

The Rank of Eigenvalue of a Four-Order Ordinary Differential Operator under Coupled Boundary Condition

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Abstract

In this paper, a class of four-order ordinary differential operators with coupled boundary conditions is discussed. By using the Lagrange equation of the operator's corresponding formula, the relation between the rank of the eigenvalue of the problem and the zero point weight of an entire function is obtained.

Keywords

Eigenvalue, Eigenfunction, Rank, Ordinary Differential Operators

某耦合边界条件下四阶常微分算子特征值的秩

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摘要

本文讨论了一类具有耦合边界条件的四阶常微分算子, 利用该算子相应算式的拉格朗日等式, 得到问题特征值的秩与某整函数的零点重数之间的联系。

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关键词

特征值, 特征函数, 秩, 常微分算子

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1. 问题的提出

在微分算子的领域中, 许多学者对特征值问题进行了研究, 如文献[1]-[7], 这些文献的研究方法均为将特征值问题的讨论转化为一个整函数零点的讨论, 本文研究了一类具有耦合边界条件的四阶微分算子, 得到问题特征值的秩与某整函数的零点重数之间的联系。

记 $L \equiv -\frac{d^4}{dx^4} + q(x)$, $x \in (0, 1)$, $q(x) \in C[0, 1]$ 。考虑以下耦合边界条件下的特征值问题:

$$Ly = -y^{(4)} + q(x)y = \lambda y \quad (1)$$

$$y(0) = y'(1), y'(0) = y''(1), y''(0) = y^{(3)}(1), y^{(3)}(0) = y(1) \quad (2)$$

设 $Q_i(x, \lambda)$ ($i=1, 2, 3, 4$) 是四个满足(1)式的解, 它们分别在 $x=0$ 处满足

$$\begin{pmatrix} Q_1(0, \lambda) \\ Q_1'(0, \lambda) \\ Q_1''(0, \lambda) \\ Q_1^{(3)}(0, \lambda) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} Q_2(0, \lambda) \\ Q_2'(0, \lambda) \\ Q_2''(0, \lambda) \\ Q_2^{(3)}(0, \lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} Q_3(0, \lambda) \\ Q_3'(0, \lambda) \\ Q_3''(0, \lambda) \\ Q_3^{(3)}(0, \lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} Q_4(0, \lambda) \\ Q_4'(0, \lambda) \\ Q_4''(0, \lambda) \\ Q_4^{(3)}(0, \lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

显然, Wronsky 行列式 $W[Q_1(x), Q_2(x), Q_3(x), Q_4(x)] = 1$, $Q_1(x), Q_2(x), Q_3(x), Q_4(x)$ 是(1)的四个线性无关的解, 且可以构成(1)的基础解系。

设 $y(x) = \sum_{i=1}^4 c_i Q_i(x, \lambda_0)$ 是问题(1) (2)的特征函数, 其中 λ_0 是特征值, 则 $y(x)$ 满足(2)式, 即:

$$\begin{cases} c_1 = c_1 Q_1'(1, \lambda_0) + c_2 Q_2'(1, \lambda_0) + c_3 Q_3'(1, \lambda_0) + c_4 Q_4'(1, \lambda_0) \\ c_2 = c_1 Q_1''(1, \lambda_0) + c_2 Q_2''(1, \lambda_0) + c_3 Q_3''(1, \lambda_0) + c_4 Q_4''(1, \lambda_0) \\ c_3 = c_1 Q_1^{(3)}(1, \lambda_0) + c_2 Q_2^{(3)}(1, \lambda_0) + c_3 Q_3^{(3)}(1, \lambda_0) + c_4 Q_4^{(3)}(1, \lambda_0) \\ c_4 = c_1 Q_1(1, \lambda_0) + c_2 Q_2(1, \lambda_0) + c_3 Q_3(1, \lambda_0) + c_4 Q_4(1, \lambda_0) \end{cases} \quad (3)$$

方程组(3)的系数行列式为 $k(\lambda_0)$, 即:

$$k(\lambda_0) = \begin{vmatrix} Q_1'(1, \lambda_0) - 1 & Q_1''(1, \lambda_0) & Q_1^{(3)}(1, \lambda_0) & Q_1(1, \lambda_0) \\ Q_2'(1, \lambda_0) & Q_2''(1, \lambda_0) - 1 & Q_2^{(3)}(1, \lambda_0) & Q_2(1, \lambda_0) \\ Q_3'(1, \lambda_0) & Q_3''(1, \lambda_0) & Q_3^{(3)}(1, \lambda_0) - 1 & Q_3(1, \lambda_0) \\ Q_4'(1, \lambda_0) & Q_4''(1, \lambda_0) & Q_4^{(3)}(1, \lambda_0) & Q_4(1, \lambda_0) - 1 \end{vmatrix} \quad (4)$$

方程组(3)可以看作是未知数 c_1, c_2, c_3, c_4 的方程组, 由特征函数的定义知, 此方程组存在非零解, 故系数行列式 $k(\lambda_0) = 0$ 。类似于文献[1]中命题 1, 2 的证明, 我们可以得到如下两个命题。

命题 1: 记

$$\Omega(\lambda) = \begin{pmatrix} Q_1'(1, \lambda_0) - 1 & Q_1''(1, \lambda_0) & Q_1^{(3)}(1, \lambda_0) & Q_1(1, \lambda_0) \\ Q_2'(1, \lambda_0) & Q_2''(1, \lambda_0) - 1 & Q_2^{(3)}(1, \lambda_0) & Q_2(1, \lambda_0) \\ Q_3'(1, \lambda_0) & Q_3''(1, \lambda_0) & Q_3^{(3)}(1, \lambda_0) - 1 & Q_3(1, \lambda_0) \\ Q_4'(1, \lambda_0) & Q_4''(1, \lambda_0) & Q_4^{(3)}(1, \lambda_0) & Q_4(1, \lambda_0) - 1 \end{pmatrix}$$

显然, $k(\lambda) = |\Omega(\lambda)|$ 。 $R(\lambda)$ 是特征值 λ_0 的秩, 则以下两结论成立:

- i) λ_0 为问题(1)(2)的特征值 $\Leftrightarrow k(\lambda_0) = 0$ 。
- ii) $R(\lambda_0) + R(\Omega(\lambda_0)) = 4$ 。

命题 2: 设 $H_i(x, \lambda) (i=1, 2, 3, 4)$ 是四个满足(1)式的解, 记为

$$\begin{pmatrix} H_1(x, \lambda) \\ H_2(x, \lambda) \\ H_3(x, \lambda) \\ H_4(x, \lambda) \end{pmatrix} = A \begin{pmatrix} Q_1(x, \lambda) \\ Q_2(x, \lambda) \\ Q_3(x, \lambda) \\ Q_4(x, \lambda) \end{pmatrix}, A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}. \tag{5}$$

记 $T_i = (a_{i1}, a_{i2}, a_{i3}, a_{i4})^T, i=1, 2, 3, 4$, 其中 $T_i \neq 0$ 。如果 $k(\lambda_0) = 0$ 且 $\Omega^T(\lambda)T_i = 0$, 则:

- i) $H_i(x, \lambda_0)$ 是问题(1)(2)的特征函数, 则满足

$$\begin{cases} H_i(0, \lambda_0) = H_i'(1, \lambda_0) = a_{i1} \\ H_i'(0, \lambda_0) = H_i''(1, \lambda_0) = a_{i2} \\ H_i''(0, \lambda_0) = H_i^{(3)}(1, \lambda_0) = a_{i3} \\ H_i^{(3)}(0, \lambda_0) = H_i(1, \lambda_0) = a_{i4} \end{cases} \quad (i=1, 2, 3, 4)$$

- ii) H_1, H_2, H_3, H_4 的极大无关组个数为 $R(A)$ 。

引理 1: 算式 $L \equiv -\frac{d^4}{dx^4} + q(x)$ 有以下的 Lagrange 恒等式

$$\int_a^b g(Lf) dx - \int_a^b f(Lg) dx = \left[\begin{vmatrix} f & g \\ f''' & g''' \end{vmatrix} - \begin{vmatrix} f' & g' \\ f'' & g'' \end{vmatrix} \right]_a^b$$

引理 2: 设 $H_i(x, \lambda_0) (i=1, 2, 3, 4)$ 是(5)式所给出的特征函数。 λ_0 是 $k(\lambda)$ 的零点, 则有:

$$\Omega(\lambda_0)B = \begin{pmatrix} -a_{12} - a_{14} & -a_{22} - a_{24} & -a_{32} - a_{34} & -a_{42} - a_{44} \\ a_{11} + a_{13} & a_{21} + a_{23} & a_{31} + a_{33} & a_{41} + a_{43} \\ -a_{12} - a_{14} & -a_{22} - a_{24} & -a_{32} - a_{34} & -a_{42} - a_{44} \\ a_{11} + a_{13} & a_{21} + a_{23} & a_{31} + a_{33} & a_{41} + a_{43} \end{pmatrix}$$

其中

$$B = \begin{pmatrix} a_{12} & a_{22} & a_{32} & a_{42} \\ -a_{11} & -a_{21} & -a_{31} & -a_{41} \\ a_{14} & a_{24} & a_{34} & a_{44} \\ -a_{13} & -a_{23} & -a_{33} & -a_{43} \end{pmatrix}$$

证明: 利用引理 1 有

$$\begin{aligned}
 0 &= \int_0^1 Q_j(x, \lambda_0)(LH_i(x, \lambda_0)) dx - \int_0^1 (LQ_j(x, \lambda_0))H_i(x, \lambda_0) dx \\
 &= \left[\begin{array}{cc} H_i(x, \lambda_0) & Q_j(x, \lambda_0) \\ H_i^{(3)}(x, \lambda_0) & Q_j^{(3)}(x, \lambda_0) \end{array} \right]_0^1 - \left[\begin{array}{cc} H_i'(x, \lambda_0) & Q_j'(x, \lambda_0) \\ H_i''(x, \lambda_0) & Q_j''(x, \lambda_0) \end{array} \right]_0^1 \\
 &= H_i(1, \lambda_0)Q_j^{(3)}(1, \lambda_0) - H_i^{(3)}(1, \lambda_0)Q_j(1, \lambda_0) - H_i(0, \lambda_0)Q_j^{(3)}(0, \lambda_0) + H_i^{(3)}(0, \lambda_0)Q_j(0, \lambda_0) \\
 &\quad - H_i'(1, \lambda_0)Q_j''(1, \lambda_0) + H_i''(1, \lambda_0)Q_j'(1, \lambda_0) + H_i'(0, \lambda_0)Q_j''(0, \lambda_0) - H_i''(0, \lambda_0)Q_j'(0, \lambda_0) \\
 &= a_{i4}Q_j^{(3)}(1, \lambda_0) - a_{i3}Q_j(1, \lambda_0) - a_{i1}Q_j^{(3)}(0, \lambda_0) + a_{i4}Q_j(0, \lambda_0) - a_{i1}Q_j''(1, \lambda_0) \\
 &\quad + a_{i2}Q_j'(1, \lambda_0) + a_{i2}Q_j''(0, \lambda_0) - a_{i3}Q_j'(0, \lambda_0)
 \end{aligned}$$

分别取 $i, j=1, 2, 3, 4$, 即得结论。

引理 3: 设 $H_i(x, \lambda_0)$ 是(5)式所给出的特征函数 ($i=1, 2, 3, 4$), λ_0 是特征值。简记 $H_i(x, \lambda_0) = H_i$, 记一元函数 $H_i^{(j-1)}(1, \lambda)$ 关于 λ 求导后在 $\lambda = \lambda_0$ 处的取值为 $\dot{H}_i^{(j-1)}(1, \lambda_0)$ ($i, j=1, 2, 3, 4$), 记两个函数 H_i, H_j 的内积为 $\langle H_i, H_j \rangle$, 其定义为

$$\langle H_i, H_j \rangle = \int_0^1 H_i(x)H_j(x) dx,$$

这样可以得到 16 个恒等式, 写成矩阵形式如下:

$$\begin{aligned}
 &\begin{pmatrix} \langle H_1, H_1 \rangle & \langle H_1, H_2 \rangle & \langle H_1, H_3 \rangle & \langle H_1, H_4 \rangle \\ \langle H_2, H_1 \rangle & \langle H_2, H_2 \rangle & \langle H_2, H_3 \rangle & \langle H_2, H_4 \rangle \\ \langle H_3, H_1 \rangle & \langle H_3, H_2 \rangle & \langle H_3, H_3 \rangle & \langle H_3, H_4 \rangle \\ \langle H_4, H_1 \rangle & \langle H_4, H_2 \rangle & \langle H_4, H_3 \rangle & \langle H_4, H_4 \rangle \end{pmatrix} \\
 &= \begin{pmatrix} \dot{H}'_1 & \dot{H}''_1 & \dot{H}^{(3)}_1 & \dot{H}_1 \\ \dot{H}'_2 & \dot{H}''_2 & \dot{H}^{(3)}_2 & \dot{H}_2 \\ \dot{H}'_3 & \dot{H}''_3 & \dot{H}^{(3)}_3 & \dot{H}_3 \\ \dot{H}'_4 & \dot{H}''_4 & \dot{H}^{(3)}_4 & \dot{H}_4 \end{pmatrix} \begin{pmatrix} -a_{12} & -a_{22} & -a_{32} & -a_{42} \\ a_{11} & a_{21} & a_{31} & a_{41} \\ -a_{14} & -a_{24} & -a_{34} & -a_{44} \\ a_{13} & a_{23} & a_{33} & a_{43} \end{pmatrix}
 \end{aligned}$$

证明:

$$\begin{aligned}
 \langle H_i, H_j \rangle &= \lim_{\lambda \rightarrow \lambda_0} \frac{1}{\lambda - \lambda_0} (\lambda - \lambda_0) \int_0^1 H_i(x, \lambda)H_j(x, \lambda_0) dx \\
 &= \lim_{\lambda \rightarrow \lambda_0} \frac{1}{\lambda - \lambda_0} \left[\int_0^1 H_j(x, \lambda_0)(LH_i(x, \lambda)) dx - \int_0^1 H_i(x, \lambda)(LH_j(x, \lambda_0)) dx \right] \\
 &= \lim_{\lambda \rightarrow \lambda_0} \frac{1}{\lambda - \lambda_0} \left\{ \left[\begin{array}{cc} H_i(x, \lambda) & H_j(x, \lambda_0) \\ H_i^{(3)}(x, \lambda) & H_j^{(3)}(x, \lambda_0) \end{array} \right]_0^1 - \left[\begin{array}{cc} H_i'(x, \lambda) & H_j'(x, \lambda_0) \\ H_i''(x, \lambda) & H_j''(x, \lambda_0) \end{array} \right]_0^1 \right\} \\
 &= \lim_{\lambda \rightarrow \lambda_0} \frac{1}{\lambda - \lambda_0} \left\{ \left[H_i(1, \lambda)H_j''(1, \lambda_0) - H_j(1, \lambda_0)H_i''(1, \lambda) - H_i(0, \lambda)H_j'''(0, \lambda_0) + H_j(0, \lambda_0)H_i'''(0, \lambda) \right] \right. \\
 &\quad \left. + \left[-H_i'(1, \lambda)H_j''(1, \lambda_0) + H_j'(1, \lambda_0)H_i''(1, \lambda) + H_i'(0, \lambda)H_j''(0, \lambda_0) - H_j'(0, \lambda_0)H_i''(0, \lambda) \right] \right\} \\
 &= \lim_{\lambda \rightarrow \lambda_0} \frac{1}{\lambda - \lambda_0} \left[a_{j3}H_i(1, \lambda) - a_{j4}H_i^{(3)}(1, \lambda) - a_{j2}H_i'(1, \lambda) + a_{j1}H_i''(1, \lambda) + h \right]
 \end{aligned}$$

其中 $h = -a_{i1}a_{j4} + a_{i4}a_{j1} + a_{i2}a_{j3} - a_{i3}a_{j2}$, 接着利用洛必达法则, 可得

$$\langle H_i, H_j \rangle = a_{j3} \dot{H}_i(1, \lambda_0) - a_{j4} \dot{H}_i^{(3)}(1, \lambda_0) - a_{j2} \dot{H}_i'(1, \lambda_0) + a_{j1} \dot{H}_i''(1, \lambda_0)$$

当 i, j 分别取 1, 2, 3, 4 时, 就可以得到定理的结论。

2. 主要结果与证明

定理 1: 设 λ_0 为问题(1)(2)的特征值, 其秩记为 $R(\lambda_0)$, 则

- 1) $R(\lambda_0) = 4$ 的充要条件是 $k(\lambda_0) = k'(\lambda_0) = k''(\lambda_0) = k^{(3)}(\lambda_0) = 0$ 且 $k^{(4)}(\lambda_0) \neq 0$ 。
- 2) 当满足条件 $a_{i2} + a_{i4} = 0$ 或 $a_{i1} + a_{i3} = 0$ ($i = 1, 2, 3, 4$) 时, $R(\lambda_0) = 3$ 的充要条件是 $k(\lambda_0) = k'(\lambda_0) = k''(\lambda_0) = 0, k^{(3)}(\lambda_0) \neq 0$
- 3) 当满足条件 $a_{i2} + a_{i4} = 0$ 和 $a_{i1} + a_{i3} = 0$ ($i = 1, 2, 3, 4$) 时, $R(\lambda_0) = 2$ 的充要条件是 $k(\lambda_0) = k'(\lambda_0) = 0, k''(\lambda_0) \neq 0$
- 4) 当满足条件 $a_{i2} + a_{i4} = 0$ 和 $a_{i1} + a_{i3} = 0$ ($i = 1, 2, 3, 4$) 时, $R(\lambda_0) = 1$ 的充要条件是 $k(\lambda_0) = 0, k'(\lambda_0) \neq 0$

证明: (1)~(4)的充分性显然成立。下面证明必要性。

- 1) $R(A) = 4$, λ_0 对应 4 个线性无关的特征函数 $H_i(x, \lambda_0), i = 1, 2, 3, 4$ 。

$$|A|k(\lambda_0) = |A\Omega(\lambda_0)| = \begin{vmatrix} H_1'(1, \lambda_0) - a_{11} & H_1''(1, \lambda_0) - a_{12} & H_1'''(1, \lambda_0) - a_{13} & H_1(1, \lambda_0) - a_{14} \\ H_2'(1, \lambda_0) - a_{21} & H_2''(1, \lambda_0) - a_{22} & H_2'''(1, \lambda_0) - a_{23} & H_2(1, \lambda_0) - a_{24} \\ H_3'(1, \lambda_0) - a_{31} & H_3''(1, \lambda_0) - a_{32} & H_3'''(1, \lambda_0) - a_{33} & H_3(1, \lambda_0) - a_{34} \\ H_4'(1, \lambda_0) - a_{41} & H_4''(1, \lambda_0) - a_{42} & H_4'''(1, \lambda_0) - a_{43} & H_4(1, \lambda_0) - a_{44} \end{vmatrix} = 0$$

因为 $H_i'(1, \lambda_0) = a_{i1}, H_i''(1, \lambda_0) = a_{i2}, H_i'''(1, \lambda_0) = a_{i3}, H_i(1, \lambda_0) = a_{i4}$, 且 $|A| \neq 0$, 故 $k(\lambda_0) = 0$ 。

又因为求导后展开的每个行列式中至少有一行未被求导, 从而行列式 $|A|k'(\lambda_0) = |A|k''(\lambda_0) = |A|k'''(\lambda_0) = 0$ 。

同理可得 $k'(\lambda_0) = k''(\lambda_0) = k'''(\lambda_0) = 0$ 。

对 $|A|k(\lambda)$ 关于 λ 求四阶导数后, 在 $\lambda = \lambda_0$ 处只剩一项。

$$|A|k^{(4)}(\lambda_0) = \begin{vmatrix} \dot{H}_1'(1, \lambda_0) & \dot{H}_1''(1, \lambda_0) & \dot{H}_1^{(3)}(1, \lambda_0) & \dot{H}_1(1, \lambda_0) \\ \dot{H}_2'(1, \lambda_0) & \dot{H}_2''(1, \lambda_0) & \dot{H}_2^{(3)}(1, \lambda_0) & \dot{H}_2(1, \lambda_0) \\ \dot{H}_3'(1, \lambda_0) & \dot{H}_3''(1, \lambda_0) & \dot{H}_3^{(3)}(1, \lambda_0) & \dot{H}_3(1, \lambda_0) \\ \dot{H}_4'(1, \lambda_0) & \dot{H}_4''(1, \lambda_0) & \dot{H}_4^{(3)}(1, \lambda_0) & \dot{H}_4(1, \lambda_0) \end{vmatrix}$$

$$|A|k^{(4)}(\lambda_0)|B| = \begin{vmatrix} \langle H_1, H_1 \rangle & \langle H_1, H_2 \rangle & \langle H_1, H_3 \rangle & \langle H_1, H_4 \rangle \\ \langle H_2, H_1 \rangle & \langle H_2, H_2 \rangle & \langle H_2, H_3 \rangle & \langle H_2, H_4 \rangle \\ \langle H_3, H_1 \rangle & \langle H_3, H_2 \rangle & \langle H_3, H_3 \rangle & \langle H_3, H_4 \rangle \\ \langle H_4, H_1 \rangle & \langle H_4, H_2 \rangle & \langle H_4, H_3 \rangle & \langle H_4, H_4 \rangle \end{vmatrix}$$

上式右端是 Crame 行列式, 故由文献[2]知, $|A|k^{(4)}(\lambda_0)|B| \neq 0$, 故 $k^{(4)}(\lambda_0) \neq 0$ 。

- 2) 先证在 $a_{i2} + a_{i4} = 0$ 条件下的结论, 另一种情况类似证明。

$R(A) = 3$, 可以生成 3 个线性无关的特征函数, 不妨取为:

$$\begin{pmatrix} H_1 \\ H_2 \\ H_4 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} Q_1(1, \lambda_0) \\ Q_2(1, \lambda_0) \\ Q_3(1, \lambda_0) \\ Q_4(1, \lambda_0) \end{pmatrix}$$

则右端 3×4 阶矩阵至少存在一个三阶子式非零, 不妨取为:

$$\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} \neq 0$$

此行列式展开的六项中至少有一项非零, 为讨论方便取为 $a_{11}a_{22}a_{44} \neq 0$ 。

$$k(\lambda) = \frac{1}{a_{11}a_{22}a_{44}} \begin{vmatrix} a_{11}[Q_1'(1, \lambda) - 1] & a_{11}Q_1''(1, \lambda) & a_{11}Q_1^{(3)}(1, \lambda) & a_{11}Q_1(1, \lambda) \\ a_{22}Q_2'(1, \lambda) & a_{22}[Q_2''(1, \lambda) - 1] & a_{22}Q_2^{(3)}(1, \lambda) & a_{22}Q_2(1, \lambda) \\ Q_3'(1, \lambda) & Q_3''(1, \lambda) & Q_3^{(3)}(1, \lambda) - 1 & Q_3(1, \lambda) \\ a_{44}Q_4'(1, \lambda) & a_{44}Q_4''(1, \lambda) & a_{44}Q_4^{(3)}(1, \lambda) & a_{44}(Q_4(1, \lambda) - 1) \end{vmatrix}$$

经过初等变换得

$$k(\lambda) = \frac{1}{a_{11}a_{22}a_{44}} \begin{vmatrix} H_1'(1, \lambda) - a_{11} & H_1''(1, \lambda) - a_{12} & H_1^{(3)}(1, \lambda) - a_{13} & H_1(1, \lambda) - a_{14} \\ H_2'(1, \lambda) - a_{21} & H_2''(1, \lambda) - a_{22} & H_2^{(3)}(1, \lambda) - a_{23} & H_2(1, \lambda) - a_{24} \\ Q_3'(1, \lambda) & Q_3''(1, \lambda) & Q_3^{(3)}(1, \lambda) - 1 & Q_3(1, \lambda) \\ H_4'(1, \lambda) - a_{41} & H_4''(1, \lambda) - a_{42} & H_4^{(3)}(1, \lambda) - a_{43} & H_4(1, \lambda) - a_{44} \end{vmatrix}$$

显然 $k(\lambda_0) = 0$, $k'(\lambda_0) = 0$, $k''(\lambda_0) = 0$ 。

$$k^{(3)}(\lambda_0) = \frac{1}{a_{11}a_{22}a_{44}} \begin{vmatrix} \dot{H}_1'(1, \lambda_0) & \dot{H}_1''(1, \lambda_0) & \dot{H}_1^{(3)}(1, \lambda_0) & \dot{H}_1(1, \lambda_0) \\ \dot{H}_2'(1, \lambda_0) & \dot{H}_2''(1, \lambda_0) & \dot{H}_2^{(3)}(1, \lambda_0) & \dot{H}_2(1, \lambda_0) \\ Q_3'(1, \lambda_0) & Q_3''(1, \lambda_0) & Q_3^{(3)}(1, \lambda_0) - 1 & Q_3(1, \lambda_0) \\ \dot{H}_4'(1, \lambda_0) & \dot{H}_4''(1, \lambda_0) & \dot{H}_4^{(3)}(1, \lambda_0) & \dot{H}_4(1, \lambda_0) \end{vmatrix}$$

经过等值变换得

$$k^{(3)}(\lambda_0) = \frac{1}{a_{11}^2 a_{22}^2 a_{44}^2} \begin{vmatrix} \langle H_1, H_1 \rangle & \langle H_1, H_2 \rangle & \langle H_1, H_4 \rangle & \dot{H}_1(1, \lambda_0) \\ \langle H_2, H_1 \rangle & \langle H_2, H_2 \rangle & \langle H_2, H_4 \rangle & \dot{H}_2(1, \lambda_0) \\ -a_{12} - a_{14} & -a_{22} - a_{24} & -a_{42} - a_{44} & Q_3(1, \lambda_0) \\ \langle H_4, H_1 \rangle & \langle H_4, H_2 \rangle & \langle H_4, H_4 \rangle & \dot{H}_4(1, \lambda_0) \end{vmatrix}$$

当 $a_{i2} + a_{i4} = 0$ ($i=1, 2, 4$) 时,

$$k^{(3)}(\lambda_0) = -\frac{Q_3(1, \lambda_0)}{a_{11}^2 a_{22}^2 a_{33}^2} \begin{vmatrix} \langle H_1, H_1 \rangle & \langle H_1, H_2 \rangle & \langle H_1, H_4 \rangle \\ \langle H_2, H_1 \rangle & \langle H_2, H_2 \rangle & \langle H_2, H_4 \rangle \\ \langle H_4, H_1 \rangle & \langle H_4, H_2 \rangle & \langle H_4, H_4 \rangle \end{vmatrix}$$

下面说明 $Q_3(1, \lambda_0) \neq 0$, 由以上假设知:

$$W[H_1, H_2, H_4, Q_3] = \begin{vmatrix} a_{11} & a_{21} & a_{41} & 0 \\ a_{12} & a_{22} & a_{42} & 0 \\ a_{13} & a_{23} & a_{43} & 1 \\ a_{14} & a_{24} & a_{44} & 0 \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{21} & a_{41} \\ a_{12} & a_{22} & a_{42} \\ a_{14} & a_{24} & a_{44} \end{vmatrix} \neq 0$$

因此 H_1, H_2, H_4, Q_3 是线性无关的。

另一方面, 由引理 2 以及 $a_{i2} + a_{i4} = 0$ ($i=1, 2, 3, 4$) 知,

$$\begin{pmatrix} a_{12} & -a_{11} & a_{14} & -a_{13} \\ a_{22} & -a_{21} & a_{24} & -a_{23} \\ a_{32} & -a_{31} & a_{34} & -a_{33} \\ a_{42} & -a_{41} & a_{44} & -a_{43} \end{pmatrix} \begin{pmatrix} Q_3'(1, \lambda_0) \\ Q_3''(1, \lambda_0) \\ Q_3^{(3)}(1, \lambda_0) - 1 \\ Q_3(1, \lambda_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

假设 $Q_3(1, \lambda_0) = 0$ ，由上式可得到

$$\begin{pmatrix} a_{12} & -a_{11} & a_{14} \\ a_{22} & -a_{21} & a_{24} \\ a_{42} & -a_{41} & a_{44} \end{pmatrix} \begin{pmatrix} Q_3'(1, \lambda_0) \\ Q_3''(1, \lambda_0) \\ Q_3^{(3)}(1, \lambda_0) - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

因为系数行列式不等于零，故有 $Q_3'(1, \lambda_0) = Q_3''(1, \lambda_0) = 0$ ， $Q_3^{(3)}(1, \lambda_0) = 1$ ，即 $Q_3(1, \lambda_0)$ 也是特征函数，这与已知矛盾，故 $k^{(3)}(\lambda_0) \neq 0$ 。

(3) (4) 仿照 $R(\lambda_0) = 3$ 的方法，进行证明。

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