a'_{mik}(t) w_k\right)$$
$$= \left(f_i(\boldsymbol{u}_{mt}), a'_{mik}(t) w_k\right),$$

关于 k 从 1 到 m 作和得

$$\left(u_{mitt}, u_{mit}\right) + \left((-1)^M \Delta^M u_{mi}, u_{mit}\right) + \left((-1)^M \Delta^M u_{mit}, u_{mit}\right) + \left((-1)^M \Delta^M u_{mitt}, u_{mit}\right) = \left(f_i(\boldsymbol{u}_{mt}), u_{mit}\right),$$

关于 i 从 1 到 N 作和得

$$\left(\boldsymbol{u}_{mtt}, \boldsymbol{u}_{mt}\right) + \left((-1)^M \Delta^M \boldsymbol{u}_m, \boldsymbol{u}_{mt}\right) + \left((-1)^M \Delta^M \boldsymbol{u}_{mt}, \boldsymbol{u}_{mt}\right) + \left((-1)^M \Delta^M \boldsymbol{u}_{mtt}, \boldsymbol{u}_{mt}\right) = \left(\boldsymbol{f}(\boldsymbol{u}_{mt}), \boldsymbol{u}_{mt}\right),$$

而由条件(1.5)知

$$\left(\boldsymbol{f}(\boldsymbol{u}_{mt}), \boldsymbol{u}_{mt}\right) = \left(\boldsymbol{f}(\boldsymbol{u}_{mt}) - \boldsymbol{f}(\boldsymbol{0}), \boldsymbol{u}_{mt}\right) = \left(\frac{\partial \boldsymbol{f}(\boldsymbol{u}_{mt})}{\partial \boldsymbol{u}_{mt}}\Big|_{\theta \boldsymbol{u}_{mt}} \boldsymbol{u}_{mt}, \boldsymbol{u}_{mt}\right) \leq k_0\left(\boldsymbol{u}_{mt}, \boldsymbol{u}_{mt}\right) \quad (0 < \theta < 1),$$

首先利用格林公式，然后两端从 0 到 t 积分 $(0 \leq t \leq T)$，得

$$\frac{1}{2}\left(\boldsymbol{u}_{mt}, \boldsymbol{u}_{mt}\right) - \frac{1}{2}\left(\boldsymbol{\psi}_{m}, \boldsymbol{\psi}_{m}\right) + \frac{1}{2}\left(\nabla^{M} \boldsymbol{u}_{m}, \nabla^{M} \boldsymbol{u}_{m}\right) - \frac{1}{2}\left(\nabla^{M} \boldsymbol{\varphi}_{m}, \nabla^{M} \boldsymbol{\varphi}_{m}\right)$$

$$+ \left[\nabla^{M} \boldsymbol{u}_{mt}, \nabla^{M} \boldsymbol{u}_{mt}\right] + \frac{1}{2}\left(\nabla^{M} \boldsymbol{u}_{mt}, \nabla^{M} \boldsymbol{u}_{mt}\right) - \frac{1}{2}\left(\nabla^{M} \boldsymbol{\psi}_{m}, \nabla^{M} \boldsymbol{\psi}_{m}\right) \leq k_{0}\left[\boldsymbol{u}_{mt}, \boldsymbol{u}_{mt}\right],$$

这里，$\boldsymbol{\varphi}_{m}(x) = \left(\varphi_{m1}(x), \varphi_{m2}(x), \cdots, \varphi_{mN}(x)\right)^{\mathrm{T}}$，$\boldsymbol{\psi}_{m}(x) = \left(\psi_{m1}(x), \psi_{m2}(x), \cdots, \psi_{mN}(x)\right)^{\mathrm{T}}$，从而有不等式

$$\frac{1}{2}\left(\boldsymbol{u}_{mt}, \boldsymbol{u}_{mt}\right) + \frac{1}{2}\left(\nabla^{M} \boldsymbol{u}_{m}, \nabla^{M} \boldsymbol{u}_{m}\right) + \left[\nabla^{M} \boldsymbol{u}_{mt}, \nabla^{M} \boldsymbol{u}_{mt}\right] + \frac{1}{2}\left(\nabla^{M} \boldsymbol{u}_{mt}, \nabla^{M} \boldsymbol{u}_{mt}\right)$$

$$\leq k_{0}\left[\boldsymbol{u}_{mt}, \boldsymbol{u}_{mt}\right] + \frac{1}{2}\left(\boldsymbol{\psi}_{m}, \boldsymbol{\psi}_{m}\right) + \frac{1}{2}\left(\nabla^{M} \boldsymbol{\varphi}_{m}, \nabla^{M} \boldsymbol{\varphi}_{m}\right) + \frac{1}{2}\left(\nabla^{M} \boldsymbol{\psi}_{m}, \nabla^{M} \boldsymbol{\psi}_{m}\right),$$

两边加上 $\left[\boldsymbol{u}_{m}, \boldsymbol{u}_{mt}\right]$，

左边将它估计为 $\frac{1}{2}\left(\boldsymbol{u}_{m}, \boldsymbol{u}_{m}\right) - \frac{1}{2}\left(\boldsymbol{\varphi}_{m}, \boldsymbol{\varphi}_{m}\right)$，

右边将它化为 $\left[\boldsymbol{u}_{m}, \boldsymbol{u}_{mt}\right] \leq \left[\boldsymbol{u}_{m}, \boldsymbol{u}_{m}\right] + \left[\boldsymbol{u}_{mt}, \boldsymbol{u}_{mt}\right]$，

从而有

$$\frac{1}{2}\left(\boldsymbol{u}_{m}, \boldsymbol{u}_{m}\right) + \frac{1}{2}\left(\boldsymbol{u}_{mt}, \boldsymbol{u}_{mt}\right) + \frac{1}{2}\left(\nabla^{M} \boldsymbol{u}_{m}, \nabla^{M} \boldsymbol{u}_{m}\right) + \left[\nabla^{M} \boldsymbol{u}_{mt}, \nabla^{M} \boldsymbol{u}_{mt}\right] + \frac{1}{2}\left(\nabla^{M} \boldsymbol{u}_{mt}, \nabla^{M} \boldsymbol{u}_{mt}\right)$$

$$\leq \frac{1}{2}\left(\boldsymbol{\varphi}_{m}, \boldsymbol{\varphi}_{m}\right) + \frac{1}{2}\left(\boldsymbol{\psi}_{m}, \boldsymbol{\psi}_{m}\right) + \frac{1}{2}\left(\nabla^{M} \boldsymbol{\varphi}_{m}, \nabla^{M} \boldsymbol{\varphi}_{m}\right) + \frac{1}{2}\left(\nabla^{M} \boldsymbol{\psi}_{m}, \nabla^{M} \boldsymbol{\psi}_{m}\right) + (k_{0} + 1)\left[\boldsymbol{u}_{mt}, \boldsymbol{u}_{mt}\right] + \left[\boldsymbol{u}_{m}, \boldsymbol{u}_{m}\right]$$

由于当 $m \to +\infty$ 时，$\boldsymbol{\varphi}_{m} \to \boldsymbol{\varphi}$ 及 $\boldsymbol{\psi}_{m} \to \boldsymbol{\psi}$ 在 $H_{0}^{M}(\Omega)$ 中强收敛，所以当 $m \to +\infty$ 时，$\left(\boldsymbol{\varphi}_{m}, \boldsymbol{\varphi}_{m}\right) \to \left(\boldsymbol{\varphi}, \boldsymbol{\varphi}\right)$，$\left(\boldsymbol{\psi}_{m}, \boldsymbol{\psi}_{m}\right) \to \left(\boldsymbol{\psi}, \boldsymbol{\psi}\right)$，$\left(\nabla^{M} \boldsymbol{\varphi}_{m}, \nabla^{M} \boldsymbol{\varphi}_{m}\right) \to \left(\nabla^{M} \boldsymbol{\varphi}, \nabla^{M} \boldsymbol{\varphi}\right)$ 及 $\left(\nabla^{M} \boldsymbol{\psi}_{m}, \nabla^{M} \boldsymbol{\psi}_{m}\right) \to \left(\nabla^{M} \boldsymbol{\psi}, \nabla^{M} \boldsymbol{\psi}\right)$。以上不等式右边前四项能用一与 m 无关的正常数来控制。由 Gronwall 不等式得

$$\left|\boldsymbol{u}_{m}\right|_{L^{2}(\Omega)}^{2} + \left|\boldsymbol{u}_{mt}\right|_{L^{2}(\Omega)}^{2} + \left|\nabla^{M} \boldsymbol{u}_{m}\right|_{L^{2}(\Omega)}^{2} + \left|\nabla^{M} \boldsymbol{u}_{mt}\right|_{L^{2}(\Omega)}^{2} + \left\|\nabla^{M} \boldsymbol{u}_{mt}\right\|_{L^{2}(\Omega)}^{2} \leq C_{1}$$

引理 3.1 证毕！

引理 3.2 的证明：

对方程组(2.4)两边关于 t 求导得

$$\left(u_{mittt}, w_{k}\right) + \left((-1)^{M} \Delta^{M} u_{mit}, w_{k}\right) + \left((-1)^{M} \Delta^{M} u_{mitt}, w_{k}\right) + \left((-1)^{M} \Delta^{M} u_{mittt}, w_{k}\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t} f_{i}\left(\boldsymbol{u}_{mt}\right), w_{k}\right)$$

两边同乘以 $a''_{mik}(t)$ 得

$$\left(u_{mittt}, a''_{mik}(t) w_{k}\right) + \left((-1)^{M} \Delta^{M} u_{mit}, a''_{mik}(t) w_{k}\right) + \left((-1)^{M} \Delta^{M} u_{mitt}, a''_{mik}(t) w_{k}\right)$$

$$+ \left((-1)^{M} \Delta^{M} u_{mittt}, a''_{mik}(t) w_{k}\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t} f_{i}\left(\boldsymbol{u}_{mt}\right), a''_{mik}(t) w_{k}\right)$$

关于 k 从 1 到 m 作和得

$$\left(u_{mittt}, u_{mitt}\right) + \left((-1)^{M} \Delta^{M} u_{mit}, u_{mitt}\right) + \left((-1)^{M} \Delta^{M} u_{mitt}, u_{mitt}\right) + \left((-1)^{M} \Delta^{M} u_{mittt}, u_{mitt}\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t} f_{i}\left(\boldsymbol{u}_{mt}\right), u_{mitt}\right)$$

关于 i 从 1 到 N 作和得

$$\left(\boldsymbol{u}_{mttt}, \boldsymbol{u}_{mtt}\right) + \left((-1)^{M} \Delta^{M} \boldsymbol{u}_{mt}, \boldsymbol{u}_{mtt}\right) + \left((-1)^{M} \Delta^{M} \boldsymbol{u}_{mtt}, \boldsymbol{u}_{mtt}\right) + \left((-1)^{M} \Delta^{M} \boldsymbol{u}_{mttt}, \boldsymbol{u}_{mtt}\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{f}\left(\boldsymbol{u}_{mt}\right), \boldsymbol{u}_{mtt}\right)$$

而由条件(1.5)知 $\left(\dfrac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{f}\left(\boldsymbol{u}_{mt}\right), \boldsymbol{u}_{mtt} \right) = \left(\dfrac{\partial \boldsymbol{f}\left(\boldsymbol{u}_{mt}\right)}{\partial \boldsymbol{u}_{mt}} \cdot \boldsymbol{u}_{mtt}, \boldsymbol{u}_{mtt} \right) \le k_0 \left(\boldsymbol{u}_{mtt}, \boldsymbol{u}_{mtt} \right),$

从而有

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\nabla^M \boldsymbol{u}_{mt}, \nabla^M \boldsymbol{u}_{mt}\right) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\nabla^M \boldsymbol{u}_{mtt}, \nabla^M \boldsymbol{u}_{mtt}\right) + \left(\nabla^M \boldsymbol{u}_{mtt}, \nabla^M \boldsymbol{u}_{mtt}\right) \le k_0 \left(\boldsymbol{u}_{mtt}, \boldsymbol{u}_{mtt}\right)$$

两边从 0 到 $t\left(0 \le t \le T\right)$ 积分得

$$\frac{1}{2}\left(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right) - \frac{1}{2}\left(\boldsymbol{u}_{mtt}\left(x,0\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right) + \frac{1}{2}\left(\nabla^M \boldsymbol{u}_{mt}, \nabla^M \boldsymbol{u}_{mt}\right) - \frac{1}{2}\left(\nabla^M \boldsymbol{\psi}_m, \nabla^M \boldsymbol{\psi}_m\right)$$

$$+ \frac{1}{2}\left(\nabla^M \boldsymbol{u}_{mtt}, \nabla^M \boldsymbol{u}_{mtt}\right) - \frac{1}{2}\left(\nabla^M \boldsymbol{u}_{mtt}\left(x,0\right), \nabla^M \boldsymbol{u}_{mtt}\left(x,0\right)\right) + \left[\nabla^M \boldsymbol{u}_{mtt}, \nabla^M \boldsymbol{u}_{mtt}\right] \le k_0 \left[\boldsymbol{u}_{mtt}, \boldsymbol{u}_{mtt}\right]$$

下证 $\left(\boldsymbol{u}_{mtt}\left(x,0\right), \boldsymbol{u}_{mtt}\left(x,0\right)\right) + \left(\nabla^M \boldsymbol{u}_{mtt}\left(x,0\right), \nabla^M \boldsymbol{u}_{mtt}\left(x,0\right)\right)$ 有界(能用一与 m 无关正常数控制住)。

方程组(2.4)两边同乘以 $a''_{mik}\left(t\right)$ 得

$$\left(u_{mitt}, a''_{mik}\left(t\right)w_k\right) + \left((-1)^M \Delta^M u_{mi}, a''_{mik}\left(t\right)w_k\right) + \left((-1)^M \Delta^M u_{mit}, a''_{mik}\left(t\right)w_k\right) + \left((-1)^M \Delta^M u_{mitt}, a''_{mik}\left(t\right)w_k\right)$$

$$= \left(f_i\left(\boldsymbol{u}_{mt}\right), a''_{mik}\left(t\right)w_k\right),$$

关于 k 从 1 到 m 作和得

$$\left(u_{mitt}, u_{mitt}\right) + \left((-1)^M \Delta^M u_{mi}, u_{mitt}\right) + \left((-1)^M \Delta^M u_{mit}, u_{mitt}\right) + \left((-1)^M \Delta^M u_{mitt}, u_{mitt}\right) = \left(f_i\left(\boldsymbol{u}_{mt}\right), u_{mitt}\right)$$

关于 i 从 1 到 N 作和得

$$\left(\boldsymbol{u}_{mtt}, \boldsymbol{u}_{mtt}\right) + \left((-1)^M \Delta^M \boldsymbol{u}_m, \boldsymbol{u}_{mtt}\right) + \left((-1)^M \Delta^M \boldsymbol{u}_{mt}, \boldsymbol{u}_{mtt}\right) + \left((-1)^M \Delta^M \boldsymbol{u}_{mtt}, \boldsymbol{u}_{mtt}\right) = \left(\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right), \boldsymbol{u}_{mtt}\right)$$

令 $t = 0$ 得

$$\left(\boldsymbol{u}_{mtt}\left(x,0\right), \boldsymbol{u}_{mtt}\left(x,0\right)\right) + \left((-1)^M \Delta^M \boldsymbol{u}_m\left(x,0\right), \boldsymbol{u}_{mtt}\left(x,0\right)\right) + \left((-1)^M \Delta^M \boldsymbol{u}_{mt}\left(x,0\right), \boldsymbol{u}_{mtt}\left(x,0\right)\right)$$

$$+ \left((-1)^M \Delta^M \boldsymbol{u}_{mtt}\left(x,0\right), \boldsymbol{u}_{mtt}\left(x,0\right)\right) = \left(\boldsymbol{f}\left(\boldsymbol{\psi}_m\right), \boldsymbol{u}_{mtt}\left(x,0\right)\right) \tag{**}$$

$$\left(\boldsymbol{f}\left(\boldsymbol{\psi}_m\right), \boldsymbol{u}_{mtt}\left(x,0\right)\right) \le \frac{1}{2\varepsilon_1}\left(\boldsymbol{f}\left(\boldsymbol{\psi}_m\right), \boldsymbol{f}\left(\boldsymbol{\psi}_m\right)\right) + \frac{\varepsilon_1}{2}\left(\boldsymbol{u}_{mtt}\left(x,0\right), \boldsymbol{u}_{mtt}\left(x,0\right)\right)$$

(其中 $\boldsymbol{\psi}_m = \left(\psi_{m1}\left(x\right), \cdots, \psi_{mN}\left(x\right)\right)^{\mathrm{T}}$),

下面证明 $\left(\boldsymbol{f}\left(\boldsymbol{\psi}_m\right), \boldsymbol{f}\left(\boldsymbol{\psi}_m\right)\right)$ 有界。

由于

$$\left(\boldsymbol{f}\left(\boldsymbol{\psi}_m\right), \boldsymbol{f}\left(\boldsymbol{\psi}_m\right)\right) \le \left(a_1 + b_1 \left|\boldsymbol{\psi}_m\right|^{\frac{p}{2}}, a_1 + b_1 \left|\boldsymbol{\psi}_m\right|^{\frac{p}{2}}\right) = a_1^2 \left|\Omega\right| + 2a_1 b_1 \int_\Omega \left|\boldsymbol{\psi}_m\right|^{\frac{p}{2}} \mathrm{d}x + b_1^2 \int_\Omega \left|\boldsymbol{\psi}_m\right|^p \mathrm{d}x.$$

当 $m \to +\infty$ 时, $\boldsymbol{\psi}_m \to \boldsymbol{\psi}$ 在 $H^{2M}\left(\Omega\right) \bigcap H_0^M\left(\Omega\right)$ 中强收敛,由 Sobolev 嵌入定理[5]可知 $\left|\boldsymbol{\psi}_m\right|_{L^p\left(\Omega\right)} \le \text{const}$,

其中当 $2M < n$ 时, $2 \le p < \dfrac{2n}{n-2M}$; 当 $2M \ge n$ 时, $2 \le p < +\infty$ 。

所以 $\int_\Omega \left|\boldsymbol{\psi}_m\right|^p \mathrm{d}x = \left|\boldsymbol{\psi}_m\right|_{L^p\left(\Omega\right)}^p \le \text{const}$,

从而有

$$\int_\Omega \left|\boldsymbol{\psi}_m\right|^{\frac{p}{2}} \mathrm{d}x \le \left(\int_\Omega \left|\boldsymbol{\psi}_m\right|^{\frac{p}{2}\times 2} \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega 1^2 \mathrm{d}x\right)^{\frac{1}{2}} = \left|\Omega\right|^{\frac{1}{2}} \left|\boldsymbol{\psi}_m\right|_{L^p\left(\Omega\right)}^{\frac{p}{2}} \le \text{const},$$

因此，$\left(f\left(\psi_m \right), f\left(\psi_m \right) \right)$ 有界。

将 $(**)$ 中的项 $\left((-1)^M \Delta^M u_m\left(x,0 \right), u_{mtt}\left(x,0 \right) \right)$ 移至等式右边得

$$-\left((-1)^M \Delta^M u_m\left(x,0 \right), u_{mtt}\left(x,0 \right) \right) = \left((-1)^{M+1} \Delta^M u_m\left(x,0 \right), u_{mtt}\left(x,0 \right) \right),$$

应用带 ε 的柯西不等式得

$$\left((-1)^{M+1} \Delta^M u_m\left(x,0 \right), u_{mtt}\left(x,0 \right) \right) \leq \frac{1}{2\varepsilon_2}\left((-1)^{M+1} \Delta^M u_m\left(x,0 \right), (-1)^{M+1} \Delta^M u_m\left(x,0 \right) \right) + \frac{\varepsilon_2}{2}\left(u_{mtt}\left(x,0 \right), u_{mtt}\left(x,0 \right) \right)$$

故 $(**)$ 可变为

$$\left(u_{mtt}\left(x,0 \right), u_{mtt}\left(x,0 \right) \right) + \left((-1)^M \Delta^M u_{mt}\left(x,0 \right), u_{mtt}\left(x,0 \right) \right) + \left((-1)^M \Delta^M u_{mtt}\left(x,0 \right), u_{mtt}\left(x,0 \right) \right)$$

$$\leq \left(f\left(\psi_m \right), u_{mtt}\left(x,0 \right) \right) + \frac{1}{2\varepsilon_2}\left((-1)^{M+1} \Delta^M u_m\left(x,0 \right), (-1)^{M+1} \Delta^M u_m\left(x,0 \right) \right) + \frac{\varepsilon_2}{2}\left(u_{mtt}\left(x,0 \right), u_{mtt}\left(x,0 \right) \right),$$

再将项 $\left((-1)^M \Delta^M u_{mt}\left(x,0 \right), u_{mtt}\left(x,0 \right) \right)$ 移至等式右边得

$$-\left((-1)^M \Delta^M u_{mt}\left(x,0 \right), u_{mtt}\left(x,0 \right) \right) = \left((-1)^{M+1} \Delta^M u_{mt}\left(x,0 \right), u_{mtt}\left(x,0 \right) \right),$$

应用带 ε 的柯西不等式得

$$\left((-1)^{M+1} \Delta^M u_{mt}\left(x,0 \right), u_{mtt}\left(x,0 \right) \right) \leq \frac{1}{2\varepsilon_3}\left((-1)^{M+1} \Delta^M u_{mt}\left(x,0 \right), (-1)^{M+1} \Delta^M u_{mt}\left(x,0 \right) \right) + \frac{\varepsilon_3}{2}\left(u_{mtt}\left(x,0 \right), u_{mtt}\left(x,0 \right) \right)$$

故 $(**)$ 可变为

$$\left(u_{mtt}\left(x,0 \right), u_{mtt}\left(x,0 \right) \right) + \left((-1)^M \Delta^M u_{mtt}\left(x,0 \right), u_{mtt}\left(x,0 \right) \right)$$

$$\leq \frac{1}{2\varepsilon_1}\left(f\left(\psi_m \right), f\left(\psi_m \right) \right) + \frac{\varepsilon_1}{2}\left(u_{mtt}\left(x,0 \right), u_{mtt}\left(x,0 \right) \right)$$

$$+ \frac{1}{2\varepsilon_2}\left((-1)^{M+1} \Delta^M u_m\left(x,0 \right), (-1)^{M+1} \Delta^M u_m\left(x,0 \right) \right) + \frac{\varepsilon_2}{2}\left(u_{mtt}\left(x,0 \right), u_{mtt}\left(x,0 \right) \right)$$

$$+ \frac{1}{2\varepsilon_3}\left((-1)^{M+1} \Delta^M u_{mt}\left(x,0 \right), (-1)^{M+1} \Delta^M u_{mt}\left(x,0 \right) \right) + \frac{\varepsilon_3}{2}\left(u_{mtt}\left(x,0 \right), u_{mtt}\left(x,0 \right) \right),$$

因为 $\Delta^M u_m\left(x,0 \right) = \Delta^M \varphi_m$，$\Delta^M u_{mt}\left(x,0 \right) = \Delta^M \psi_m$ 所以在上式中

$$\left((-1)^{M+1} \Delta^M u_m\left(x,0 \right), (-1)^{M+1} \Delta^M u_m\left(x,0 \right) \right) = \left((-1)^{M+1} \Delta^M \varphi_m, (-1)^{M+1} \Delta^M \varphi_m \right),$$

$$\left((-1)^{M+1} \Delta^M u_{mt}\left(x,0 \right), (-1)^{M+1} \Delta^M u_{mt}\left(x,0 \right) \right) = \left((-1)^{M+1} \Delta^M \psi_m, (-1)^{M+1} \Delta^M \psi_m \right),$$

从而有

$$\left(u_{mtt}\left(x,0 \right), u_{mtt}\left(x,0 \right) \right) + \left((-1)^M \Delta^M u_{mtt}\left(x,0 \right), u_{mtt}\left(x,0 \right) \right)$$

$$\leq \frac{1}{2\varepsilon_1}\left(f\left(\psi_m \right), f\left(\psi_m \right) \right) + \frac{\varepsilon_1}{2}\left(u_{mtt}\left(x,0 \right), u_{mtt}\left(x,0 \right) \right)$$

$$+ \frac{1}{2\varepsilon_2}\left((-1)^{M+1} \Delta^M \varphi_m\left(x,0 \right), (-1)^{M+1} \Delta^M \varphi_m\left(x,0 \right) \right) + \frac{\varepsilon_2}{2}\left(u_{mtt}\left(x,0 \right), u_{mtt}\left(x,0 \right) \right)$$

$$+ \frac{1}{2\varepsilon_3}\left((-1)^{M+1} \Delta^M \psi_m\left(x,0 \right), (-1)^{M+1} \Delta^M \psi_m\left(x,0 \right) \right) + \frac{\varepsilon_3}{2}\left(u_{mtt}\left(x,0 \right), u_{mtt}\left(x,0 \right) \right),$$

移项整理得

$$\left(1-\frac{\varepsilon_1}{2}-\frac{\varepsilon_2}{2}-\frac{\varepsilon_3}{2}\right)\left(\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)+\left(\nabla^M \boldsymbol{u}_{mtt}(x,0),\nabla^M \boldsymbol{u}_{mtt}(x,0)\right)$$

$$\leq \frac{1}{2\varepsilon_2}\left(\Delta^M \boldsymbol{\varphi}_m,\Delta^M \boldsymbol{\varphi}_m\right)+\frac{1}{2\varepsilon_3}\left(\Delta^M \boldsymbol{\psi}_m,\Delta^M \boldsymbol{\psi}_m\right)+\frac{1}{2\varepsilon_1}\left(\boldsymbol{f}(\boldsymbol{\psi}_m),\boldsymbol{f}(\boldsymbol{\psi}_m)\right),$$

取 $\varepsilon_1,\varepsilon_2,\varepsilon_3$ 充分小，使得 $\left(1-\dfrac{\varepsilon_1}{2}-\dfrac{\varepsilon_2}{2}-\dfrac{\varepsilon_3}{2}\right)\geq\dfrac{1}{2}$。

由于当 $m\to+\infty$ 时，$\boldsymbol{\psi}_m\to\boldsymbol{\psi}$ 在 $H^{2M}(\Omega)\bigcap H_0^M(\Omega)$ 中强收敛，故当 $m\to+\infty$ 时，$\left(\Delta^M\boldsymbol{\psi}_m,\Delta^M\boldsymbol{\psi}_m\right)\to\left(\Delta^M\boldsymbol{\psi},\Delta^M\boldsymbol{\psi}\right)$，故 $\left(\Delta^M\boldsymbol{\psi}_m,\Delta^M\boldsymbol{\psi}_m\right)$ 能用一与 m 无关的正常数控制住，同样道理 $\left(\Delta^M\boldsymbol{\varphi}_m,\Delta^M\boldsymbol{\varphi}_m\right)$ 也能用一与 m 无关的正常数控制住，所以 $\left(\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)+\left(\nabla^M\boldsymbol{u}_{mtt}(x,0),\nabla^M\boldsymbol{u}_{mtt}(x,0)\right)$ 有界。

又由已证不等式

$$\frac{1}{2}\left(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right)+\frac{1}{2}\left(\nabla^M\boldsymbol{u}_{mt},\nabla^M\boldsymbol{u}_{mt}\right)+\frac{1}{2}\left(\nabla^M\boldsymbol{u}_{mtt},\nabla^M\boldsymbol{u}_{mtt}\right)+\left[\nabla^M\boldsymbol{u}_{mtt},\nabla^M\boldsymbol{u}_{mtt}\right]$$

$$\leq\frac{1}{2}\left(\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)+\frac{1}{2}\left(\nabla^M\boldsymbol{\psi}_m,\nabla^M\boldsymbol{\psi}_m\right)+\frac{1}{2}\left(\nabla^M\boldsymbol{u}_{mtt}(x,0),\nabla^M\boldsymbol{u}_{mtt}(x,0)\right)+k_0\left[\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right],$$

因为 $\left(\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)+\left(\nabla^M\boldsymbol{u}_{mtt}(x,0),\nabla^M\boldsymbol{u}_{mtt}(x,0)\right)$ 有界，$\dfrac{1}{2}\left(\nabla^M\boldsymbol{\psi}_m,\nabla^M\boldsymbol{\psi}_m\right)$ 又能用一与 m 无关的正常数控制住，所以

$$\frac{1}{2}\left(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right)+\frac{1}{2}\left(\nabla^M\boldsymbol{u}_{mt},\nabla^M\boldsymbol{u}_{mt}\right)+\frac{1}{2}\left(\nabla^M\boldsymbol{u}_{mtt},\nabla^M\boldsymbol{u}_{mtt}\right)+\left[\nabla^M\boldsymbol{u}_{mtt},\nabla^M\boldsymbol{u}_{mtt}\right]\leq k_0\left[\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right]+C,$$

其中 C 为与 m 无关的正常数，再根据 Gronwall 不等式得

$$\left|\boldsymbol{u}_{mtt}\right|^2_{L^2(\Omega)}+\left|\nabla^M\boldsymbol{u}_{mtt}\right|^2_{L^2(\Omega)}+\left|\nabla^M\boldsymbol{u}_{mt}\right|^2_{L^2(\Omega)}+\left\|\nabla^M\boldsymbol{u}_{mtt}\right\|^2_{L^2(\Omega)}\leq C_2.$$

引理 3.2 证毕！

引理 3.3 的证明：

方程(2.4)两边同乘 $\lambda_k a'_{mik}(t)$ 得

$$\left(u_{mitt},\lambda_k a'_{mik}(t)w_k\right)+\left((-1)^M\Delta^M u_{mi},\lambda_k a'_{mik}(t)w_k\right)+\left((-1)^M\Delta^M u_{mit},\lambda_k a'_{mik}(t)w_k\right)$$

$$+\left((-1)^M\Delta^M u_{mitt},\lambda_k a'_{mik}(t)w_k\right)=\left(f_i(\boldsymbol{u}_{mt}),\lambda_k a'_{mik}(t)w_k\right),$$

关于 k 从 1 到 m 作和得

$$\left(u_{mitt},(-1)^M\Delta^M u_{mit}\right)+\left((-1)^M\Delta^M u_{mi},(-1)^M\Delta^M u_{mit}\right)+\left((-1)^M\Delta^M u_{mit},(-1)^M\Delta^M u_{mit}\right)$$

$$+\left((-1)^M\Delta^M u_{mitt},(-1)^M\Delta^M u_{mit}\right)=\left(f_i(\boldsymbol{u}_{mt}),(-1)^M\Delta^M u_{mit}\right),$$

关于 i 从 1 到 N 作和得

$$\left(\boldsymbol{u}_{mtt},(-1)^M\Delta^M \boldsymbol{u}_{mt}\right)+\left((-1)^M\Delta^M \boldsymbol{u}_m,(-1)^M\Delta^M \boldsymbol{u}_{mt}\right)+\left((-1)^M\Delta^M \boldsymbol{u}_{mt},(-1)^M\Delta^M \boldsymbol{u}_{mt}\right)$$

$$+\left((-1)^M\Delta^M \boldsymbol{u}_{mtt},(-1)^M\Delta^M \boldsymbol{u}_{mt}\right)=\left(\boldsymbol{f}(\boldsymbol{u}_{mt}),(-1)^M\Delta^M \boldsymbol{u}_{mt}\right),$$

对于右边项经计算得

$$\left(f\left(\boldsymbol{u}_{mt}\right),(-1)^M \Delta^M \boldsymbol{u}_{mt}\right) \le \frac{1}{2\varepsilon_1}\left(f\left(\boldsymbol{u}_{mt}\right),f\left(\boldsymbol{u}_{mt}\right)\right)+\frac{\varepsilon_1}{2}\left((-1)^M \Delta^M \boldsymbol{u}_{mt},(-1)^M \Delta^M \boldsymbol{u}_{mt}\right),$$

取 ε_1 充分小，使得 $1-\dfrac{\varepsilon_1}{2} \ge \dfrac{1}{2}$ 故成立如下不等式

$$\left(\boldsymbol{u}_{mtt},(-1)^M \Delta^M \boldsymbol{u}_{mt}\right)+\left((-1)^M \Delta^M \boldsymbol{u}_m,(-1)^M \Delta^M \boldsymbol{u}_{mt}\right)+\frac{1}{2}\left((-1)^M \Delta^M \boldsymbol{u}_{mt},(-1)^M \Delta^M \boldsymbol{u}_{mt}\right)$$

$$+\left((-1)^M \Delta^M \boldsymbol{u}_{mtt},(-1)^M \Delta^M \boldsymbol{u}_{mt}\right) \le \frac{1}{2\varepsilon_1}\left(f\left(\boldsymbol{u}_{mt}\right),f\left(\boldsymbol{u}_{mt}\right)\right),$$

由格林公式得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\nabla^M \boldsymbol{u}_{mt},\nabla^M \boldsymbol{u}_{mt}\right)+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\Delta^M \boldsymbol{u}_m,\Delta^M \boldsymbol{u}_m\right)+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\Delta^M \boldsymbol{u}_{mt},\Delta^M \boldsymbol{u}_{mt}\right)$$

$$+\frac{1}{2}\left(\Delta^M \boldsymbol{u}_{mt},\Delta^M \boldsymbol{u}_{mt}\right) \le \frac{1}{2\varepsilon_1}\left(f\left(\boldsymbol{u}_{mt}\right),f\left(\boldsymbol{u}_{mt}\right)\right),$$

两边从 0 到 $t\left(0 \le t \le T\right)$ 积分并移项得

$$\frac{1}{2}\left(\nabla^M \boldsymbol{u}_{mt},\nabla^M \boldsymbol{u}_{mt}\right)+\frac{1}{2}\left(\Delta^M \boldsymbol{u}_m,\Delta^M \boldsymbol{u}_m\right)+\frac{1}{2}\left[\Delta^M \boldsymbol{u}_{mt},\Delta^M \boldsymbol{u}_{mt}\right]+\frac{1}{2}\left(\Delta^M \boldsymbol{u}_{mt},\Delta^M \boldsymbol{u}_{mt}\right)$$

$$\le \frac{1}{2}\left(\nabla^M \boldsymbol{\psi}_m,\nabla^M \boldsymbol{\psi}_m\right)+\frac{1}{2}\left(\Delta^M \boldsymbol{\varphi}_m,\Delta^M \boldsymbol{\varphi}_m\right)+\frac{1}{2}\left(\Delta^M \boldsymbol{\psi}_m,\Delta^M \boldsymbol{\psi}_m\right)+\frac{1}{2\varepsilon_1}\left[f\left(\boldsymbol{u}_{mt}\right),f\left(\boldsymbol{u}_{mt}\right)\right],$$

由引理 3.1 知 $\left|\boldsymbol{u}_{mt}\right|_{H^M(\Omega)}^2 \le \mathrm{const}\left(0 \le t \le T\right)$，

$$\left(f\left(\boldsymbol{u}_{mt}\right),f\left(\boldsymbol{u}_{mt}\right)\right) \le \left(a_1+b_1\left|\boldsymbol{u}_{mt}\right|^{\frac{p}{2}},a_1+b_1\left|\boldsymbol{u}_{mt}\right|^{\frac{p}{2}}\right)=a_1\left|\Omega\right|+2a_1 b_1 \int_\Omega\left|\boldsymbol{u}_{mt}\right|^{\frac{p}{2}}\mathrm{d}x+b_1^2 \int_\Omega\left|\boldsymbol{u}_{mt}\right|^p \mathrm{d}x,$$

由 Sobolev 嵌入定理[5]得

$$\int_\Omega\left|\boldsymbol{u}_{mt}\right|^p \mathrm{d}x=\left|\boldsymbol{u}_{mt}\right|_{L^p(\Omega)}^p \le \mathrm{const},\ \int_\Omega\left|\boldsymbol{u}_{mt}\right|^{\frac{p}{2}}\mathrm{d}x \le \left(\int_\Omega\left|\boldsymbol{u}_{mt}\right|^{\frac{p}{2}\times 2}\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega 1^2 \mathrm{d}x\right)^{\frac{1}{2}}=\left|\Omega\right|^{\frac{1}{2}}\left|\boldsymbol{u}_{mt}\right|_{L^p(\Omega)}^{\frac{p}{2}} \le \mathrm{const}$$

其中当 $2M < n$ 时，$2 \le p < \dfrac{2n}{n-2M}$；当 $2M \ge n$ 时，$2 \le p < +\infty$。

所以 $\left(f\left(\boldsymbol{u}_{mt}\right),f\left(\boldsymbol{u}_{mt}\right)\right) \le \mathrm{const}$。在下面的不等式中，

$$\frac{1}{2}\left(\nabla^M \boldsymbol{u}_{mt},\nabla^M \boldsymbol{u}_{mt}\right)+\frac{1}{2}\left(\Delta^M \boldsymbol{u}_m,\Delta^M \boldsymbol{u}_m\right)+\frac{1}{2}\left[\Delta^M \boldsymbol{u}_{mt},\Delta^M \boldsymbol{u}_{mt}\right]+\frac{1}{2}\left(\Delta^M \boldsymbol{u}_{mt},\Delta^M \boldsymbol{u}_{mt}\right)$$

$$\le \frac{1}{2}\left(\nabla^M \boldsymbol{\psi}_m,\nabla^M \boldsymbol{\psi}_m\right)+\frac{1}{2}\left(\Delta^M \boldsymbol{\varphi}_m,\Delta^M \boldsymbol{\varphi}_m\right)+\frac{1}{2}\left(\Delta^M \boldsymbol{\psi}_m,\Delta^M \boldsymbol{\psi}_m\right)+\frac{1}{2\varepsilon_1}\left[f\left(\boldsymbol{u}_{mt}\right),f\left(\boldsymbol{u}_{mt}\right)\right],$$

由于当 $m \to +\infty$ 时，$\boldsymbol{\varphi}_m \to \boldsymbol{\varphi}$，$\boldsymbol{\psi}_m \to \boldsymbol{\psi}$ 在 $H^{2M}(\Omega)\bigcap H_0^M(\Omega)$ 中强收敛，所以当 $m \to +\infty$ 时，

$$\frac{1}{2}\left(\nabla^M \boldsymbol{\psi}_m,\nabla^M \boldsymbol{\psi}_m\right)+\frac{1}{2}\left(\Delta^M \boldsymbol{\varphi}_m,\Delta^M \boldsymbol{\varphi}_m\right)+\frac{1}{2}\left(\Delta^M \boldsymbol{\psi}_m,\Delta^M \boldsymbol{\psi}_m\right)$$

收敛于 $\dfrac{1}{2}\left(\nabla^M \boldsymbol{\psi},\nabla^M \boldsymbol{\psi}\right)+\dfrac{1}{2}\left(\Delta^M \boldsymbol{\varphi},\Delta^M \boldsymbol{\varphi}\right)+\dfrac{1}{2}\left(\Delta^M \boldsymbol{\psi},\Delta^M \boldsymbol{\psi}\right)$，

$\dfrac{1}{2}\left(\nabla^M \boldsymbol{\psi}_m,\nabla^M \boldsymbol{\psi}_m\right)+\dfrac{1}{2}\left(\Delta^M \boldsymbol{\varphi}_m,\Delta^M \boldsymbol{\varphi}_m\right)+\dfrac{1}{2}\left(\Delta^M \boldsymbol{\psi}_m,\Delta^M \boldsymbol{\psi}_m\right)$ 能用一与 m 无关的正常数控制住，因此由以上不等式得

$$\frac{1}{2}\left(\nabla^M \boldsymbol{u}_{mt}, \nabla^M \boldsymbol{u}_{mt}\right) + \frac{1}{2}\left(\Delta^M \boldsymbol{u}_m, \Delta^M \boldsymbol{u}_m\right) + \frac{1}{2}\left[\Delta^M \boldsymbol{u}_{mt}, \Delta^M \boldsymbol{u}_{mt}\right] + \frac{1}{2}\left(\Delta^M \boldsymbol{u}_{mt}, \Delta^M \boldsymbol{u}_{mt}\right) \leq \text{const},$$

即

$$\left|\nabla^M \boldsymbol{u}_{mt}\right|_{L^2(\Omega)}^2 + \left|\Delta^M \boldsymbol{u}_m\right|_{L^2(\Omega)}^2 + \left|\Delta^M \boldsymbol{u}_{mt}\right|_{L^2(\Omega)}^2 + \left\|\Delta^M \boldsymbol{u}_{mt}\right\|_{L^2(\Omega)}^2 \leq C_3 \quad (C_3\text{为与}\boldsymbol{u}_m\text{无关的正常数}).$$

引理 3.3 证毕！

引理 3.4 的证明：

方程(2.4)两边同乘以 $\lambda_k a_{mik}''(t)$，得

$$\left(u_{mitt}, \lambda_k a_{mik}''(t) w_k\right) + \left((-1)^M \Delta^M u_{mi}, \lambda_k a_{mik}''(t) w_k\right) + \left((-1)^M \Delta^M u_{mit}, \lambda_k a_{mik}''(t) w_k\right)$$
$$+ \left((-1)^M \Delta^M u_{mitt}, \lambda_k a_{mik}''(t) w_k\right) = \left(f_i(\boldsymbol{u}_{mt}), \lambda_k a_{mik}''(t) w_k\right),$$

关于 k 从 1 到 m 作和得

$$\left(u_{mitt}, (-1)^M \Delta^M u_{mitt}\right) + \left((-1)^M \Delta^M u_{mi}, (-1)^M \Delta^M u_{mitt}\right) + \left((-1)^M \Delta^M u_{mit}, (-1)^M \Delta^M u_{mitt}\right)$$
$$+ \left((-1)^M \Delta^M u_{mitt}, (-1)^M \Delta^M u_{mitt}\right) = \left(f_i(\boldsymbol{u}_{mt}), (-1)^M \Delta^M u_{mitt}\right),$$

关于 i 从 1 到 N 作和得

$$\left(\boldsymbol{u}_{mtt}, (-1)^M \Delta^M \boldsymbol{u}_{mtt}\right) + \left((-1)^M \Delta^M \boldsymbol{u}_m, (-1)^M \Delta^M \boldsymbol{u}_{mtt}\right) + \left((-1)^M \Delta^M \boldsymbol{u}_{mt}, (-1)^M \Delta^M \boldsymbol{u}_{mtt}\right)$$
$$+ \left((-1)^M \Delta^M \boldsymbol{u}_{mtt}, (-1)^M \Delta^M \boldsymbol{u}_{mtt}\right) = \left(\boldsymbol{f}(\boldsymbol{u}_{mt}), (-1)^M \Delta^M \boldsymbol{u}_{mtt}\right),$$

移项得如下不等式

$$\left((-1)^M \Delta^M \boldsymbol{u}_{mtt}, (-1)^M \Delta^M \boldsymbol{u}_{mtt}\right) = -\left(\boldsymbol{u}_{mtt}, (-1)^M \Delta^M \boldsymbol{u}_{mtt}\right) - \left((-1)^M \Delta^M \boldsymbol{u}_m, (-1)^M \Delta^M \boldsymbol{u}_{mtt}\right)$$
$$-\left((-1)^M \Delta^M \boldsymbol{u}_{mt}, (-1)^M \Delta^M \boldsymbol{u}_{mtt}\right) + \left(\boldsymbol{f}(\boldsymbol{u}_{mt}), (-1)^M \Delta^M \boldsymbol{u}_{mtt}\right),$$

而

$$-\left(\boldsymbol{u}_{mtt}, (-1)^M \Delta^M \boldsymbol{u}_{mtt}\right) \leq \frac{\varepsilon_1}{2}\left((-1)^M \Delta^M \boldsymbol{u}_{mtt}, (-1)^M \Delta^M \boldsymbol{u}_{mtt}\right) + \frac{1}{2\varepsilon_1}\left(\boldsymbol{u}_{mtt}, \boldsymbol{u}_{mtt}\right)$$

$$= \frac{\varepsilon_1}{2}\left(\Delta^M \boldsymbol{u}_{mtt}, \Delta^M \boldsymbol{u}_{mtt}\right) + \frac{1}{2\varepsilon_1}\left(\boldsymbol{u}_{mtt}, \boldsymbol{u}_{mtt}\right),$$

$$-\left((-1)^M \Delta^M \boldsymbol{u}_m, (-1)^M \Delta^M \boldsymbol{u}_{mtt}\right)$$
$$\leq \frac{\varepsilon_2}{2}\left((-1)^M \Delta^M \boldsymbol{u}_{mtt}, (-1)^M \Delta^M \boldsymbol{u}_{mtt}\right) + \frac{1}{2\varepsilon_2}\left((-1)^M \Delta^M \boldsymbol{u}_m, (-1)^M \Delta^M \boldsymbol{u}_m\right)$$

$$= \frac{\varepsilon_2}{2}\left(\Delta^M \boldsymbol{u}_{mtt}, \Delta^M \boldsymbol{u}_{mtt}\right) + \frac{1}{2\varepsilon_2}\left(\Delta^M \boldsymbol{u}_m, \Delta^M \boldsymbol{u}_m\right),$$

$$-\left((-1)^M \Delta^M \boldsymbol{u}_{mt}, (-1)^M \Delta^M \boldsymbol{u}_{mtt}\right)$$
$$\leq \frac{\varepsilon_3}{2}\left((-1)^M \Delta^M \boldsymbol{u}_{mtt}, (-1)^M \Delta^M \boldsymbol{u}_{mtt}\right) + \frac{1}{2\varepsilon_3}\left((-1)^M \Delta^M \boldsymbol{u}_{mt}, (-1)^M \Delta^M \boldsymbol{u}_{mt}\right)$$

$$= \frac{\varepsilon_3}{2}\left(\Delta^M \boldsymbol{u}_{mtt}, \Delta^M \boldsymbol{u}_{mtt}\right) + \frac{1}{2\varepsilon_3}\left(\Delta^M \boldsymbol{u}_{mt}, \Delta^M \boldsymbol{u}_{mt}\right),$$

$$\left(\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right),(-1)^{M}\Delta^{M}\boldsymbol{u}_{mtt}\right)\leq\frac{\varepsilon_{4}}{2}\left((-1)^{M}\Delta^{M}\boldsymbol{u}_{mtt},(-1)^{M}\Delta^{M}\boldsymbol{u}_{mtt}\right)+\frac{1}{2\varepsilon_{4}}\left(\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right),\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right)\right),$$

所以

$$\left(\Delta^{M}\boldsymbol{u}_{mtt},\Delta^{M}\boldsymbol{u}_{mtt}\right)=\left((-1)^{M}\Delta^{M}\boldsymbol{u}_{mtt},(-1)^{M}\Delta^{M}\boldsymbol{u}_{mtt}\right)$$

$$\leq\frac{\varepsilon_{1}}{2}\left(\Delta^{M}\boldsymbol{u}_{mtt},\Delta^{M}\boldsymbol{u}_{mtt}\right)+\frac{1}{2\varepsilon_{1}}\left(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right)+\frac{\varepsilon_{2}}{2}\left(\Delta^{M}\boldsymbol{u}_{mtt},\Delta^{M}\boldsymbol{u}_{mtt}\right)$$

$$+\frac{1}{2\varepsilon_{2}}\left(\Delta^{M}\boldsymbol{u}_{m},\Delta^{M}\boldsymbol{u}_{m}\right)+\frac{\varepsilon_{3}}{2}\left(\Delta^{M}\boldsymbol{u}_{mtt},\Delta^{M}\boldsymbol{u}_{mtt}\right)+\frac{1}{2\varepsilon_{3}}\left(\Delta^{M}\boldsymbol{u}_{mt},\Delta^{M}\boldsymbol{u}_{mt}\right)$$

$$+\frac{\varepsilon_{4}}{2}\left(\Delta^{M}\boldsymbol{u}_{mtt},\Delta^{M}\boldsymbol{u}_{mtt}\right)+\frac{1}{2\varepsilon_{4}}\left(\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right),\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right)\right),$$

经计算得

$$\left(1-\frac{\varepsilon_{1}}{2}-\frac{\varepsilon_{2}}{2}-\frac{\varepsilon_{3}}{2}-\frac{\varepsilon_{4}}{2}\right)\left(\Delta^{M}\boldsymbol{u}_{mtt},\Delta^{M}\boldsymbol{u}_{mtt}\right)$$

$$\leq\frac{1}{2\varepsilon_{1}}\left(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right)+\frac{1}{2\varepsilon_{2}}\left(\Delta^{M}\boldsymbol{u}_{m},\Delta^{M}\boldsymbol{u}_{m}\right)+\frac{1}{2\varepsilon_{3}}\left(\Delta^{M}\boldsymbol{u}_{mt},\Delta^{M}\boldsymbol{u}_{mt}\right)+\frac{1}{2\varepsilon_{4}}\left(\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right),\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right)\right)$$

由引理 3.2 和引理 3.3 的结论知

$$\left(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right)\leq\text{const},\quad\left(\Delta^{M}\boldsymbol{u}_{m},\Delta^{M}\boldsymbol{u}_{m}\right)\leq\text{const},\quad\left(\Delta^{M}\boldsymbol{u}_{mt},\Delta^{M}\boldsymbol{u}_{mt}\right)\leq\text{const}$$

类似于引理 3.2 的证明过程可证 $\left(\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right),\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right)\right)\leq\text{const}$，

取 $\varepsilon_{1},\varepsilon_{2},\varepsilon_{3},\varepsilon_{4}$ 充分小，使得 $\left(1-\frac{\varepsilon_{1}}{2}-\frac{\varepsilon_{2}}{2}-\frac{\varepsilon_{3}}{2}-\frac{\varepsilon_{4}}{2}\right)\geq\frac{1}{2}$，所以

$$\left(\Delta^{M}\boldsymbol{u}_{mtt},\Delta^{M}\boldsymbol{u}_{mtt}\right)\leq\text{const},\ \text{即}\ \left|\Delta^{M}\boldsymbol{u}_{mtt}\right|_{L^{2}(\Omega)}^{2}\leq C_{4}\quad\left(C_{4}\text{为与}\boldsymbol{u}_{m}\text{无关的正常数}\right).$$

引理 3.4 证毕！

下面在引理 3.1~3.4 的基础上给出定理 1 的证明。

定理 1 的证明：将(2.1)~(2.3)代入(2.4)及其初值条件得

$$\begin{cases}\sum_{j=1}^{m}a_{mij}''(t)\left(w_{j},w_{k}\right)+\sum_{j=1}^{m}\left((-1)^{M}\Delta^{M}w_{j},w_{k}\right)a_{mij}(t)+\sum_{j=1}^{m}\left((-1)^{M}\Delta^{M}w_{j},w_{k}\right)a_{mij}'(t)\\ \qquad+\sum_{j=1}^{m}\left((-1)^{M}\Delta^{M}w_{j},w_{k}\right)a_{mij}''(t)=\left(f_{i}\left(\boldsymbol{u}_{mt}\right),w_{k}\right)\\ a_{mij}(0)=a_{mij}\\ a_{mij}'(0)=b_{mij}\\ k=1,2,\cdots,m;\ i=1,2,\cdots,N.\end{cases}$$

由引理 3.1~引理 3.4 的证明过程知 $\left|\left(f_{i}\left(\boldsymbol{u}_{mt}\right),w_{k}\right)\right|$ 有界，因此由常微分方程理论知上述常微分方程组在 $[0,T]$ 上有整体解，从而常微分方程组 $(*)$ 在 $[0,T]$ 上有整体解 $u_{mi}(x,t)$，且有下列估计式成立：

$$\left|\boldsymbol{u}_{m}\right|_{L^{2}(\Omega)}^{2}\leq\text{const},\quad\left|\boldsymbol{u}_{mt}\right|_{L^{2}(\Omega)}^{2}\leq\text{const},\quad\left|\boldsymbol{u}_{mtt}\right|_{L^{2}(\Omega)}^{2}\leq\text{const},\quad\left\|\nabla^{M}\boldsymbol{u}_{mtt}\right\|_{L^{2}(\Omega)}^{2}\leq\text{const},$$

$$\left|\nabla^{M}\boldsymbol{u}_{mt}\right|_{L^{2}(\Omega)}^{2}\leq\text{const},\quad\left|\Delta^{M}\boldsymbol{u}_{mt}\right|_{L^{2}(\Omega)}^{2}\leq\text{const},\quad\left|\Delta^{M}\boldsymbol{u}_{m}\right|_{L^{2}(\Omega)}^{2}\leq\text{const}.$$

由列紧性原理知存在 $\{u_{mi}\}$ 的一个子序列，不妨设为 $\{u_{vi}\}$，使得当 $v \to \infty$ 时，u_{vi} 弱收敛到 u_i 于 $L^{\infty}\left([0,T], H^{2M}(\Omega) \cap H_0^M(\Omega)\right)$，$u_{vit}$ 弱收敛到 u_{it} 于

$L^{\infty}\left([0,T], H^{2M}(\Omega) \cap H_0^M(\Omega)\right) \subset L^{\infty}\left([0,T], L^2(\Omega)\right)$，$u_{vitt}$ 弱收敛到 u_{itt} 于

$L^2\left([0,T], H_0^M(\Omega)\right) \cap L^{\infty}\left([0,T], L^2(\Omega)\right)$，另一方面，由内插不等式知 $|\nabla u_{vt}|_{L^2(\Omega)}$ 可被 $|u_{vt}|_{L^2(\Omega)}$ 与 $\left|\nabla^M u_{vt}\right|_{L^2(\Omega)}$ 控制住，而 $|u_{vt}|_{L^2(\Omega)}$ 与 $\left|\nabla^M u_{vt}\right|_{L^2(\Omega)}$ 是有界的，所以 $|\nabla u_{vt}|_{L^2(\Omega)}$ 有界。故由引理 3.1~引理 3.4 知 $u_{vt}, u_{vtt}, \nabla u_{vtt}$ 于 $L^{\infty}\left(0,T; L^2(\Omega)\right) \subset L^2\left(0,T; L^2(\Omega)\right) = L^2(Q_T)\left(Q_T = \Omega \times (0,T)\right)$ 中有界，因此 u_{vt} 在 $H^1(Q_T)$ 中有界，由 $H^1(Q_T)$ 紧致嵌入到 $L^2(Q_T)$ 知从 u_{vt} 可选取一子序列(仍记为 u_{vt})在 $L^2(Q_T)$ 中强收敛且几乎处处收敛到 u_t。又由引理 3.3 的证明过程知 $\left(f(u_{vt}), f(u_{vt})\right) = \left|f(u_{vt})\right|^2_{L^2(\Omega)} \leq \text{const}$，由 [6] 中引理 1.3 知 $f(u_{vt}) \to f(u_t)$ 在 $L^2(Q_T)$ 中弱收敛。在(2.4)中令 $m = v$ 将

$$\left(u_{mitt}, w_k\right) + \left((-1)^M \Delta^M u_{mi}, w_k\right) + \left((-1)^M \Delta^M u_{mit}, w_k\right) + \left((-1)^M \Delta^M u_{mitt}, w_k\right) = \left(f_i(u_{mt}), w_k\right),$$

改为

$$\left(u_{vitt}, w_k\right) + \left((-1)^M \Delta^M u_{vi}, w_k\right) + \left((-1)^M \Delta^M u_{vit}, w_k\right) + \left((-1)^M \Delta^M u_{vitt}, w_k\right) = \left(f_i(u_{vt}), w_k\right),$$

两边同乘 $d_{ki}(t)$，其中 $d_{ki}(t) \in C[0,T], k = 1, 2, \cdots; i = 1, 2, \cdots, N$，得

$$\left(u_{vitt}, d_{ki}(t) w_k\right) + \left((-1)^M \Delta^M u_{vi}, d_{ki}(t) w_k\right) + \left((-1)^M \Delta^M u_{vit}, d_{ki}(t) w_k\right) + \left((-1)^M \Delta^M u_{vitt}, d_{ki}(t) w_k\right)$$
$$= \left(f_i(u_{vt}), d_{ki}(t) w_k\right),$$

对 $k = 1, 2, \cdots, v' (v' \leq v)$ 求和得

$$\left(u_{vitt}, \sum_{k=1}^{v'} d_{ki}(t) w_k(x)\right) + \left((-1)^M \Delta^M u_{vi}, \sum_{k=1}^{v'} d_{ki}(t) w_k(x)\right) + \left((-1)^M \Delta^M u_{vit}, \sum_{k=1}^{v'} d_{ki}(t) w_k(x)\right)$$
$$+ \left((-1)^M \Delta^M u_{vitt}, \sum_{k=1}^{v'} d_{ki}(t) w_k(x)\right) = \left(f_i(u_{vt}), \sum_{k=1}^{v'} d_{ki}(t) w_k(x)\right),$$

对 t 从 0 到 T 积分得

$$\int_0^T \left(u_{vitt}, \sum_{k=1}^{v'} d_{ki}(t) w_k(x)\right) dt + \int_0^T \left((-1)^M \Delta^M u_{vi}, \sum_{k=1}^{v'} d_{ki}(t) w_k(x)\right) dt + \int_0^T \left((-1)^M \Delta^M u_{vit}, \sum_{k=1}^{v'} d_{ki}(t) w_k(x)\right) dt$$
$$+ \int_0^T \left((-1)^M \Delta^M u_{vitt}, \sum_{k=1}^{v'} d_{ki}(t) w_k(x)\right) dt = \int_0^T \left(f_i(u_{vt}), \sum_{k=1}^{v'} d_{ki}(t) w_k(x)\right) dt,$$

对应当 $v \to +\infty$ 时，$f(u_{vt}) \to f(u_t)$ 在 $L^2(Q_T)$ 中弱收敛，有当 $v \to +\infty$ 时，$f_i(u_{vt}) \to f_i(u_t)$ 在 $L^2(Q_T)$ 中弱收敛 $(i = 1, 2, \cdots, N)$，令 $v \to +\infty$ 得

$$\int_0^T \left(u_{itt}, \sum_{k=1}^{v'} d_{ki}(t) w_k(x)\right) dt + \int_0^T \left((-1)^M \Delta^M u_i, \sum_{k=1}^{v'} d_{ki}(t) w_k(x)\right) dt + \int_0^T \left((-1)^M \Delta^M u_{it}, \sum_{k=1}^{v'} d_{ki}(t) w_k(x)\right) dt$$
$$+ \int_0^T \left((-1)^M \Delta^M u_{itt}, \sum_{k=1}^{v'} d_{ki}(t) w_k(x)\right) dt = \int_0^T \left(f_i(u_t), \sum_{k=1}^{v'} d_{ki}(t) w_k(x)\right) dt,$$

因为 $\{w_k(x)\}_{k=1}^{+\infty}$ 构成 $L^2(\Omega)$ 的一组标准正交基，

$$d_{ki}(t) \in C[0,T], k = 1, 2, \cdots; i = 1, 2, \cdots, N$$

$\left\{ \sum\limits_{k=1}^{v'} d_{ki}(t) w_k(x) \Big| v'=1,2,\cdots; i=1,2,\cdots,N \right\}$ 在空间 $C\left([0,T]; L^2(\Omega)\right)$ 中稠密，所以对于任意的 $d_i(x,t) \in C\left([0,T]; L^2(\Omega)\right)$ $(i=1,2,\cdots,N)$ 成立

$$\int_0^T (u_{itt}, d_i(x,t)) \mathrm{d}t + \int_0^T \left((-1)^M \Delta^M u_i, d_i(x,t)\right) \mathrm{d}t + \int_0^T \left((-1)^M \Delta^M u_{it}, d_i(x,t)\right) \mathrm{d}t$$

$$+ \int_0^T \left((-1)^M \Delta^M u_{itt}, d_i(x,t)\right) \mathrm{d}t = \int_0^T (f_i(\boldsymbol{u}_t), d_i(x,t)) \mathrm{d}t,$$

上式关于 $i=1,2,\cdots,N$ 求和得对任意

$$\boldsymbol{d}(x,t) = \left(d_1(x,t), d_2(x,t), \cdots, d_N(x,t)\right)^{\mathrm{T}} \in C\left([0,T]; L^2(\Omega)\right) \text{ 成立}$$

$$\int_0^T (\boldsymbol{u}_{tt}, \boldsymbol{d}(x,t)) \mathrm{d}t + \int_0^T \left((-1)^M \Delta^M \boldsymbol{u}, \boldsymbol{d}(x,t)\right) \mathrm{d}t + \int_0^T \left((-1)^M \Delta^M \boldsymbol{u}_t, \boldsymbol{d}(x,t)\right) \mathrm{d}t$$

$$+ \int_0^T \left((-1)^M \Delta^M \boldsymbol{u}_{tt}, \boldsymbol{d}(x,t)\right) \mathrm{d}t = \int_0^T (\boldsymbol{f}(\boldsymbol{u}_t), \boldsymbol{d}(x,t)) \mathrm{d}t.$$

下面验证初始条件 $\boldsymbol{u}(x,0) = \boldsymbol{\varphi}(x)$。因为当 $v \to \infty$ 时，u_{vi} 弱收敛到 u_i 于 $L^\infty\left([0,T], H^{2M}(\Omega) \cap H_0^M(\Omega)\right)$，$u_{vit}$ 弱收敛到 u_{it} 于

$$L^\infty\left([0,T], H^{2M}(\Omega) \cap H_0^M(\Omega)\right) \subset L^\infty\left([0,T], L^2(\Omega)\right),$$

因此 $u_{vi}(x,t), u_i(x,t) \in C\left([0,T], H^{2M}(\Omega) \cap H_0^M(\Omega)\right)$，

所以，当 $v \to +\infty$ 时 $u_{vi}(x,0)$ 弱收敛到 $u_i(x,0)$ 于 $H^{2M}(\Omega) \cap H_0^M(\Omega)$ 中，而当 $v \to +\infty$ 时，$u_{vi}(x,0) \to \varphi_i(x)$ $(i=1,2,\cdots,N)$ 在 $H^{2M}(\Omega) \cap H_0^M(\Omega)$ 中强收敛，故

$$u_i(x,0) = \varphi_i(x) (1,2,\cdots,N), \quad \text{即} \ \boldsymbol{u}(x,0) = \boldsymbol{\varphi}(x).$$

再证初始条件 $\boldsymbol{u}_t(x,0) = \boldsymbol{\psi}(x)$。因为当 $v \to \infty$ 时 u_{vit} 弱收敛到 u_{it} 于 $L^\infty\left([0,T], H^{2M}(\Omega) \cap H_0^M(\Omega)\right) \subset L^\infty\left([0,T], L^2(\Omega)\right)$，$u_{vitt}$ 弱收敛到 u_{itt} 于 $L^2\left([0,T], H_0^M(\Omega)\right) \cap L^\infty\left([0,T], L^2(\Omega)\right)$，因此 $u_{vit}(x,t), u_{it}(x,t) \in C\left([0,T], L^2(\Omega)\right)$，所以，$u_{vit}(x,0)$ 弱收敛到 $u_{it}(x,0)$ $(i=1,2,\cdots,N)$ 于 $L^2(\Omega)$ 中，而当 $v \to +\infty$ 时，$u_{vit}(x,0) \to \psi_i(x)$ $(i=1,2,\cdots,N)$ 在 $L^2(\Omega)$ 中强收敛，故 $u_{it}(x,0) = \psi_i(x) (i=1,2,\cdots,N)$，即 $\boldsymbol{u}_t(x,0) = \boldsymbol{\psi}(x)$。

综上所述由定义 $\boldsymbol{u} = \boldsymbol{u}(x,t)$ 为问题(1.1)~(1.3)的整体强解。

下面给出定理 2 的证明，定理 2 是关于问题(1.1)~(1.3)强解唯一性的讨论。

定理 2 的证明：设 $\boldsymbol{u}^1, \boldsymbol{u}^2$ 为非线性问题(1.1)~(1.3)的两个整体强解，

令 $\boldsymbol{w} = \boldsymbol{u}^1 - \boldsymbol{u}^2$，则 $\boldsymbol{w}$ 满足

$$\begin{cases} \boldsymbol{w}_{tt} + (-1)^M \Delta^M \boldsymbol{w} + (-1)^M \Delta^M \boldsymbol{w}_t + (-1)^M \Delta^M \boldsymbol{w}_{tt} = \boldsymbol{f}(\boldsymbol{u}_t^1) - \boldsymbol{f}(\boldsymbol{u}_t^2), \\ \boldsymbol{w}(x,0) = 0, \boldsymbol{w}_t(x,0) = 0, \\ D^\gamma \boldsymbol{w}(x,t)\big|_{\partial\Omega \times [0,T]} = 0, \qquad 0 \leq |\gamma| \leq M-1. \end{cases}$$

两边用 $\boldsymbol{w}_t$ 做内积，得

$$(\boldsymbol{w}_{tt}, \boldsymbol{w}_t) + \left((-1)^M \Delta^M \boldsymbol{w}, \boldsymbol{w}_t\right) + \left((-1)^M \Delta^M \boldsymbol{w}_t, \boldsymbol{w}_t\right) + \left((-1)^M \Delta^M \boldsymbol{w}_{tt}, \boldsymbol{w}_t\right)$$

$$= \left(\frac{\partial \boldsymbol{f}(\boldsymbol{u}_t)}{\partial \boldsymbol{u}_t}\bigg|_{\boldsymbol{u}_t^2 + \theta \boldsymbol{w}_t} \boldsymbol{w}_t, \boldsymbol{w}_t\right) \quad (0 < \theta < 1),$$

由条件(1.5)知 $\left(\left. \dfrac{\partial \boldsymbol{f}(\boldsymbol{u}_t)}{\partial \boldsymbol{u}_t} \right|_{u_t^2 + \theta w_t} \boldsymbol{w}_t, \boldsymbol{w}_t \right) \le k_0 (\boldsymbol{w}_t, \boldsymbol{w}_t) \ \ (0 < \theta < 1)$，

因此

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{w}_t, \boldsymbol{w}_t) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\nabla^M \boldsymbol{w}, \nabla^M \boldsymbol{w}) + (\nabla^M \boldsymbol{w}_t, \nabla^M \boldsymbol{w}_t) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\nabla^M \boldsymbol{w}_t, \nabla^M \boldsymbol{w}_t) \le k_0 (\boldsymbol{w}_t, \boldsymbol{w}_t),$$

两边从 0 到 $t(0 \le t \le T)$ 积分得

$$\frac{1}{2}(\boldsymbol{w}_t, \boldsymbol{w}_t) + \frac{1}{2}(\nabla^M \boldsymbol{w}, \nabla^M \boldsymbol{w}) + \left[\nabla^M \boldsymbol{w}_t, \nabla^M \boldsymbol{w}_t \right] + \frac{1}{2}(\nabla^M \boldsymbol{w}_t, \nabla^M \boldsymbol{w}_t) \le k_0 \left[\boldsymbol{w}_t, \boldsymbol{w}_t \right],$$

两边同时加上 $\left[\boldsymbol{w}, \boldsymbol{w}_t \right] + \left[\nabla^M \boldsymbol{w}, \nabla^M \boldsymbol{w}_t \right]$，左边将它化为 $\dfrac{1}{2}(\boldsymbol{w}, \boldsymbol{w}) + \dfrac{1}{2}(\nabla^M \boldsymbol{w}, \nabla^M \boldsymbol{w})$，右边将它估计为

$$\left[\boldsymbol{w}, \boldsymbol{w}_t \right] + \left[\nabla^M \boldsymbol{w}, \nabla^M \boldsymbol{w}_t \right] \le \left[\boldsymbol{w}, \boldsymbol{w} \right] + \left[\boldsymbol{w}_t, \boldsymbol{w}_t \right] + \frac{1}{2\varepsilon}\left[\nabla^M \boldsymbol{w}, \nabla^M \boldsymbol{w} \right] + \frac{\varepsilon}{2}\left[\nabla^M \boldsymbol{w}_t, \nabla^M \boldsymbol{w}_t \right],$$

因此有

$$\frac{1}{2}(\boldsymbol{w}, \boldsymbol{w}) + \frac{1}{2}(\boldsymbol{w}_t, \boldsymbol{w}_t) + (\nabla^M \boldsymbol{w}, \nabla^M \boldsymbol{w}) + \left[\nabla^M \boldsymbol{w}_t, \nabla^M \boldsymbol{w}_t \right] + \frac{1}{2}(\nabla^M \boldsymbol{w}_t, \nabla^M \boldsymbol{w}_t)$$

$$\le \left[\boldsymbol{w}, \boldsymbol{w} \right] + (k_0 + 1)\left[\boldsymbol{w}_t, \boldsymbol{w}_t \right] + \frac{\varepsilon}{2}\left[\nabla^M \boldsymbol{w}_t, \nabla^M \boldsymbol{w}_t \right] + \frac{1}{2\varepsilon}\left[\nabla^M \boldsymbol{w}, \nabla^M \boldsymbol{w} \right],$$

取 ε 充分小使得 $1 - \dfrac{\varepsilon}{2} \ge \dfrac{1}{2}$ 整理得

$$\frac{1}{2}(\boldsymbol{w}, \boldsymbol{w}) + \frac{1}{2}(\boldsymbol{w}_t, \boldsymbol{w}_t) + (\nabla^M \boldsymbol{w}, \nabla^M \boldsymbol{w}) + \frac{1}{2}\left[\nabla^M \boldsymbol{w}_t, \nabla^M \boldsymbol{w}_t \right] + \frac{1}{2}(\nabla^M \boldsymbol{w}_t, \nabla^M \boldsymbol{w}_t)$$

$$\le \left[\boldsymbol{w}, \boldsymbol{w} \right] + (k_0 + 1)\left[\boldsymbol{w}_t, \boldsymbol{w}_t \right] + \frac{1}{2\varepsilon}\left[\nabla^M \boldsymbol{w}, \nabla^M \boldsymbol{w} \right],$$

由 Gronwall 不等式得 $|\boldsymbol{w}|^2_{L^2(\Omega)} + |\boldsymbol{w}_t|^2_{L^2(\Omega)} + |\nabla^M \boldsymbol{w}|^2_{L^2(\Omega)} = 0$。

所以 $\boldsymbol{w} \equiv \boldsymbol{0}$ *a.e* $\boldsymbol{u}^1 = \boldsymbol{u}^2$。

唯一性证毕！

4. 结论

本文主要研究了在粘弹性力学中具有实际背景的一类高阶 *n* 维非线性波动方程组。要求方程的非线性项 *f* 一次连续可微且非线性项 *f* 满足条件：*f* 的 Jacobi 矩阵半有界。首先选取高阶调和算子的特征函数系作为一组基，用 Galerkin 方法构造初边值问题的近似解，用 4 个引理对近似解作出一系列的先验估计，在先验估计的基础上，由列紧性原理证明了初边值问题整体强解的存在性，最后证明了初边值问题整体强解的唯一性。其中，选取高阶调和算子的特征函数系作为一组基，该特征函数系分别构成 $L^2(\Omega)$，$H_0^M(\Omega), H^{2M} \cap H_0^M(\Omega)$ 的正交基底，这样对近似解作先验估计时会更方便。

参考文献 (References)

[1] 朱位秋 (1980) 弹性杆中的非线性波. *固体力学学报*, **2**, 247-253.

[2] 尚亚东 (2000) 一类四阶非线性波动方程的初边值问题. *应用数学学报*, **1**, 7-11.

[3] 丁丽娟 (2013) 一类非线性发展方程组的初边值问题. *理论数学*, **3**, 72-80.

[4] 严曼 (2012) 一类高阶 n 维非线性伪双曲方程. *应用数学进展*, **2**, 91-97.

[5] Adams, R.A., 著 (1983) 叶其孝, 等, 译. 索伯列夫空间. 人民教育出版社, 北京.

[6] Lions, J.L., 著 (1992) 郭柏灵, 等, 译. 非线性边值问题的一些解法. 中山大学出版社, 广州.

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