Pure Mathematics 理论数学, 2017, 7(6), 422-430
Published Online November 2017 in Hans. http://www.hanspub.org/journal/pm
https://doi.org/10.12677/pm.2017.76055

Hans汉斯

Conditions That Subdirect Sums of *MB*-Matrices Is Still *MB*-Matrices

Yi Luo
School of Mathematics and Statistics, Yunnan University, Kunming Yunnan
Email: 649097377@qq.com

Received: Oct. 14th, 2017; accepted: Oct. 28th, 2017; published: Nov. 2nd, 2017

Abstract

By splitting an *MB*-matrix *A* into a sum of a nonsingular *M*-matrix and a nonnegative rank 1 matrix, some sufficient and necessary conditions and some sufficient conditions are given such that the subdirect sum of two *MB*-matrices is still an *MB*-matrix. Some examples are also given to illustrate the results.

Keywords

***Z*-Matrix, Nonsingular *M*-Matrix, *MB*-Matrix, Subdirect Sum**

MB-矩阵子直和仍为*MB*-矩阵的条件

骆 毅
云南大学数学与统计学院，云南 昆明
Email: 649097377@qq.com

收稿日期：2017年10月14日；录用日期：2017年10月28日；发布日期：2017年11月2日

摘 要

通过将*MB*-矩阵分裂成一个非奇异*M*-矩阵和一个秩1非负矩阵之和，获得*MB*-矩阵的子直和仍为*MB*-矩阵的一些充要条件和充分条件，最后用数值例子对所给结论进行了说明和解释。

关键词

***Z*-矩阵，非奇异*M*-矩阵，*MB*-矩阵，子直和**

文章引用: 骆毅. *MB*-矩阵子直和仍为 *MB*-矩阵的条件[J]. 理论数学, 2017, 7(6): 422-430.
DOI: 10.12677/pm.2017.76055

1. 引言

1999 年 Fallat 和 Johnson 首先在文献[1]中提出矩阵的子直和的概念，由于其在诸如马可夫链的递增许瓦兹迭代及分裂和重叠的递增许瓦兹迭代等研究中的重要性，引起了学者的关注和研究，并取得了一些重要研究成果，如文献[2] [3] [4]分别对非奇异 *M*-矩阵及其逆的子直和、*H*-矩阵和双对角占优矩阵的子直和等进行了研究。本文在文献[2]和[5]的基础上对 *MB*-矩阵的子直和进行研究，试图得到 *MB*-矩阵的子直和仍为 *MB*-矩阵的一些新的条件。

2. 预备知识

本节先给出一些基本概念、定理与符号，以备后用。

设 $A=\left(a_{ij}\right)\in R^{m\times n}$，如果对于所有的 $i=1,\cdots,m;\ j=1,\cdots,n$ 都有 $a_{ij}>0\left(a_{ij}\geq 0\right)$，则称 A 为正(非负)矩阵，记为 $A>O\left(A\geq O\right)$。

定义 2.1. [6]设 $A=\left(a_{ij}\right)\in R^{n\times n}$，如果对于所有的 $1\leq i,j\leq n$，$i\neq j$ 都有 $a_{ij}\leq 0$，则 A 称为 *Z*-矩阵。如果 A 是 *Z*-矩阵且 $A^{-1}\geq O$，则称 A 为 *M*-矩阵。

定义 2.2. [7]设 $A=\left(a_{ij}\right)\in R^{n\times n}$，将 A 分裂为 $A=A^z+A^r$，其中

$$A^z=\begin{bmatrix} a_{11}-\beta_1^A & a_{12}-\beta_1^A & \cdots & a_{1n}-\beta_1^A \\ a_{21}-\beta_2^A & a_{22}-\beta_2^A & \cdots & a_{2n}-\beta_2^A \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}-\beta_n^A & a_{n2}-\beta_n^A & \cdots & a_{nn}-\beta_n^A \end{bmatrix},\quad A^r=\begin{bmatrix} \beta_1^A & \beta_1^A & \cdots & \beta_1^A \\ \beta_2^A & \beta_2^A & \cdots & \beta_2^A \\ \vdots & \vdots & \ddots & \vdots \\ \beta_n^A & \beta_n^A & \cdots & \beta_n^A \end{bmatrix} \tag{1}$$

$\beta_i^A=\max\left\{0,a_{ij}\mid\forall j\neq i\right\}$。显然 A^z 是 *Z*-矩阵，A^r 是秩 1 非负矩阵。若 A^z 为 *M*-矩阵，则称 A 为 *MB*-矩阵。

定义 2.3. [2]设 $A\in R^{n_1\times n_1},B\in R^{n_2\times n_2}$，$k$ 是整数且 $1\leq k\leq\min\left\{n_1,n_2\right\}$，$n=n_1+n_2-k$，$A,B$ 分块如下：

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\quad B=\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \tag{2}$$

其中 A_{22}，B_{11} 都是 k 阶方阵。定义矩阵

$$M=\begin{bmatrix} A_{11} & A_{12} & O \\ A_{21} & A_{22}+B_{11} & B_{12} \\ O & B_{21} & B_{22} \end{bmatrix}\in R^{n\times n} \tag{3}$$

称其为 A 和 B 的 n $\left(n=n_1+n_2-k\right)$ 阶 k-子直和，记为 $M=A\oplus_k B$。

将 $A=\left(a_{ij}\right)\in R^{n_1\times n_1},B=\left(b_{ij}\right)\in R^{n_2\times n_2}$ 和 $M=\left(m_{ij}\right)\in R^{n\times n}$ 按定义 2.2 中的(1)式分别分裂为：

$$A=A^z+A^r,\quad B=B^z+B^r,\quad M=A\oplus_k B=M^z+M^r$$

将 A^z,A^r,B^z,B^r 按(2)式分块为：

$$A^z=\begin{bmatrix} A_{11}^z & A_{12}^z \\ A_{21}^z & A_{22}^z \end{bmatrix},\quad A^r=\begin{bmatrix} A_{11}^r & A_{12}^r \\ A_{21}^r & A_{22}^r \end{bmatrix},\quad B^z=\begin{bmatrix} B_{11}^z & B_{12}^z \\ B_{21}^z & B_{22}^z \end{bmatrix},\quad B^r=\begin{bmatrix} B_{11}^r & B_{12}^r \\ B_{21}^r & B_{22}^r \end{bmatrix}$$

定义矩阵

$$\overline{M}=\begin{bmatrix} A_{11}^z & A_{12}^z & -A_{13}^r \\ A_{21}^z-B_{13}^r & A_{22}^z+B_{11}^z & B_{12}^z-A_{23}^r \\ -B_{23}^r & B_{21}^z & B_{22}^z \end{bmatrix}\in R^{n\times n} \tag{4}$$

其中

$A_{11}^z\in R^{(n_1-k)\times(n_1-k)}$，$A_{12}^z\in R^{(n_1-k)\times k}$，$A_{21}^z\in R^{k\times(n_1-k)}$，$A_{22}^z\in R^{k\times k}$，

$B_{11}^z\in R^{k\times k}$，$B_{12}^z\in R^{k\times(n-n_1)}$，$B_{21}^z\in R^{(n-n_1)\times k}$，$B_{22}^z\in R^{(n-n_1)\times(n-n_1)}$，

$A_{11}^r\in R^{(n_1-k)\times(n_1-k)}$，$A_{12}^r\in R^{(n_1-k)\times k}$，$A_{13}^r\in R^{(n_1-k)(n-n_1)}$ 的第 i 行为 $\left(\beta_i^A,\beta_i^A,\cdots,\beta_i^A\right)$，

$A_{21}^r\in R^{k\times(n_1-k)}$，$A_{22}^r\in R^{k\times k}$，$A_{23}^r\in R^{k\times(n-n_1)}$ 的第 i 行为 $\left(\beta_i^A,\beta_i^A,\cdots,\beta_i^A\right)$，

$B_{11}^r\in R^{k\times k}$，$B_{12}^r\in R^{k\times(n-n_1)}$，$B_{13}^r\in R^{k\times(n_1-k)}$ 的第 i 行为 $\left(\beta_i^B,\beta_i^B,\cdots,\beta_i^B\right)$，

$B_{21}^r\in R^{(n-n_1)\times k}$，$B_{22}^r\in R^{(n-n_1)\times(n-n_1)}$，$B_{23}^r\in R^{(n-n_1)\times(n_1-k)}$ 的第 i 行为 $\left(\beta_i^B,\beta_i^B,\cdots,\beta_i^B\right)$。

容易验证这里的 $\overline{M}$ 就是[5]中的 $\overline{M}$，于是由文献[5]知 $M^z\geq\overline{M}$，且都为 Z-矩阵。

当 A^z,B^z 为非奇异矩阵时，将 $\left(A^z\right)^{-1},\left(B^z\right)^{-1}$ 按(2)分块为：

$$\left(A^z\right)^{-1}=\begin{bmatrix}\widehat{A_{11}^z} & \widehat{A_{12}^z}\\ \widehat{A_{21}^z} & \widehat{A_{22}^z}\end{bmatrix},\quad \left(B^z\right)^{-1}=\begin{bmatrix}\widehat{B_{11}^z} & \widehat{B_{12}^z}\\ \widehat{B_{21}^z} & \widehat{B_{22}^z}\end{bmatrix} \tag{*}$$

其中 $\widehat{A_{11}^z}\in R^{(n_1-k)\times(n_1-k)}$，$\widehat{A_{12}^z}\in R^{(n_1-k)\times k}$，$\widehat{A_{21}^z}\in R^{k\times(n_1-k)}$，$\widehat{A_{22}^z}\in R^{k\times k}$，$\widehat{B_{11}^z}\in R^{k\times k}$，$\widehat{B_{12}^z}\in R^{k\times(n-n_1)}$，$\widehat{B_{21}^z}\in R^{(n-n_1)\times k}$，$\widehat{B_{22}^z}\in R^{(n-n_1)\times(n-n_1)}$。

定理 2.1. [2]设 $A=\left[a_{ij}\right]\in R^{n\times n}$，则如下 3 款成立：

1) 当 A 为非奇异 M-矩阵时，其主对角元为正。

2) 当 A 为非奇异 M-矩阵，$B=\left[b_{ij}\right]$ 为 Z-矩阵且 $B\geq A$ 时，B 为非奇异 M-矩阵。

3) A 为非奇异 M-矩阵的充要条件为 A 的每一个主子矩阵为非奇异 M-矩阵。

引理 2.1. [8]设 $A=\begin{bmatrix}D & E\\ F & G\end{bmatrix}$ 非奇异，其中 $D\in R^{(n-k)\times(n-k)}$，$G\in R^{k\times k}$，$E\in R^{(n-k)\times k}$，$F\in R^{k\times(m-k)}$。则

1) 若 D 非奇异且 $D^{-1}\geq 0$，$-E\geq 0$，$-F\geq 0$，则 $A^{-1}\geq 0$ 当且仅当 $\left(A/D\right)^{-1}\geq 0$；

2) 若 $G^{-1}\geq 0$，$-E\geq 0$，$-F\geq 0$，则 $A^{-1}\geq 0$ 当且仅当 $\left(A/G\right)^{-1}\geq 0$。

这里 A/D 表示矩阵 A 内 D 的 Schur 补。

3. *MB*-矩阵的 *k*-子直和

先给出 $\overline{M}$ 为非奇异的 Z-矩阵的充要条件。

定理 3.1. 设 A^z,B^z 为 M-矩阵，

$$\det\widehat{H}=\det\left\{\left(\widehat{B_{11}^z}+\widehat{A_{22}^z}\right)+\widehat{A_{22}^z}B_{13}^r\left(A_{12}^z\widehat{B_{11}^z}-A_{13}^r\widehat{B_{21}^z}\right)-\left(\widehat{A_{21}^z}A_{13}^r+\widehat{A_{22}^z}A_{23}^r\right)\widehat{B_{21}^z}\right\}\neq 0$$

且 $B_{23}^r=O$，则 $\overline{M}$ 为非奇异的 Z-矩阵。

证明： 由(*)式得

$$\left(A^z\right)^{-1}\begin{bmatrix}I_{n_1-k} & O\\ B_{13}^r & I_k\end{bmatrix}=\begin{bmatrix}A_{11}^z+\widehat{A_{12}^z}B_{13}^r & \widehat{A_{12}^z}\\ \widehat{A_{21}^z}+\widehat{A_{22}^z}B_{13}^r & \widehat{A_{22}^z}\end{bmatrix}$$

再由 $\det\left(A^z\right)^{-1}\det\begin{bmatrix}I_{n_1-k} & O\\ B_{13}^r & I_k\end{bmatrix}\neq 0$ 得

$$\det\begin{bmatrix} A_{11}^z+\widehat{A_{12}^z}B_{13}^r & \widehat{A_{12}^z} \\ \widehat{A_{21}^z}+\widehat{A_{22}^z}B_{13}^r & \widehat{A_{22}^z} \end{bmatrix} \neq 0$$

于是由 $A^z\left(A^z\right)^{-1}=I_{n_1}$，得

$$A_{11}^z\widehat{A_{11}^z}+A_{12}^z\widehat{A_{12}^z}=I_{n_1-k}=I_{n-n_2}\text{，}\quad A_{11}^z\widehat{A_{12}^z}+A_{12}^z\widehat{A_{22}^z}=0\text{，}$$

$$A_{21}^z\widehat{A_{11}^z}+A_{22}^z\widehat{A_{21}^z}=0\text{，}\quad A_{21}^z\widehat{A_{12}^z}+A_{22}^z\widehat{A_{22}^z}=I_k$$

由 $\left(A^z\right)^{-1}A^z=I_{n_1}$，得

$$\widehat{A_{11}^z}A_{11}^z+\widehat{A_{12}^z}A_{12}^z=I_{n-n_2}\text{，}\quad \widehat{A_{11}^z}A_{12}^z+\widehat{A_{12}^z}A_{22}^z=0$$

$$\widehat{A_{21}^z}A_{11}^z+\widehat{A_{22}^z}A_{21}^z=0\text{，}\quad \widehat{A_{21}^z}A_{12}^z+\widehat{A_{22}^z}A_{22}^z=I_k$$

由 $B^z\left(B^z\right)^{-1}=I_{n_2}$，得

$$B_{11}^z\widehat{B_{11}^z}+B_{12}^z\widehat{B_{21}^z}=I_k\text{，}\quad B_{11}^z\widehat{B_{12}^z}+B_{12}^z\widehat{B_{22}^z}=0\text{，}$$

$$B_{21}^z\widehat{B_{11}^z}+B_{22}^z\widehat{B_{21}^z}=0\text{，}\quad B_{21}^z\widehat{B_{12}^z}+B_{22}^z\widehat{B_{22}^z}=I_{n-n_1}$$

由 $\left(B^z\right)^{-1}B^z=I_{n_2}$，得

$$\widehat{B_{11}^z}B_{11}^z+\widehat{B_{12}^z}B_{21}^z=I_k\text{，}\quad \widehat{B_{11}^z}B_{12}^z+\widehat{B_{12}^z}B_{22}^z=0\text{，}$$

$$\widehat{B_{21}^z}B_{11}^z+\widehat{B_{22}^z}B_{21}^z=0\text{，}\quad \widehat{B_{21}^z}B_{12}^z+\widehat{B_{22}^z}B_{22}^z=I_{n-n_1}$$

故有

$$\begin{bmatrix} \widehat{A_{11}^z}+\widehat{A_{12}^z}B_{13}^r & \widehat{A_{12}^z} & 0 \\ \widehat{A_{21}^z}+\widehat{A_{22}^z}B_{13}^r & \widehat{A_{22}^z} & 0 \\ 0 & 0 & I_{n-n_1} \end{bmatrix} \bar{M} \begin{bmatrix} I_{n-n_2} & 0 & 0 \\ 0 & \widehat{B_{11}^z} & \widehat{B_{12}^z} \\ 0 & \widehat{B_{21}^z} & \widehat{B_{22}^z} \end{bmatrix}$$

$$=\begin{bmatrix} \widehat{A_{11}^z}+\widehat{A_{12}^z}B_{13}^r & \widehat{A_{12}^z} & 0 \\ \widehat{A_{21}^z}+\widehat{A_{22}^z}B_{13}^r & \widehat{A_{22}^z} & 0 \\ 0 & 0 & I_{n-n_1} \end{bmatrix} \begin{bmatrix} A_{11}^z & A_{12}^z & -A_{13}^r \\ A_{21}^z-B_{13}^r & A_{22}^z+B_{11}^z & B_{12}^z-A_{23}^r \\ -B_{23}^r & B_{21}^z & B_{22}^z \end{bmatrix} \begin{bmatrix} I_{n-n_2} & 0 & 0 \\ 0 & \widehat{B_{11}^z} & \widehat{B_{12}^z} \\ 0 & \widehat{B_{21}^z} & \widehat{B_{22}^z} \end{bmatrix}$$

$$=\begin{bmatrix} \left(\widehat{A_{11}^z}+\widehat{A_{12}^z}B_{13}^r\right)A_{11}^z+\widehat{A_{12}^z}\left(A_{21}^z-B_{13}^r\right) & \left(\widehat{A_{11}^z}+\widehat{A_{12}^z}B_{13}^r\right)A_{12}^z+\widehat{A_{12}^z}\left(A_{22}^z+B_{11}^z\right) & -\left(\widehat{A_{11}^z}+\widehat{A_{12}^z}B_{13}^r\right)A_{13}^r+\widehat{A_{12}^z}\left(B_{12}^z-A_{23}^r\right) \\ \left(\widehat{A_{21}^z}+\widehat{A_{22}^z}B_{13}^r\right)A_{11}^z+\widehat{A_{22}^z}\left(A_{21}^z-B_{13}^r\right) & \left(\widehat{A_{21}^z}+\widehat{A_{22}^z}B_{13}^r\right)A_{12}^z+\widehat{A_{22}^z}\left(A_{22}^z+B_{11}^z\right) & -\left(\widehat{A_{21}^z}+\widehat{A_{22}^z}B_{13}^r\right)A_{13}^r+\widehat{A_{22}^z}\left(B_{12}^z-A_{23}^r\right) \\ -B_{23}^r & B_{21}^z & B_{22}^z \end{bmatrix} \begin{bmatrix} I_{n-n_2} & 0 & 0 \\ 0 & \widehat{B_{11}^z} & \widehat{B_{12}^z} \\ 0 & \widehat{B_{21}^z} & \widehat{B_{22}^z} \end{bmatrix}$$

$$
=\begin{bmatrix}
\left(\widehat{A}^z_{11}+\widehat{A}^z_{12}B^r_{13}\right)A^z_{11}+\widehat{A}^z_{12}\left(A^z_{21}-B^r_{13}\right) & \left(\left(\widehat{A}^z_{11}+\widehat{A}^z_{12}B^r_{13}\right)A^z_{12}+\widehat{A}^z_{12}\left(A^z_{22}+B^z_{11}\right)\right)\widehat{B}^z_{11}+\left(-\left(\widehat{A}^z_{11}+\widehat{A}^z_{12}B^r_{13}\right)A^r_{13}+\widehat{A}^z_{12}\left(B^z_{12}-A^r_{23}\right)\right)\widehat{B}^z_{21} & \left(\left(\widehat{A}^z_{11}+\widehat{A}^z_{12}B^r_{13}\right)A^z_{12}+\widehat{A}^z_{12}\left(A^z_{22}+B^z_{11}\right)\right)\widehat{B}^z_{12}+\left(-\left(\widehat{A}^z_{11}+\widehat{A}^z_{12}B^r_{13}\right)A^r_{13}+\widehat{A}^z_{12}\left(B^z_{12}-A^r_{23}\right)\right)\widehat{B}^z_{22} \\
\left(\widehat{A}^z_{21}+\widehat{A}^z_{22}B^r_{13}\right)A^z_{11}+\widehat{A}^z_{22}\left(A^z_{21}-B^r_{13}\right) & \left(\left(\widehat{A}^z_{21}+\widehat{A}^z_{22}B^r_{13}\right)A^z_{12}+\widehat{A}^z_{22}\left(A^z_{22}+B^z_{11}\right)\right)\widehat{B}^z_{11}+\left(-\left(\widehat{A}^z_{21}+\widehat{A}^z_{22}B^r_{13}\right)A^r_{13}+\widehat{A}^z_{22}\left(B^z_{12}-A^r_{23}\right)\right)\widehat{B}^z_{21} & \left(\left(\widehat{A}^z_{21}+\widehat{A}^z_{22}B^r_{13}\right)A^z_{12}+\widehat{A}^z_{22}\left(A^z_{22}+B^z_{11}\right)\right)\widehat{B}^z_{12}+\left(-\left(\widehat{A}^z_{21}+\widehat{A}^z_{22}B^r_{13}\right)A^r_{13}+\widehat{A}^z_{22}\left(B^z_{12}-A^r_{23}\right)\right)\widehat{B}^z_{22} \\
-B^r_{23} & B^z_{21}\widehat{B}^z_{11}+B^z_{22}\widehat{B}^z_{21} & B^z_{21}\widehat{B}^z_{12}+B^z_{22}\widehat{B}^z_{22}
\end{bmatrix}
$$

$$
=\begin{bmatrix}
\widehat{A}^z_{11}A^z_{11}+\widehat{A}^z_{12}B^r_{13}A^z_{11}+\widehat{A}^z_{12}A^z_{21}-\widehat{A}^z_{12}B^r_{13} & \widehat{A}^z_{11}A^z_{12}\widehat{B}^z_{11}+\widehat{A}^z_{12}B^r_{13}A^z_{12}\widehat{B}^z_{11}+\widehat{A}^z_{12}A^z_{22}\widehat{B}^z_{11}+\widehat{A}^z_{12}B^z_{11}\widehat{B}^z_{11}-\widehat{A}^z_{11}A^r_{13}\widehat{B}^z_{21}-\widehat{A}^z_{12}B^r_{13}A^r_{13}\widehat{B}^z_{21}+\widehat{A}^z_{12}B^z_{12}\widehat{B}^z_{21}-\widehat{A}^z_{12}A^r_{23}\widehat{B}^z_{21} & \widehat{A}^z_{11}A^z_{12}\widehat{B}^z_{12}+\widehat{A}^z_{12}B^r_{13}A^z_{12}\widehat{B}^z_{12}+\widehat{A}^z_{12}A^z_{22}\widehat{B}^z_{12}+\widehat{A}^z_{12}B^z_{11}\widehat{B}^z_{12}-\widehat{A}^z_{11}A^r_{13}\widehat{B}^z_{22}-\widehat{A}^z_{12}B^r_{13}A^r_{13}\widehat{B}^z_{22}+\widehat{A}^z_{12}B^z_{12}\widehat{B}^z_{22}-\widehat{A}^z_{12}A^r_{23}\widehat{B}^z_{22} \\
\widehat{A}^z_{21}A^z_{11}+\widehat{A}^z_{22}B^r_{13}A^z_{11}+A^z_{22}A^z_{21}-\widehat{A}^z_{22}\widehat{B}^r_{13} & \widehat{A}^z_{21}A^z_{12}\widehat{B}^z_{11}+\widehat{A}^z_{22}B^r_{13}A^z_{12}\widehat{B}^z_{11}+\widehat{A}^z_{22}A^z_{22}\widehat{B}^z_{11}+\widehat{A}^z_{22}B^z_{11}\widehat{B}^z_{11}-\widehat{A}^z_{21}A^r_{13}\widehat{B}^z_{21}-\widehat{A}^z_{22}B^r_{13}A^r_{13}\widehat{B}^z_{21}+\widehat{A}^z_{22}B^z_{12}\widehat{B}^z_{21}-\widehat{A}^z_{22}A^r_{23}\widehat{B}^z_{21} & \widehat{A}^z_{21}A^z_{12}\widehat{B}^z_{12}+\widehat{A}^z_{22}B^r_{13}A^z_{12}\widehat{B}^z_{12}+\widehat{A}^z_{22}A^z_{22}\widehat{B}^z_{12}+\widehat{A}^z_{22}B^z_{11}\widehat{B}^z_{12}-\widehat{A}^z_{21}A^r_{13}\widehat{B}^z_{22}-\widehat{A}^z_{22}B^r_{13}A^r_{13}\widehat{B}^z_{22}+\widehat{A}^z_{22}B^z_{12}\widehat{B}^z_{22}-\widehat{A}^z_{22}A^r_{23}\widehat{B}^z_{22} \\
-B^z_{23} & B^z_{21}\widehat{B}^z_{11}+B^z_{22}\widehat{B}^z_{21} & B^z_{21}\widehat{B}^z_{12}+B^z_{22}\widehat{B}^z_{22}
\end{bmatrix}
$$

$$
=\begin{bmatrix}
\widehat{A}^z_{11}A^z_{11}+\widehat{A}^z_{12}A^z_{21} & \left(\widehat{A}^z_{11}A^z_{12}+\widehat{A}^z_{12}A^z_{22}\right)\widehat{B}^z_{11}+\widehat{A}^z_{12}B^r_{13}\left(A^z_{12}\widehat{B}^z_{11}-A^r_{13}\widehat{B}^z_{21}\right)+\widehat{A}^z_{12}\left(B^z_{11}\widehat{B}^z_{11}+B^z_{12}\widehat{B}^z_{21}\right)-\left(\widehat{A}^z_{11}A^r_{13}+\widehat{A}^z_{12}A^r_{23}\right)\widehat{B}^z_{21} & \left(\widehat{A}^z_{11}A^z_{12}+\widehat{A}^z_{12}A^z_{22}\right)\widehat{B}^z_{12}+\widehat{A}^z_{12}B^r_{13}\left(A^z_{12}\widehat{B}^z_{12}-A^r_{13}\widehat{B}^z_{22}\right)+\widehat{A}^z_{12}\left(B^z_{11}\widehat{B}^z_{12}+B^z_{12}\widehat{B}^z_{22}\right)-\left(\widehat{A}^z_{11}A^r_{13}+\widehat{A}^z_{12}A^r_{23}\right)\widehat{B}^z_{22} \\
\widehat{A}^z_{21}A^z_{11}+\widehat{A}^z_{22}A^z_{21} & \left(\widehat{A}^z_{21}A^z_{12}+\widehat{A}^z_{22}A^z_{22}\right)\widehat{B}^z_{11}+\widehat{A}^z_{22}B^r_{13}\left(A^z_{12}\widehat{B}^z_{11}-A^r_{13}\widehat{B}^z_{21}\right)+\widehat{A}^z_{22}\left(B^z_{11}\widehat{B}^z_{11}+B^z_{12}\widehat{B}^z_{21}\right)-\left(\widehat{A}^z_{21}A^r_{13}+\widehat{A}^z_{22}A^r_{23}\right)\widehat{B}^z_{21} & \left(\widehat{A}^z_{21}A^z_{12}+\widehat{A}^z_{22}A^z_{22}\right)\widehat{B}^z_{12}+\widehat{A}^z_{22}B^r_{13}\left(A^z_{12}\widehat{B}^z_{12}-A^r_{13}\widehat{B}^z_{22}\right)+\widehat{A}^z_{22}\left(B^z_{11}\widehat{B}^z_{12}+B^z_{12}\widehat{B}^z_{22}\right)-\left(\widehat{A}^z_{21}A^r_{13}+\widehat{A}^z_{22}A^r_{23}\right)\widehat{B}^z_{22} \\
-B^z_{23} & 0 & I_{n-n_1}
\end{bmatrix}
$$

$$
=\begin{bmatrix}
I_{n-n_2} & \widehat{A}^z_{12}+\widehat{A}^z_{12}B^r_{13}\left(A^z_{12}\widehat{B}^z_{11}-A^r_{13}\widehat{B}^z_{21}\right)-\left(\widehat{A}^z_{11}A^r_{13}+\widehat{A}^z_{12}A^r_{23}\right)\widehat{B}^z_{21} & \widehat{A}^z_{12}B^r_{13}\left(A^z_{12}\widehat{B}^z_{12}-A^r_{13}\widehat{B}^z_{22}\right)-\left(\widehat{A}^z_{11}A^r_{13}+\widehat{A}^z_{12}A^r_{23}\right)\widehat{B}^z_{22} \\
0 & \widehat{H} & \widehat{B}^z_{12}+\widehat{A}^z_{22}B^r_{13}\left(A^z_{12}\widehat{B}^z_{12}-A^r_{13}\widehat{B}^z_{22}\right)-\left(\widehat{A}^z_{21}A^r_{13}+\widehat{A}^z_{22}A^r_{23}\right)\widehat{B}^z_{22} \\
-B^r_{23} & 0 & I_{n-n_1}
\end{bmatrix}
$$

由此式知，当 $\det\widehat{H}\neq 0$ 且 $B_{23}^r=O$ 时，$\bar{M}$ 为非奇异的。

现在讨论 *MB*-矩阵的子直和为 *MB*-矩阵的充分条件。

容易验证

$$\begin{bmatrix} I_{n-n_2} & F & Y \\ 0 & \hat{H} & Q \\ 0 & 0 & I_{n-n_1} \end{bmatrix}^{-1}\begin{bmatrix} I_{n-n_2} & C & D \\ O & \hat{H}^{-1} & E \\ O & O & I_{n-n_1} \end{bmatrix}=\begin{bmatrix} I_{n-n_2} & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & I_{n-n_1} \end{bmatrix}$$

其中

$$C=-\left[\widehat{A_{12}^z}+\widehat{A_{12}^z}B_{13}^r\left(A_{12}^z\widehat{B_{11}^z}-A_{13}^r\widehat{B_{21}^z}\right)-\left(\widehat{A_{11}^z}A_{13}^r+\widehat{A_{12}^z}A_{23}^r\right)\widehat{B_{21}^z}\right]\times\hat{H}^{-1}$$

$$\begin{aligned} D=&-\left[\widehat{A_{12}^z}B_{13}^r\left(A_{12}^z\widehat{B_{12}^z}-A_{13}^r\widehat{B_{22}^z}\right)-\left(\widehat{A_{11}^z}A_{13}^r+\widehat{A_{12}^z}A_{23}^r\right)\widehat{B_{22}^z}\right] \\ &+\left[\widehat{A_{12}^z}+\widehat{A_{12}^z}B_{13}^r\left(A_{12}^z\widehat{B_{11}^z}-A_{13}^r\widehat{B_{21}^z}\right)-\left(\widehat{A_{11}^z}A_{13}^r+\widehat{A_{12}^z}A_{23}^r\right)\widehat{B_{21}^z}\right] \\ &\times\hat{H}^{-1}\times\left[\widehat{B_{12}^z}+\widehat{A_{22}^z}B_{13}^r\left(A_{12}^z\widehat{B_{12}^z}-A_{13}^r\widehat{B_{22}^z}\right)-\left(\widehat{A_{21}^z}A_{13}^r+\widehat{A_{22}^z}A_{23}^r\right)\widehat{B_{22}^z}\right] \end{aligned}$$

$$E=-\hat{H}^{-1}\times\left[\widehat{B_{12}^z}+\widehat{A_{22}^z}B_{13}^r\left(A_{12}^z\widehat{B_{12}^z}-A_{13}^r\widehat{B_{22}^z}\right)-\left(\widehat{A_{21}^z}A_{13}^r+\widehat{A_{22}^z}A_{23}^r\right)\widehat{B_{22}^z}\right]$$

$$F=\widehat{A_{12}^z}+\widehat{A_{12}^z}B_{13}^r\left(A_{12}^z\widehat{B_{11}^z}-A_{13}^r\widehat{B_{21}^z}\right)-\left(\widehat{A_{11}^z}A_{13}^r+\widehat{A_{12}^z}A_{23}^r\right)\widehat{B_{21}^z}$$

$$Y=\widehat{A_{12}^z}B_{13}^r\left(A_{12}^z\widehat{B_{12}^z}-A_{13}^r\widehat{B_{22}^z}\right)-\left(\widehat{A_{11}^z}A_{13}^r+\widehat{A_{12}^z}A_{23}^r\right)\widehat{B_{22}^z}$$

$$Q=\widehat{B_{12}^z}+\widehat{A_{22}^z}B_{13}^r\left(A_{12}^z\widehat{B_{12}^z}-A_{13}^r\widehat{B_{22}^z}\right)-\left(\widehat{A_{21}^z}A_{13}^r+\widehat{A_{22}^z}A_{23}^r\right)\widehat{B_{22}^z}$$

从而有

$$\begin{aligned} &\begin{bmatrix} \widehat{A_{11}^z}+\widehat{A_{12}^z}B_{13}^r & \widehat{A_{12}^z} & 0 \\ \widehat{A_{21}^z}+\widehat{A_{22}^z}B_{13}^r & \widehat{A_{22}^z} & 0 \\ 0 & 0 & I_{n-n_1} \end{bmatrix}\bar{M}\begin{bmatrix} I_{n-n_2} & 0 & 0 \\ 0 & \widehat{B_{11}^z} & \widehat{B_{12}^z} \\ 0 & \widehat{B_{21}^z} & \widehat{B_{22}^z} \end{bmatrix} \\ &=\begin{bmatrix} I_{n-n_2} & \widehat{A_{12}^z}+\widehat{A_{12}^z}B_{13}^r\left(A_{12}^z\widehat{B_{11}^z}-A_{13}^r\widehat{B_{21}^z}\right)-\left(\widehat{A_{11}^z}A_{13}^r+\widehat{A_{12}^z}A_{23}^r\right)\widehat{B_{21}^z} & \widehat{A_{12}^z}B_{13}^r\left(A_{12}^z\widehat{B_{12}^z}-A_{13}^r\widehat{B_{22}^z}\right)-\left(\widehat{A_{11}^z}A_{13}^r+\widehat{A_{12}^z}A_{23}^r\right)\widehat{B_{22}^z} \\ 0 & \hat{H} & \widehat{B_{12}^z}+\widehat{A_{22}^z}B_{13}^r\left(A_{12}^z\widehat{B_{12}^z}-A_{13}^r\widehat{B_{22}^z}\right)-\left(\widehat{A_{21}^z}A_{13}^r+\widehat{A_{22}^z}A_{23}^r\right)\widehat{B_{22}^z} \\ -B_{23}^r & 0 & I_{n-n_1} \end{bmatrix} \end{aligned}$$

对上式两边同时取逆得：

$$\begin{bmatrix} I_{n-n_2} & 0 & 0 \\ 0 & \widehat{B_{11}^z} & \widehat{B_{12}^z} \\ 0 & \widehat{B_{21}^z} & \widehat{B_{22}^z} \end{bmatrix}^{-1} \left(\bar{M}\right)^{-1} \begin{bmatrix} \widehat{A_{11}^z}+\widehat{A_{12}^z}B_{13}^r & \widehat{A_{12}^z} & 0 \\ \widehat{A_{21}^z}+\widehat{A_{22}^z}B_{13}^r & \widehat{A_{22}^z} & 0 \\ 0 & 0 & I_{n-n_1} \end{bmatrix}^{-1}$$

$$= \left\{ \begin{bmatrix} I_{n-n_2} & \widehat{A_{12}^z}+\widehat{A_{12}^z}B_{13}^r\left(\widehat{A_{12}^z}\widehat{B_{11}^z}-A_{13}^r\widehat{B_{21}^z}\right)-\left(\widehat{A_{11}^z}A_{13}^r+\widehat{A_{12}^z}A_{23}^r\right)\widehat{B_{21}^z} & \widehat{A_{12}^z}B_{13}^r\left(\widehat{A_{12}^z}\widehat{B_{12}^z}-A_{13}^r\widehat{B_{22}^z}\right)-\left(\widehat{A_{11}^z}A_{13}^r+\widehat{A_{12}^z}A_{23}^r\right)\widehat{B_{22}^z} \\ 0 & \hat{H} & \widehat{B_{12}^z}+\widehat{A_{22}^z}B_{13}^r\left(\widehat{A_{12}^z}\widehat{B_{12}^z}-A_{13}^r\widehat{B_{22}^z}\right)-\left(\widehat{A_{21}^z}A_{13}^r+\widehat{A_{22}^z}A_{23}^r\right)\widehat{B_{22}^z} \\ -B_{23}^r & 0 & I_{n-n_1} \end{bmatrix} \right\}^{-1}$$

$$\begin{matrix} \widehat{A_{12}^z}B_{13}^r\left(\widehat{A_{12}^z}\widehat{B_{12}^z}-A_{13}^r\widehat{B_{22}^z}\right)-\left(\widehat{A_{11}^z}A_{13}^r+\widehat{A_{12}^z}A_{23}^r\right)\widehat{B_{22}^z} \\ \widehat{B_{12}^z}+\widehat{A_{22}^z}B_{13}^r\left(\widehat{A_{12}^z}\widehat{B_{12}^z}-A_{13}^r\widehat{B_{22}^z}\right)-\left(\widehat{A_{21}^z}A_{13}^r+\widehat{A_{22}^z}A_{23}^r\right)\widehat{B_{22}^z} \\ I_{n-n_1} \end{matrix} \Bigg]\Bigg\}^{-1}$$

进而可得：

$$\left(\bar{M}\right)^{-1} = \begin{bmatrix} I_{n-n_2} & O & O \\ O & \widehat{B_{11}^z} & \widehat{B_{12}^z} \\ O & \widehat{B_{21}^z} & \widehat{B_{22}^z} \end{bmatrix} \begin{bmatrix} I_{n-n_2} & C & D \\ O & \hat{H}^{-1} & E \\ O & O & I_{n-n_1} \end{bmatrix} \begin{bmatrix} \widehat{A_{11}^z}+\widehat{A_{12}^z}B_{13}^r & \widehat{A_{12}^z} & O \\ \widehat{A_{21}^z}+\widehat{A_{22}^z}B_{13}^r & \widehat{A_{22}^z} & O \\ O & O & I_{n-n_1} \end{bmatrix}$$

即

$$\left(\bar{M}\right)^{-1} = \begin{bmatrix} \widehat{A_{11}^z}+\widehat{A_{12}^z}B_{13}^r+C\left(\widehat{A_{21}^z}+\widehat{A_{22}^z}B_{13}^r\right) & \widehat{A_{12}^z}+C\widehat{A_{22}^z} & D \\ \widehat{B_{11}^z}\hat{H}\left(\widehat{A_{21}^z}+\widehat{A_{22}^z}B_{13}^r\right) & \widehat{B_{11}^z}\hat{H}^{-1}\widehat{A_{22}^z} & \widehat{B_{11}^z}E+\widehat{B_{12}^z} \\ \widehat{B_{21}^z}\hat{H}^{-1}\left(\widehat{A_{21}^z}+\widehat{A_{22}^z}B_{13}^r\right) & \widehat{B_{21}^z}\hat{H}^{-1}\widehat{A_{22}^z} & \widehat{B_{21}^z}E+\widehat{B_{22}^z} \end{bmatrix} \tag{5}$$

定理 3.2. 设 A^z, B^z 为 M-矩阵，

$$\det\hat{H} = \det\left\{\left(\widehat{B_{11}^z}+\widehat{A_{22}^z}\right)+\widehat{A_{22}^z}B_{13}^r\left(\widehat{A_{12}^z}\widehat{B_{11}^z}-A_{13}^r\widehat{B_{21}^z}\right)-\left(\widehat{A_{21}^z}A_{13}^r+\widehat{A_{22}^z}A_{23}^r\right)\widehat{B_{21}^z}\right\} \neq 0$$

且 $B_{23}^r = O$，则当(5)式中的每一个分块为非负矩阵时，$M = A \oplus_k B$ 为 MB-矩阵。

证明：当 $\bar{M}$ 满足上述条件时，由定义 2.1 知 $\bar{M}$ 为非奇异 M-矩阵。再由 $M^z \ge \bar{M}$ 及定理 2.1 的性质(2)知 M^z 为非奇异 M-矩阵，于是由定义 2.2 知 $M = A \oplus_k B$ 为 MB-矩阵。

下面对 $B_{23}^r \neq O$ 时的情况进行讨论。设 $A = \begin{bmatrix} D & E \\ F & G \end{bmatrix} \in R^{n\times n}$，其中 $D \in R^{k\times k}$ 且非奇异，矩阵 A 内 D 的

Schur 补记为 A/D，即 $A/D = G - FD^{-1}E$ 。同样如果 G 为非奇异时，$A/G = D - EG^{-1}F$ 。

定理 3.4. 设 A, B 为 MB-矩阵，若

$$\left(\tilde{D}\right)^{-1} \ge 0，\left(\tilde{G} - \tilde{F}\left(\tilde{D}\right)^{-1}\tilde{E}\right)^{-1} \ge 0$$

则 $M = A \oplus_k B$ 为 MB-矩阵，其中

$$\tilde{D} = A_{11}^z - A_{13}^r\left(B_{22}^z\right)^{-1}B_{23}^r，\ \tilde{E} = A_{12}^z + A_{13}^r\left(B_{22}^z\right)^{-1}B_{21}^z \le 0，$$

$$\tilde{F} = A_{21}^z - B_{13}^r + B_{12}^z\left(B_{22}^z\right)^{-1}B_{23}^r - A_{23}^r\left(B_{22}^z\right)^{-1}B_{23}^r \le 0，\ \tilde{G} = A_{22}^z + B_{11}^z - B_{12}^z\left(B_{22}^z\right)^{-1}B_{21}^z + A_{23}^r\left(B_{22}^z\right)^{-1}B_{21}^z$$

证明： 将 $\bar{M}$ 分块为 $\bar{M} = \begin{bmatrix} X & \tilde{Q} \\ \tilde{S} & T \end{bmatrix}$，其中 $X = \begin{pmatrix} A_{11}^z & A_{12}^z \\ A_{21}^z - B_{13}^r & A_{22}^z + B_{11}^z \end{pmatrix}$，$\tilde{Q} = \begin{bmatrix} -A_{13}^r & B_{12}^z - A_{23}^r \end{bmatrix}^{\mathrm{T}}$，$\tilde{S} = \begin{bmatrix} -B_{23}^r & B_{21}^z \end{bmatrix}$，$T = B_{22}^z$。由 A^z, B^z 为 M-矩阵，定义 2.2 及定理 2.1 得 $\tilde{Q} \le 0$，$T^{-1} \ge 0$，$\tilde{S} \le 0$。由引理 2.1 得 $\left(\bar{M}\right)^{-1} \ge 0$ 当且仅当 $\left(\bar{M}/T\right)^{-1} \ge 0$，其中 $\bar{M}/T = X - \tilde{Q}T^{-1}\tilde{S}$，即

$$\bar{M}/T = \begin{bmatrix} A_{11}^z - A_{13}^r\left(B_{22}^z\right)^{-1}B_{23}^r & A_{12}^z + A_{13}^r\left(B_{22}^z\right)^{-1}B_{21}^z \\ A_{21}^z - B_{13}^r + B_{12}^z\left(B_{22}^z\right)^{-1}B_{23}^r - A_{23}^r\left(B_{22}^z\right)^{-1}B_{23}^r & A_{22}^z + B_{11}^z - B_{12}^z\left(B_{22}^z\right)^{-1}B_{21}^z + A_{23}^r\left(B_{22}^z\right)^{-1}B_{21}^z \end{bmatrix}$$

令 $\bar{M}/T = \begin{bmatrix} \tilde{D} & \tilde{E} \\ \tilde{F} & \tilde{G} \end{bmatrix}$，则当 $\left(\tilde{D}\right)^{-1} \ge 0$ 时，由引理 2.1 得 $\left(\bar{M}/T\right)^{-1} \ge 0$ 当且仅当

$$\left(\left(\bar{M}/T\right)/\tilde{D}\right)^{-1} = \left(\tilde{G} - \tilde{F}\left(\tilde{D}\right)^{-1}\tilde{E}\right)^{-1} \ge 0$$

从而当 $\left(\tilde{G} - \tilde{F}\left(\tilde{D}\right)^{-1}\tilde{E}\right)^{-1} \ge 0$ 时，$\left(\bar{M}\right)^{-1} \ge 0$。又因 $\bar{M}$ 为 Z-矩阵，故 $\bar{M}$ 为 M-矩阵。由 $M^z \ge \bar{M}$ 且 M^z 为 Z-矩阵得，M^z 为 M-矩阵，再由定义 2.2 得 $M = A \oplus_k B$ 为 MB-矩阵。

下面我们将讨论 A, B 均为 MB-矩阵且 $A_{22}^z = B_{11}^z$ 的特殊情况，定理 3.7 给出了它们的子直和仍为 MB-矩阵的充分条件。

定理 3.7. 设 A, B 为 MB-矩阵且 $A_{22}^z = B_{11}^z$，若 C 为非奇异 M-矩阵，则 $M = A \oplus_k B$ 为 MB-矩阵，其中

$$C = \begin{bmatrix} A_{11}^z & A_{12}^z & -A_{13}^r \\ A_{21}^z - B_{13}^r & A_{22}^z & B_{12}^z - A_{23}^r \\ -B_{23}^r & B_{21}^z & B_{22}^z \end{bmatrix}$$

证明： 构造一个 Z-矩阵 $T \in R^{n\times n}$

$$T = \begin{bmatrix} A_{11}^z & 2A_{12}^z & -A_{13}^r \\ A_{21}^z - B_{13}^r & 2A_{22}^z & B_{12}^z - A_{23}^r \\ -B_{23}^r & 2B_{21}^z & B_{22}^z \end{bmatrix}$$

则 $T = C\,diag\left(I, 2I, I\right)$ 且 $T^{-1} = diag\left(I, (1/2)I, I\right)C^{-1} \ge O$，$T$ 是一个非奇异的 M-矩阵。此时

$$\bar{M} = \begin{bmatrix} A_{11}^z & A_{12}^z & -A_{13}^r \\ A_{21}^z - B_{13}^r & 2A_{22}^z & B_{12}^z - A_{23}^r \\ -B_{23}^r & B_{21}^z & B_{22}^z \end{bmatrix}$$

故 $\bar{M} \ge T$，于是 $\bar{M}$ 为非奇异的 M-矩阵。由 $M^z \ge \bar{M}$ 得 M^z 也为非奇异的 M-矩阵。

故由定义 2.2 得 $M = A\oplus_k B$ 为 MB-矩阵。

例 3.3. 设矩阵 A, B 及按定义 2.2 分裂为

$$A=\begin{bmatrix} 4 & . & 1 & 1 \\ . & . & . & . \\ -1 & . & 11 & -3 \\ -1 & . & -13 & 8 \end{bmatrix} = A^z + A^r = \begin{bmatrix} 3 & 0 & 0 \\ -1 & 11 & -3 \\ -1 & -13 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B=\begin{bmatrix} 11 & -2 & . & -1 \\ -13 & 8 & . & -3 \\ . & . & . & . \\ 1 & 1 & . & 3 \end{bmatrix}\begin{bmatrix} 11 & -3 & -1 \\ -13 & 8 & -3 \\ 1 & 1 & 3 \end{bmatrix} = B^z + B^r = \begin{bmatrix} 11 & -3 & -1 \\ -13 & 8 & -3 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

按定理 3.7 取

$$C=\begin{bmatrix} 3 & 0 & 0 & -1 \\ -1 & 11 & -3 & -1 \\ -1 & -13 & 8 & -3 \\ -1 & 0 & 0 & 2 \end{bmatrix}$$

容易验证 A, B 为 MB-矩阵和 C 为非奇异 M-矩阵，则 A, B 的子直和

$$M = A\oplus_2 B = \begin{bmatrix} 4 & 1 & 1 & 0 \\ -1 & 22 & -6 & -1 \\ -1 & -26 & 16 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

由定理 3.7 得 $M = A\oplus_2 B$ 为 MB-矩阵。

致　谢

由衷地感谢我的导师李耀堂教授在学习和科研上对我的指导和帮助。

参考文献 (References)

[1] Fallat, S.M. and Johnson, C.R. (1999) Sub-Direct Sums and Positivity Classes of Matrices. *Linear Algebra & Its Applications*, **288**, 149－173. https://doi.org/10.1016/S0024-3795(98)10194-5

[2] Bru, R., Pedroche, F. and Szyld, D.B. (2005) Subdirect Sums of Nonsingular M-Matrices and of Their Inverse. *Electronic Journal of Linear Algebra Ela*, **13**, 162－174. https://doi.org/10.13001/1081-3810.1159

[3] Zhu, Y., Huang, T.Z. and Liu, J. (2009) Subdirect Sums of H-matrices. *International Journal of Nonlinear Science*, **1**, 50-58.

[4] Zhu, Y. and Huang, T.Z. (2007) Subdirect Sum of Doubly Diagonally Dominant Matrices. *Electronic Journal of Linear Algebra Ela*, **16**, 171-182. https://doi.org/10.13001/1081-3810.1192

[5] 骆毅，李耀堂. MB-矩阵的子直和[J]. 应用数学进展, 2017, 6(3): 338-347.

[6] Berman, A. and Plemmons, R.J. (1994) Nonnegative Matrix in the Mathematical Sciences. SIAM Publisher, Philadelphia.

[7] Pena JM. (2003) On an Alternative to Gershgorin Circles and Ovals of Cassini. *Numerische Mathematik*, **95**, 337-345. https://doi.org/10.1007/s00211-002-0427-8

[8] Jose, S. and Sivakumar, K.C. (2013) On Inverse-Positivity of Sub-Direct Sums of Matrices. *Linear Algebra and Its Applications*, **439**, 1670-1677. https://doi.org/10.1016/j.laa.2013.05.001

知网检索的两种方式：

1. 打开知网页面 http://kns.cnki.net/kns/brief/result.aspx?dbPrefix=WWJD
   下拉列表框选择：[ISSN]，输入期刊 ISSN：2160-7583，即可查询
2. 打开知网首页 http://cnki.net/
   左侧“国际文献总库”进入，输入文章标题，即可查询

投稿请点击：http://www.hanspub.org/Submission.aspx
期刊邮箱：pm@hanspub.org