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(3 + 1)维修正KdV-ZK方程和(3 + 1)维KP方程的精确行波解

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摘　要

本文讨论了(3 + 1)维修正KdV-Zakharov-Kuznetsov方程和(3 + 1)维Kadomtsev-Petviashvili方程的精确行波解，得到了(3 + 1)维修正KdV-Zakharov-Kuznetsov方程的扭状孤波解和(3 + 1)维Kadomtsev-Petviashvili方程的双曲函数奇异解，并且利用Maple软件给出了解的3D和2D图，分析了解在特殊参数值下的动力行为。

关键词

行波解，(3 + 1)维修正KdV-Zakharov-Kuznetsov方程，(3 + 1)维Kadomtsev-Petviashvili方程

Exact Traveling Wave Solutions of the (3 + 1)-Dimensional Modified KdV-ZK Equation and the (3 + 1)-Dimensional KP Equation

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Abstract

In this paper, we discuss the exact traveling wave solutions of the (3 + 1)-dimensional modified

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KdV-Zakharov-Kuznetsov equation and the (3 + 1)-dimensional Kadomtsev-Petviashvili equation, the twisted solitary wave solutions of the (3 + 1)-dimensional modified KdV-Zakharov-Kuznetsov equation and the hyperbolic function singular solutions of the (3 + 1)-dimensional Kadomtsev-Petviashvili equation were obtained; the 3D and 2D plots of the solutions were given with Maple, analyzing the dynamic behavior of the solutions under the particular parameters value.

Keywords

Traveling Wave Solution, The (3 + 1)-Dimensional Modified KdV-Zakharov-Kuznetsov Equation, The (3 + 1)-Dimensional Kadomtsev-Petviashvili Equation

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1. 引言

非线性发展方程在许多领域中有着广泛的应用，例如在流体力学、固体物理、生物和海洋工程、光纤、等离子体物理和化学物理等领域中，涌现出类似于薛定谔方程[1]、KdV 方程[2]等许多重要的非线性发展方程。如今，人们提出了许多有效的方法去构造非线性发展方程的精确行波解，如$(G'/G,1/G)$扩展法[3] [4]，正弦－余弦方法[5]，$\mathrm{e}^{(-\Phi(\xi))}$函数法[6] [7]，Jacobi 椭圆函数法[8] [9]，tanh 函数法[10]，扩展的直接代数法及新的扩展的直接代数法[11] [12] [13]，(G'/G)扩展法[14] [15]，F-扩展法[16] [17]，映射法和扩展映射法[18]，$(1/G')$扩展法[19] [20] [21] [22]等。

本文主要用$(1/G')$扩展法构造(3 + 1)维修正 KdV-Zakharov-Kuznetsov 方程[14] (简称 mKdV-ZK 方程)

$$u_t + \alpha u^2 u_x + u_{xxx} + u_{xyy} + u_{xzz} = 0 \tag{1}$$

和(3 + 1)维 Kadomtsev-Petviashvili 方程[3] (简称 KP 方程)的精确行波解。

$$\left(u_t + 6uu_x + u_{xxx}\right)_x - 3u_{yy} - 3u_{zz} = 0 \tag{2}$$

方程(1)中包含四项耗散效应$u_t, u_{xxx}, u_{xyy}, u_{xzz}$和一个对流过程$u^2u_x$。在均匀磁场存在的情况下，方程(1)控制弱非线性离子声波的行为，包括冷离子和热等温电子的等离子体。1970 年，Kadomtsev 和 Petviashvili 在研究散色和非线性介质中的非线性波动理论时提出了 KP 方程，该方程是描述浅水波和等离子声波的方程，且在很多领域都有重要的应用。Zhang Z Y [8]利用 Jacobi 椭圆函数展开法得到了方程(1)精确的行波解；Uttam Ghosh [23]利用修正分数阶子方程方法得到了方程(1)的精确解析解；Md. Nur Alam [24]利用一种(G'/G)扩展法获得了方程(1)一些新的和更一般的行波解。Ma W X [25]总结了方程(2)的行波解和有理解；Zayed E M E [26]利用$(G'/G,1/G)$扩展法得到方程(2)的孤立波解和三角周期解；Lu D [27]运用拟设法获得了方程(2)的孤立波解、冲击波解和奇异波解等。

本文结构为：在第二部分，给出$(1/G')$扩展法的具体步骤；在第三部分，利用$(1/G')$扩展法构造了(3 + 1)维修正 KdV-Zakharov-Kuznetsov 方程和(3 + 1)维 Kadomtsev-Petviashvili 方程的精确行波解；在第四部分，对行波解的图形性态进行分析；总结在第五部分。

2. $(1/G')$扩展法的步骤

考虑如下非线性偏微分方程

$$F\left(u,\frac{\partial u}{\partial t},\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x^2},\cdots\right)=0 \tag{3}$$

令行波变换

$$u=u(\xi)=u(x,y,z,t),\xi=x+y+z-ct,c\neq 0 \tag{4}$$

其中 c 是波速。在方程(3)中用行波变换(4)可得如下常微分方程

$$P(U,U',U'',\cdots)=0 \tag{5}$$

$'=\frac{\partial U}{\partial \xi}$。假定方程(5)有如下形式的解

$$U(\xi)=a_0+\sum_{i=1}^{n}a_i\left(\frac{1}{G'(\xi)}\right)^i \tag{6}$$

其中 $a_i(i=0,1,\cdots,n)$ 为待定常数，n 由齐次平衡原则确定，$G=G(\xi)$ 满足二阶常微分方程

$$G''+\kappa G'+\mu=0 \tag{7}$$

其中 κ,μ 是参数，方程(7)有如下形式的解[19]

$$\frac{1}{G'(\xi)}=\frac{1}{-\frac{\mu}{\kappa}+A\cosh(\xi\kappa)-A\sinh(\xi\kappa)} \tag{8}$$

将 $U(\xi)$ 以及 $U(\xi)$ 的各阶导数与方程(7)代入方程(5)中，合并 $\left(\frac{1}{G'}\right)^i$ $(i=0,1,\cdots,n)$ 的次数，得到一个关于 $a_0,a_1,\cdots,a_n$ 的代数方程组，再借助解(8)可得出方程(3)的精确行波解。

3. 精确行波解

在本部分利用 $(1/G')$ 扩展法构造(3 + 1)维修正 KdV-Zakharov-Kuznetsov 方程和(3 + 1)维 Kadomtsev-Petviashvili 方程的精确行波解。

3.1. mKdV-ZK 方程的精确行波解

将行波变换(4)代入方程(1)可得

$$-cu'+\alpha u^2u'+3u'''=0 \tag{9}$$

其中 c 是波速。对方程(9)积分一次，并令积分常数为零，则

$$-cu+\frac{1}{3}\alpha u^3+3u''=0 \tag{10}$$

根据最高阶导数项 u'' 与非线性项 αu^3 的平衡原则，可得 $n=1$。再由方程(6)可设方程(10)有如下形式的解

$$u(\xi)=a_0+a_1\left(\frac{1}{G'}\right) \tag{11}$$

其中 a_0,a_1 是待定常数，将方程(11)以及 $u(\xi)$ 的相关导数与方程(7)代入方程(10)，合并同类项令 $\left(\frac{1}{G'}\right)^i$ $(i=0,1,2,3)$ 的系数为 0，得到关于 a_0,a_1 与 c 的代数方程组为：

$$\begin{aligned}
&\left(\frac{1}{G'}\right)^0: \quad -a_0c+\frac{1}{3}\alpha a_0^3=0\\
&\left(\frac{1}{G'}\right)^1: \quad -a_1c+\alpha a_0^2a_1+3a_1\kappa^2=0\\
&\left(\frac{1}{G'}\right)^2: \quad \alpha a_0a_1^2+9a_1\kappa\mu=0\\
&\left(\frac{1}{G'}\right)^3: \quad \frac{1}{3}\alpha a_1^3+6a_1\mu^2=0
\end{aligned} \tag{12}$$

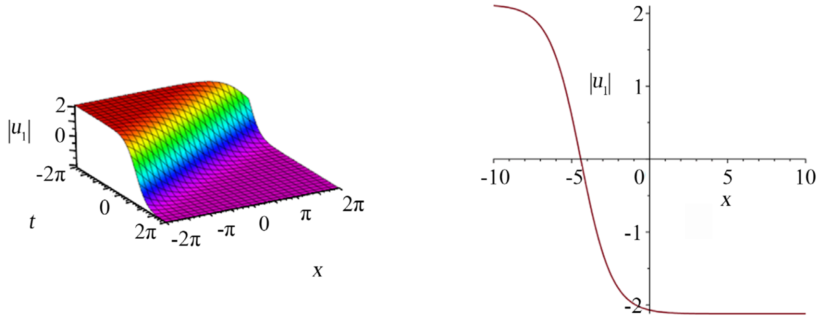
求解上述方程组可得：

$$a_0=\pm\sqrt{\frac{3c}{\alpha}}, a_1=-\frac{9\kappa\mu}{\alpha a_0}, c=-\frac{3\kappa^2}{2} \tag{13}$$

将方程(13)和方程(8)代入方程(11)，得出方程(1)有如下形式的双曲函数解：

$$u_1(\xi)=\pm\frac{3\sqrt{2\alpha}}{\alpha}i\left[\frac{\kappa}{2}+\frac{\mu}{-\frac{\mu}{\kappa}+A\cosh(\xi\kappa)-A\sinh(\xi\kappa)}\right] \tag{14}$$

由$(1/G')$扩展法得到的双曲函数解(14)的 3D、2D 图如图 1 所示，其中$\xi=x+y+z+\frac{3\kappa^2}{2}t$，$\kappa,\mu$为参数，$i$为虚数单位。



(a) 3D-图　　(b) 2D-图

Figure 1. The 3D and 2D plots of solution (14)

图 1. 解(14)的 3D 与 2D 图

3.2. KP 方程的精确行波解

将行波变换(4)代入方程(2)可得

$$\left(-cu'+6uu'+u'''\right)'-6u''=0 \tag{15}$$

其中 c 表示波速。对方程(15)积分两次，令积分常数为零，可得

$$-cu-6u+3u^2+u''=0 \tag{16}$$

考虑$\xi\to\pm\infty$时，$|u|\to0$，根据最高阶导数项u''与非线性项u^2的平衡原则，可得$n=2$，再由方程(6)可设方程(16)有如下形式的解

$$u(\xi)=a_0+a_1\left(\frac{1}{G'}\right)+a_2\left(\frac{1}{G'}\right)^2 \tag{17}$$

其中 a_0, a_1, a_2 是待定常数，将方程(17)以及 $u(\xi)$ 的相关导数与方程(7)代入方程(16)，合并同类项令 $\left(\frac{1}{G'}\right)^i$ $(i=0,1,2,3,4)$ 的系数为 0，得到关于 a_0, a_1, a_2 与 c 的代数方程组为：

$$\begin{aligned}
&\left(\frac{1}{G'}\right)^0: \quad -a_0c-6a_0+3a_0^2=0\\
&\left(\frac{1}{G'}\right)^1: \quad -a_1c-6a_1+6a_0a_1+a_1\kappa^2=0\\
&\left(\frac{1}{G'}\right)^2: \quad -a_2c-6a_2+3a_1^2+6a_0a_2+3a_1\kappa\mu+4a_2\kappa^2=0\\
&\left(\frac{1}{G'}\right)^3: \quad 3a_1a_2+a_1\mu^2+5a_2\kappa\mu=0\\
&\left(\frac{1}{G'}\right)^4: \quad a_2^2+2a_2\mu^2=0
\end{aligned} \tag{18}$$

求解上述方程组，得出如下解的情况：

$$\text{情况 1.}\quad a_0=0, a_1=-2\kappa\mu, a_2=-2\mu^2, c=\kappa^2-6 \tag{19}$$

将方程(19)和方程(8)代入方程(17)，得出方程(2)有如下形式的双曲函数解：

$$u_2(\xi)=-\frac{2\kappa\mu}{-\frac{\mu}{\kappa}+A\cosh(\xi\kappa)-A\sinh(\xi\kappa)}-\frac{2\mu^2}{\left(-\frac{\mu}{\kappa}+A\cosh(\xi\kappa)-A\sinh(\xi\kappa)\right)^2} \tag{20}$$

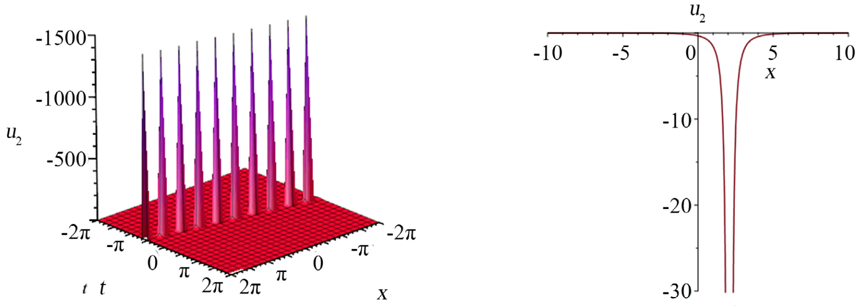
其中 $\xi=x+y+z-\left(\kappa^2-6\right)t$。

$$\text{情况 2.}\quad a_0=\frac{1}{3}(c+6), a_1=-2\kappa\mu, a_2=-2\mu^2, c=-\left(\kappa^2+6\right) \tag{21}$$

将方程(21)和方程(8)代入方程(17)，得出方程(2)有如下形式的双曲函数解：

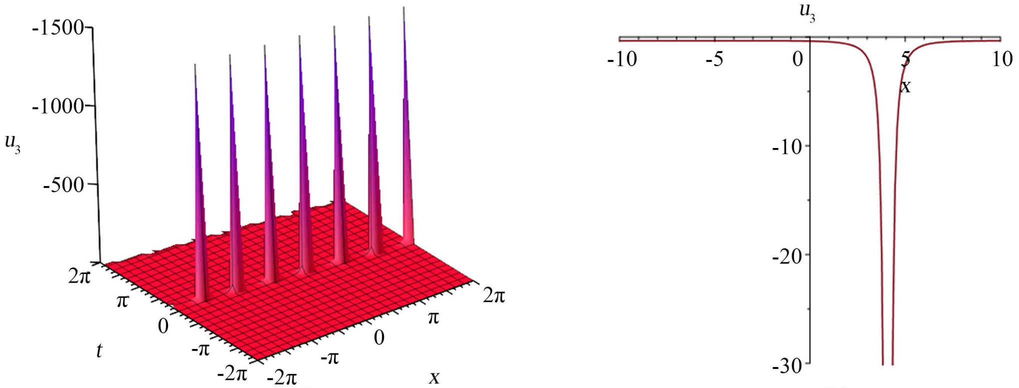
$$u_3(\xi)=\frac{1}{3}(c+6)-\frac{2\kappa\mu}{-\frac{\mu}{\kappa}+A\cosh(\xi\kappa)-A\sinh(\xi\kappa)}-\frac{2\mu^2}{\left(-\frac{\mu}{\kappa}+A\cosh(\xi\kappa)-A\sinh(\xi\kappa)\right)^2} \tag{22}$$

其中 $\xi=x+y+z+\left(\kappa^2+6\right)t$。



(a) 3D-图　　(b) 2D-图

Figure 2. The 3D and 2D plots of solution (20)
图 2. 解(20)的 3D 与 2D 图



(a) 3D-图　(b) 2D-图

Figure 3. The 3D and 2D plots of solution (22)
图 3. 解(22)的 3D 与 2D 图

4. 图像分析

图 1 是方程(1)的解 u_1 在 $-10 \le x,t \le 10$ 区间内，当参数 $\mu = -1, \alpha = \kappa = y = z = 1, A = 0.4$ 时的 3D 性态，以及 $t = 1$ 时的 2D 性态。根据波向右传播，有一个波峰，且波的形状和速度保持不变，可知解 u_1 为方程(1)的扭状孤波解。图 2 是方程(2)的解 u_2 在 $-10 \le x,t \le 10$ 区间内，当参数 $\mu = \kappa = y = z = 1, A = 0.4, c = -5$ 时的 3D 和 $t = -1$ 时的 2D 性态。可以看出解 u_2 有一个波谷，并且出现尖点，因此 u_2 为方程(2)的非光滑奇异解。图 3 是方程(2)的解 u_3 在 $-10 \le x,t \le 10$ 区间内，当参数 $\mu = \kappa = y = z = 1, A = 0.4, c = -7$ 时的 3D 和 $t = -1$ 时的 2D 性态。可以观察到解 u_3 有一个波谷和尖点，因此 u_3 为方程(2)的非光滑奇异解。

5. 总结

本文主要用 $(1/G')$ 扩展法构造了方程(1)和方程(2)的精确行波解，得到方程(1)的解为扭状孤波解，方程(2)的解为非光滑奇异解。做出了这些解在特殊参数值下的 3D 和 2D 图，并对这些解的性态进行了分析。

通过将解(14)与文献[19]中解(23)比较发现，若解(14)中令 $\alpha = -18$ 时，可得出解的形式为 $u_1(\xi) = -\frac{\kappa}{2} - \frac{\mu}{-\frac{\mu}{\kappa} + A\cosh(\xi\kappa) - A\sinh(\xi\kappa)}$，而解(23)中令 $a_1 = -\mu$，可得出相同形式的解。由此可见，解(14)比解(23)更具有一般性。

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