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The Neumann Boundary Value for the Singularly Perturbed Problem with Degenerate Equation Having Double Root

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Abstract

The Neumann boundary value for the second order singularly perturbed problem with degenerate question having double root is studied in this paper. Under certain conditions, the method of modified boundary layer function is used to construct the formal asymptotic expansion of the solution and obtain the more precise boundary layer functions which decay exponentially. The existence and uniformly valid approximation of solutions are obtained by upper and lower solutions method.

Keywords

Singular Perturbation, Double Root, Boundary Layer Method, Asymptotic Solution

具有重退化根的奇摄动方程的Neumann边值问题

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摘 要

研究具有重退化根的二阶奇摄动方程 Neumann 边值问题。在一定条件下利用修正的边界层函数法构造出解的形式渐近展开式，获得更为精确的以指数形式衰减的边界层函数。最后利用上下解方法得到了解

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的存在性和一致有效估计。

关键词

奇摄动，重根，边界层函数法，渐近解

1. 引言

对于非线性奇摄动二阶方程的 Dirichlet 边值问题[1]：

$$\varepsilon^2 \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = f(u,x,\varepsilon), 0 < x < A,$$

$$u(0,\varepsilon) = u^0, \quad u(A,\varepsilon) = u^1.$$

其中 $\varepsilon, 0 < \varepsilon \ll 1$ 为小参数，f 是充分光滑函数。近几年来，俄罗斯学派 Vasil'eva、Butuzov 等用边界层函数法[2] [3] [4] [5]研究了该退化方程 $f(u,x,0)=0$ 具有重根时解的存在性和有效性，国内学者倪明康等研究了一类具有代数衰减边界层的解的性态[5]，Songlin Chen 等更进一步研究了退化方程具有三重根时的情形[6]。本文受其启发，研究如下带有 Neumann 边值条件的二阶奇摄动方程：

$$\varepsilon^2 \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = f(u,x,\varepsilon), 0 < x < A, \tag{1}$$

$$u(0,\varepsilon) = u^0, \quad u'(A,\varepsilon) = 0. \tag{2}$$

首先给出如下假设：

[H1] 假设函数 $f(u,x,\varepsilon) \in C^\infty([0,A] \times R)$，且 $f(u,x,\varepsilon)$ 关于其变量充分光滑.

[H2] 假设函数 $f(u,x,\varepsilon)$ 有如下形式：

$$f(u,x,\varepsilon) = h(x)(u-\varphi(x))^2 - \varepsilon f_1(u,x,\varepsilon). \tag{3}$$

$$\text{且 } h(x) > 0,\ 0 < x < A \tag{4}$$

表达式(3)可知退化方程有二重根 $u=\varphi(x)$，为了简化计算，不妨设 $h(0)=1$。

[H3] 假设

$$\overline{f}_1(x) := f_1(\varphi(x),x,0) > 0. \tag{5}$$

2. 外部解的渐近展开

设原问题具有下述的形式外部解

$$U(x,\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^{\frac{i}{2}} U_i(x). \tag{6}$$

其中 $U(x,\varepsilon)$ 满足

$$\varepsilon^2 \frac{d^2 U}{dx^2} = f(U,x,\varepsilon). \tag{7}$$

将(6)代入(3)，(7)，

$$\varepsilon^2 \frac{d^2}{dx^2}\left(U_0+\varepsilon^{\frac{1}{2}}U_1+\cdots\right) = h(x)\left(U_0+\varepsilon^{\frac{1}{2}}U_1+\cdots-\varphi(x)\right)^2+\varepsilon f_1\left(U_0+\varepsilon^{\frac{1}{2}}U_1+\cdots\right). \tag{8}$$

比较上式两边ε的同次幂系数

$$\varepsilon^0: 0 = h(x)\left(U_0-\varphi(x)\right)^2; \quad \varepsilon^1: 0 = h(x)U_1^2-\overline{f_1}(x);$$

$$\varepsilon^{\frac{3}{2}}: \quad 0 = h(x)(2U_1U_2)-\overline{f_{1y}}(x)U_1; \quad \varepsilon^i: \quad \left(2h(x)U_1\right)U_i = F_i(x).$$

其中$F_i(x)(i\neq 0)$为逐次已知的函数，由上式可得到解U_i：

$$U_0=\varphi(x); \quad U_1=\left[h^{-1}(x)\overline{f_1}(x)\right]^{\frac{1}{2}}; \quad U_2=\frac{1}{2}h^{-1}(x)\overline{f_{1y}}(x); \quad U_i=\left[2h(x)U_1\right]^{-1}F_i(x).$$

因此得出形式外部解(6)。将它代入方程(1)中，将其右端在以点$(U_0,x,0)$为展开中心进行泰勒级数展开，合并ε的同次幂项之后即得

$$\varepsilon^2 \frac{d^2}{dx^2}\left(U_0+\varepsilon^{\frac{1}{2}}U_1+\cdots+\varepsilon^{\frac{i}{2}}U_i+\cdots\right) = f(U_0,x,0)+\varepsilon\left(\overline{f_u}U_1+f_1\right)+\cdots+\varepsilon^{\frac{i}{2}}\left(\overline{f_u}U_i+f_i\right)+\cdots$$

$$\overset{def}{=} \overline{f_0}+\varepsilon\overline{f_1}+\cdots+\varepsilon^{\frac{i}{2}}\overline{f_i}+\cdots$$

则可得出估计

$$\varepsilon^2 \frac{d^2 \overset{(n)}{U}}{dx^2}-f\left(\overset{(n)}{U},x,\varepsilon\right) = O\left(\varepsilon^{n/2+1}\right), \quad 0\le x\le A \tag{9}$$

其中

$$\overset{(n)}{U} \equiv \sum_{i=0}^{n}\varepsilon^{\frac{i}{2}}U_i.$$

3. 边界层校正项

[H4] 假设

$$u^o > \phi(0). \tag{10}$$

引入多重尺度变量，首先在x=0处附近构造边界层校正项，设为

$$V(\tau,\varepsilon)=\sum_{i=0}^{\infty}\varepsilon^{\frac{i}{4}}V_i(\tau,\varepsilon) \ \ \tau=\frac{x}{\varepsilon}. \tag{11}$$

则边值问题(1)~(3)的合成渐近解为

$$u(x,\varepsilon)=U(x,\varepsilon)+V(\tau,\varepsilon). \tag{12}$$

令$V(\tau,\varepsilon)$满足

$$\left(U(\varepsilon\tau,\varepsilon)+V(\tau,\varepsilon),\varepsilon\tau,\varepsilon\right)-f\left(U(\varepsilon\tau,\varepsilon),\varepsilon\tau,\varepsilon\right)=:Vf. \tag{13}$$

将(11)代入(3)，(13)

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(V_0+\varepsilon^{\frac{1}{4}}V_1+\cdots\right)=\left(h(0)+\varepsilon\tau h'(0)+\cdots\right)\left(V_0+\varepsilon^{\frac{1}{4}}V_1+\cdots\right)^2$$
$$+2\varepsilon^{\frac{1}{2}}\left(U_1(0)+\varepsilon\tau U_1'(0)+\cdots+\sqrt{\varepsilon}U_2(0)+\cdots\right)\left(V_0+\varepsilon^{\frac{1}{4}}V_1+\cdots\right)-\varepsilon Vf_1. \tag{14}$$

由(2)，(6)，(11)，(12)得

$$u(0,\varepsilon)=U_0(0)+\varepsilon^{\frac{1}{2}}U_1(0)+\varepsilon U_2(0)+\cdots+V_0(0)+\varepsilon^{\frac{1}{4}}V_1(0)+\varepsilon^{\frac{1}{2}}V_2(0)+\cdots=u^0. \tag{15}$$

比较 ε 的同次幂，得如下形式的 $V_i(0,\varepsilon)$

$$V_0(0)=u^0-\varphi(0)=:p,$$
$$V_i(0,\tau)=\begin{cases}-U_{i/2}(0) & i\text{为偶数}\\ 0 & i\text{为奇数}\end{cases},\qquad i=2,3,\cdots \tag{16}$$

考虑到 $V_i(\tau)$ 为边界层函数，所以要求

$$V_i(\infty,\varepsilon)=0. \tag{17}$$

比较(14)式两边 ε 的同次幂系数：

$$\varepsilon^0:\quad \frac{\mathrm{d}^2V_0}{\mathrm{d}\tau^2}=V_0^2.\quad V_0(0)=p,\ V_0(\infty)=0. \tag{18}$$

解得

$$V_0(\tau)=\frac{p}{\left(\sqrt{p/6}\tau+1\right)^2}. \tag{19}$$

对于任意的 $\tau>0$, 当 $\tau\to\infty$ 时，首次形式近似 V_0 具有幂率衰减的性态

$$V_0(\tau)=O\left(\frac{1}{(1+\tau)^2}\right).$$

为了得到边界层函数的正确刻画，我们将对上述满足 $V_0(\tau)$ 的方程进行修正。在(18)式的右端添加一个含未知元 V_0 的小量修正项，得

$$\frac{\mathrm{d}^2V_0}{\mathrm{d}\tau^2}=V_0^2(\tau)+2\sqrt{\varepsilon}U_1(0)V_0.\quad V_0(0)=p,\ V_0(\infty)=0. \tag{20}$$

将问题(20)化为一阶方程

$$\frac{\mathrm{d}V_0}{\mathrm{d}\tau}=-\left(\frac{2}{3}V_0+2\sqrt{\varepsilon}U_1(0)\right)^{\frac{1}{2}}V_0. \tag{21}$$

由分离变量法，可以求得

$$V_0(\tau)=\frac{12\sqrt{\varepsilon}U_1(0)c_0\exp\left(-\varepsilon^{\frac{1}{4}}k_0\tau\right)}{\left[1-c_0\exp\left(-\varepsilon^{\frac{1}{4}}k_0\tau\right)\right]^2}. \tag{22}$$

其中

$$c_0=\frac{\sqrt{p/3+\sqrt{\varepsilon}U_1(0)}-\sqrt{\varepsilon}U_1(0)}{\sqrt{p/3+\sqrt{\varepsilon}U_1(0)}+\sqrt{\varepsilon}U_1(0)},\quad k_0=\sqrt{2U_1(0)}. \tag{22'}$$

当$u^0>\varphi(0)$时，$0<c_0<1$，$V_0(\tau)>0$。此时

$$\left|V_0(\tau)\right|\le c\exp\left(-\varepsilon^{\frac{1}{4}}k_0\tau\right). \tag{23}$$

为了后面的估计，定义

$$V_\kappa(\tau)=c\exp\left(-\varepsilon^{\frac{1}{4}}\kappa\tau\right),\quad 0\le\kappa<k_0.$$

由于

$$\frac{V_0}{V_\kappa}=\frac{c_1\exp\left(-\varepsilon^{1/4}k_0\tau\right)}{c_2\exp\left(-\varepsilon^{1/4}\kappa\tau\right)}=c\exp\left(-\varepsilon^{1/4}(k_0-\kappa)\tau\right).$$

零次近似$V_0(\tau)$满足估计

$$\left|V_0(\tau)\right|\le cV_\kappa.$$

继续要求$V_1(\tau)$满足：

$$\varepsilon^{\frac{1}{4}}:\quad \frac{d^2V_1}{d\tau^2}=2\left(V_0+\sqrt{\varepsilon}U_1(0)\right)V_1.\quad V_1(0)=0,\quad V_1(\infty)=0. \tag{24}$$

解得：

$$V_1(\tau)=0. \tag{25}$$

类似地，我们得出V_2的方程：

$$\frac{d^2V_2}{d\tau^2}=2\left(V_0+\sqrt{\varepsilon}U_1(0)\right)V_2+v_2(\tau,\varepsilon).\quad V_2(0,\tau)=-U_1(0),\quad V_2(\infty)=0. \tag{26}$$

其中

$$v_2(\tau)=\sqrt{\varepsilon}\left[2U_2(0)V_0(\tau)-\overline{f_{1U}}(0)V_0(\tau)\right]\le c\left[V_\kappa^2(\tau)+\sqrt{\varepsilon}V_\kappa(\tau)\right].$$

问题(26)是一个非齐次二阶微分方程，求出其显示解为

$$V_2(\tau)=\phi(\tau)\phi^{-1}(0)V_2(0)+\phi(\tau)\int_0^\tau\phi^{-2}(\sigma)d\sigma\int_\infty^\sigma\phi(s)v_2(s,\varepsilon)ds.$$

这里

$$\phi(\tau)=\frac{dV_0}{d\tau}=-\left(\frac{2}{3}V_0+2\sqrt{\varepsilon}U_1(0)\right)^{\frac{1}{2}}V_0, \tag{27}$$

且$\phi(\tau,\varepsilon)$满足估计

$$\left|\phi(\tau,\varepsilon)\right|\le c\left[V_0(\tau,\varepsilon)+\sqrt{\varepsilon}\right]^{\frac{1}{2}}V_0(\tau,\varepsilon). \tag{28}$$

$$\begin{aligned}\left|V_2(\tau)\right| &= \left|\phi(\tau)\phi^{-1}(0)V_2(0)\right| + \left|\phi(\tau)\int_0^\tau \phi^{-2}(\sigma)\mathrm{d}\sigma\int_\infty^\sigma \phi(s)v_2(s)\mathrm{d}s\right| \\ &\le cV_\kappa(\tau) + c\left[V_0(\tau)+\sqrt{\varepsilon}\right]^{\frac{1}{2}} V_0(\tau)\int_0^\tau \left[V_0^3(\sigma)+\sqrt{\varepsilon}V_0^2(\sigma)\right]^{-1}\mathrm{d}\sigma \times \int_\infty^\sigma \frac{\mathrm{d}V_0}{\mathrm{d}\tau}\left|v_2(s)\right|\mathrm{d}s \\ &\le cV_\kappa(\tau) + \left[V_0(\tau)+\sqrt{\varepsilon}\right]^{\frac{1}{2}} V_0(\tau)\int_0^\tau \left[V_0^3(\sigma)+\sqrt{\varepsilon}V_0^2(\sigma)\right]^{-1} \times V_0(\sigma)\left[V_\kappa^2(\sigma)+\sqrt{\varepsilon}V_\kappa(\sigma)\right]\mathrm{d}\sigma \\ &\le cV_\kappa + c\left[\sqrt{V_0(\tau)}+\varepsilon^{\frac{1}{4}}\right]V_0(\tau)\int_0^\tau \frac{V_\kappa(\sigma)+\sqrt{\varepsilon}}{V_0(\sigma)+\sqrt{\varepsilon}}\cdot\frac{V_\kappa}{V_0}\mathrm{d}\sigma \\ &\le cV_\kappa + \left|c\left[\sqrt{V_0(\tau)}+\varepsilon^{\frac{1}{4}}\right]\tau V_\kappa\right| \le cV_\kappa\end{aligned}$$

类似地，我们还可以得出其余边界层项的估计

$$\left|V_i(\tau)\right| \le c\,V_\kappa,\quad i=3,4,\cdots \tag{29}$$

将(11)代入方程(1)中得出估计

$$\frac{\mathrm{d}^2 \overset{(n)}{V}}{\mathrm{d}\tau^2} - \overset{(n)}{V} f = O\left(\varepsilon^{(n+1)/4}\right)V_\kappa,\quad \tau>0. \tag{30}$$

其中

$$\overset{(n)}{V} \equiv \sum_{i=0}^{n}\varepsilon^{\frac{i}{4}}V_i.$$

接下来在 x = A 处附近构造边界层校正项，设为

$$P(\xi,\varepsilon) = \varepsilon^{\frac{3}{4}}\sum_{i=0}^{\infty}\varepsilon^{\frac{i}{4}}P_i(\xi,\varepsilon),\quad \xi=\frac{A-x}{\varepsilon^{3/4}}. \tag{31}$$

则 $P(\xi,\varepsilon)$ 满足

$$\sqrt{\varepsilon}\frac{\mathrm{d}^2 P}{\mathrm{d}\xi^2} = f\left(U\left(A-\varepsilon^{3/4}\xi,\varepsilon\right)+P\left(A-\varepsilon^{3/4}\xi,\varepsilon\right),A-\varepsilon^{3/4}\xi,\varepsilon\right) - f\left(U\left(A-\varepsilon^{3/4}\xi,\varepsilon\right),A-\varepsilon^{3/4}\xi,\varepsilon\right) =: Pf. \tag{32}$$

对于 $i>0$，关于 $P_i(\xi)$ 的方程及定解条件为：

$$\frac{\mathrm{d}^2 P_i}{\mathrm{d}\xi^2} = 2h(A)U_1(A)P_i + \rho_i. \tag{33}$$

$$\frac{\mathrm{d}P_i}{\mathrm{d}\xi}(0,\xi) = \begin{cases} \dfrac{\mathrm{d}U_{i/2}}{\mathrm{d}x}(A,\xi) & i\text{为偶数} \\ 0 & i\text{为奇数} \end{cases},\quad i=1,2,\cdots \tag{34}$$

$$P_i(\infty)=0. \tag{35}$$

其中 p_i 是关于 $P_j(j<i)$ 的递归表达式。

特别地

$$\varepsilon^{\frac{5}{4}}:\quad \frac{\mathrm{d}^2 P_0}{\mathrm{d}\xi^2} = 2h(A)U_1(A)P_0. \tag{36}$$

解得：

$$P_0(\xi)=-\frac{\mathrm{d}\varphi}{\mathrm{d}x}(A,\xi)\left[2h(A)U_1(A)\right]^{-\frac{1}{2}}\exp\left(-\left(2h(A)U_1(A)\right)^{\frac{1}{2}}\xi\right). \tag{37}$$

则 $P_0(\xi)$ 有如下指数估计

$$\left|P_0(\xi)\right|\le c_0\exp(-\kappa_0\xi).$$

同问题(30)，我们可逐次求出 $P_i(\xi)$

$$P_i(\xi)=\frac{\mathrm{d}P_i(0)}{\mathrm{d}\xi}\psi(\xi)\frac{\mathrm{d}\psi^{-1}(0)}{\mathrm{d}\xi}+\psi^{-1}(0)\int_0^{\infty}\psi(s)\rho_i(s,\varepsilon)\mathrm{d}s+\psi(\xi)\int_0^{\xi}\psi^{-2}(\sigma)\mathrm{d}\sigma\int_{\infty}^{\sigma}\psi(s)\rho_i(s,\varepsilon)\mathrm{d}s$$

这里

$$\psi(\xi)=\frac{\mathrm{d}P_0}{\mathrm{d}\xi}$$

不难看出，所有的 $P_i(\xi)$ 都有指数估计

$$\left|P_i(\xi)\right|\le c\exp(-\kappa\xi).$$

将(31)代入方程(1)中，得出估计

$$\sqrt{\varepsilon}\frac{\mathrm{d}^2\overset{(n)}{P}}{\mathrm{d}\xi^2}-\overset{(n)}{P}f=O\left(\varepsilon^{3/4+(n+1)/4}\right)\exp(-\kappa\xi),\quad \xi>0 \tag{38}$$

其中

$$\overset{(n)}{P}\equiv\varepsilon^{\frac{3}{4}}\sum_{i=0}^{n}\varepsilon^{\frac{i}{4}}P_{i.}$$

4. 形式解的一致有效性

定理：假设 H1~H4 成立，那么对于充分小的 $\varepsilon>0$，边值问题(1)，(2)存在解 $u(x,\varepsilon)$，且对于任意的自然数 $n\ge1$，有下列关系式成立：

$$u(x,\varepsilon)=u_n(x,\varepsilon)+O\left(\varepsilon^{n+1/2}\right) \tag{39}$$

这里

$$u_n=\sum_{i=0}^{2n}U_i(x)+\sum_{i=0}^{4n}\varepsilon^{i/4}V_i\left(\frac{x}{\varepsilon},\varepsilon\right)+\varepsilon^{3/4}\sum_{i=0}^{4n}\varepsilon^{i/4}P_i\left(\frac{A-x}{\varepsilon^{3/4}},\varepsilon\right)$$

证明：构造

$$\underline{u_n}(x,\varepsilon)=u_n(x,\varepsilon)-M\varepsilon^{n+1/2} \tag{40}$$

我们有

$$\begin{aligned}L_\varepsilon\underline{u_n}(x,\varepsilon)&:=\varepsilon^2\frac{\mathrm{d}^2\underline{u_n}}{\mathrm{d}x^2}-f\left(\underline{u_n},x,\varepsilon\right)=L_\varepsilon u_n+f(u_n,x,\varepsilon)-f\left(u_n-M\varepsilon^{n+1/2},x,\varepsilon\right)\\&=L_\varepsilon u_n+h(x)\left[\left(u_n-\varphi(x)\right)^2-\left(u_n-M\varepsilon^{n+1/2}-\varphi(x)\right)^2\right]+\varepsilon\left[f\left(u_n-M\varepsilon^{n+1/2},x,\varepsilon\right)-f(u_n,x,\varepsilon)\right]\\&=L_\varepsilon u_n+h(x)M\varepsilon^{n+1/2}\left[2\left(u_n-\varphi(x)\right)-M\varepsilon^{n+1/2}\right]-\varepsilon f_{1u}^{*}M\varepsilon^{n+1/2}\\&=O\left(\varepsilon^{n+1}\right)+O\left(\varepsilon^{n+1}\right)V_\kappa+2h(x)M\varepsilon^{n+1/2}\left[\left(\sqrt{\varepsilon}U_1(x)+V_0+\sqrt{\varepsilon}V_2+O\left(\varepsilon^{3/4}\right)\right)-M\varepsilon^{n+1/2}\right]-O\left(\varepsilon^{n+3/2}\right)M\\&\ge O\left(\varepsilon^{n+1}\right)+O\left(\varepsilon^{n+3/2}\right)+2h(x)M\varepsilon^{n+1}\left[U_1(x)+O\left(\varepsilon^{1/4}\right)-M\varepsilon^{n}\right]+2h(x)M\varepsilon^{n+1/2}\left(V_0+\sqrt{\varepsilon}V_2\right)\end{aligned}$$

由条件 H4，当 M 充分大时，可得

$$L_\varepsilon \underline{u_n}(x,\varepsilon) \geq 0.$$

类似地，构造函数

$$\overline{u_n}(x,\varepsilon) = u_n(x,\varepsilon) + M\varepsilon^{n+1/2} \tag{41}$$

可得

$$L_\varepsilon \overline{u_n}(x,\varepsilon) \leq 0.$$

则 $\overline{u_n}(x,\varepsilon)$，$\underline{u_n}(x,\varepsilon)$ 为边值问题(1)的上、下解。

由 Nagumo’s 定理[7]知，边值问题(1)存在解 $u(x,\varepsilon)$ 满足

$$\underline{u_n}(x,\varepsilon) \leq u(x,\varepsilon) \leq \overline{u_n}(x,\varepsilon), \qquad 0 \leq x \leq A.$$

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