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α-Bloch空间到β-Bloch空间的广义积分算子的差分

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摘 要

设φ, g是复平面$\mathbb{C}$中单位圆盘$\mathbb{D}$上的解析映射，且$\varphi(\mathbb{D}) \subset \mathbb{D}, n \in \mathbb{N}$。定义广义积分算子为

$$I_{g,\varphi}^{(n)} f(z) = \int_0^z f^{(n)}(\varphi(\zeta))g(\zeta)d\zeta.$$

本文旨在探究α-**Bloch**空间到β-**Bloch**空间上的广义积分算子差的有界性和紧性问题。

关键词

差分，广义积分算子，**Bloch**空间

Differences of Generalized Integration Operators from α-Bloch Spaces to β-Bloch Spaces

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Abstract

A generalized integration operator is defined by

$$I_{g,\varphi}^{(n)}f(z)=\int_0^z f^{(n)}(\varphi(\zeta))g(\zeta)d\zeta$$

induced by holomorphic maps g and φ of the unit disk $\mathbb{D}$, where $\varphi(\mathbb{D})\subset\mathbb{D}$ and n is a positive integer. In this paper, we investigate the boundedness and the compactness of the differences of two generalized integration operators from α-Bloch spaces to β-Bloch spaces.

Keywords

Differences, Generalized Integration Operator, Bloch Space

1. 引言

记$\mathbb{D}$为复平面$\mathbb{C}$中的单位开圆盘, $H(\mathbb{D})$为$\mathbb{D}$上的解析函数全体, $H^\infty(\mathbb{D})$ 为$\mathbb{D}$上的有界解析函数全体. 对$0<\alpha<\infty$, Bloch型空间(或α-Bloch空间)$\mathcal{B}^\alpha$是指所有满足

$$\|f\|_\alpha=\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)|<\infty$$

的解析函数全体. 易知, 在范数$\|f\|_{\mathcal{B}^\alpha}=|f(0)|+\|f\|_\alpha$意义下, $\mathcal{B}^\alpha$构成一个Banach 空间.

在本文中, 记$S(\mathbb{D})$为$\mathbb{D}$到自身的解析映射全体. 对$f\in H(\mathbb{D})$, 定义以$\varphi\in S(\mathbb{D})$为符号的复合算子为$C_\varphi f=f\circ\varphi$. 复合算子理论是近年来的研究热点之一, 特别是$\varphi$的性质与$C_\varphi$的关系问题. 对于经典解析函数空间上的复合算子理论, 详见文献 [1]和 [2].

设$g:\mathbb{D}\to\mathbb{C}$是单位开圆盘上的解析映射, 对$f\in H(\mathbb{D}), z\in\mathbb{D}$, 称算子

$$I_gf(z)=\int_0^z f(\zeta)g'(\zeta)d\zeta\ \ (z\in\mathbb{D})$$

为Riemann - Stieltjes算子(或广义Cesàro 算子). Ch. Pommerenke在文献 [3]中首次研究了Riemann - Stieltjes 算子, 其证明了I_g在H^2上有界当且仅当$g \in BMOA$. 文献 [4]和 [5]把该结论推广到了其他的$H^p(1 \le p < \infty)$空间, 其根据符号g还完全刻画了I_g在H^p的紧性和I_g在H^2上的Schatten类.

在本文里, 我们考虑积分算子

$$I_{g,\varphi}^{(n)} f(z) = \int_0^z f^{(n)}(\varphi(\zeta)) g(\zeta) d\zeta, \quad z \in \mathbb{D}.$$

称该算子为广义积分算子, 其在 [6]中首次引用并在文献 [7]和 [8]中得到进一步研究. 同时, 算子$I_{g,\varphi}^{(n)}$可以看成是由g所诱导的Rimann-Stieltjes 算子I_g 的推广. 事实上, 算子$I_{g,\varphi}^{(n)}$可以得到许多熟知的算子. 比如, 当$n = 1$时, $I_{g,\varphi}^{(n)}$就是S. Stević, S. Li, X. Zhu 和W. Yang在文献 [9–15]中所研究的积分算子. 当$n = 1, g(z) = \varphi'(z)$时, 则为复合算子$C_\varphi$. 若记$D$为微分算子, $n = m + 1, g(z) = \varphi'(z)$, 则为算子$C_\varphi D^m f(z) = f^{(m)}(\varphi(z)) - f^{(m)}(\varphi(0))$(见文献 [16–18]).

为研究H^2空间上复合算子集$\mathcal{C}(H^2)$的拓扑结构(算子范数拓扑意义下), 文献 [19]中首次探究了两个复合算子的差分性质. 随后, 不少研究者也刻画了(加权)复合算子的差分. MacCluer, Ohno和Zhao在 [20]中证明了$C_\varphi - C_\psi : H^\infty \to H^\infty$的紧性与$C_\varphi - C_\psi : \mathcal{B} \to H^\infty$ 的紧性等价. 同时, H^∞上的复合算子C_φ 和C_ψ在同一个连通分支上当且仅当$C_\varphi - C_\psi : \mathcal{B} \to H^\infty$有界. Hosokawa和Ohno 在 [21]中不仅对$\mathcal{B}$到H^∞上的两个加权复合算子差分的有界性和紧性给出了新的结果, 还给出了他们的本性范数. 到目前为止, 关于两个(加权)复合算子差分的文献有很多, 如 [22–25], 其他更多的文献在此不一一列出.

本文中, 我们将探究α-Bloch空间到β-Bloch空间上的两个广义积分算子差的有界性和紧性问题. 同时, 在此说明: 文中正常数C 在不同的地方取值不一定相同.

2. 一些引理

对$a \in \mathbb{D}$, 设σ_a为$\mathbb{D}$上把0映成a的自同构映射. 即, $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$, $z \in \mathbb{D}$. 同时, $\mathbb{D}$上的伪双曲距离记为

$$\rho(z, a) = |\sigma_a(z)| = \left| \frac{a - z}{1 - \bar{a}z} \right|, \quad z, a \in \mathbb{D}.$$

接下来, 我们给出文中主要结论证明时需要用到的几个结论.

引理1 **[26]** 对$\alpha > 0$, $f \in \mathcal{B}^\alpha$, 有

$$|f^{(n)}(z)| \le C \frac{\|f\|_{\mathcal{B}^\alpha}}{(1 - |z|^2)^{\alpha+n-1}},$$

其中C与f无关.

与([1],Pro.3.11)的证明类似, 可以得到下列引理2.

引理2 设$\varphi_1, \varphi_2 \in S(\mathbb{D})$, $g_1, g_2 \in H(\mathbb{D})$. 则$I_{g_1,\varphi_1}^{(n)} - I_{g_2,\varphi_2}^{(n)} : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ 是紧算子当且仅当$I_{g_1,\varphi_1}^{(n)} - I_{g_2,\varphi_2}^{(n)} : \mathcal{B}^\alpha \to \mathcal{B}^\beta$是有界算子, 且对$\mathcal{B}^\alpha$中的任意有界序列$\{f_k\}$, 当其在$\mathbb{D}$的任意紧子集上一致收敛于0时, 有$\|(I_{g_1,\varphi_1}^{(n)} - I_{g_2,\varphi_2}^{(n)}) f_k\|_{\mathcal{B}^\beta} \to 0$.

引理3 [27] 设$0 < \alpha < \infty$. 则对任意$z, w \in \mathbb{D}$, 存在常数$C > 0$, 使得对所有$f \in \mathcal{B}^\alpha$满足$\|f\|_{\mathcal{B}^\alpha} \le 1$时, 有

$$|(1-|z|^2)^{\alpha+n-1} f^{(n)}(z) - (1-|w|^2)^{\alpha+n-1} f^{(n)}(w)| \le C\rho(z,w).$$

引理4 [27] 对$z, w \in \mathbb{D}$且$a \neq 0$, 令

$$f_a(z) = \frac{1-|a|^2}{\alpha(\alpha+1)\cdots(\alpha+n-1)\bar{a}^n(1-\bar{a}z)^\alpha},$$

$$k_a(z) = \frac{1}{\alpha(\alpha+1)\cdots(\alpha+n)}\left(\frac{n(1-|a|^2)}{\bar{a}^{n+1}(1-\bar{a}z)^\alpha} + \alpha\cdot\frac{(a-z)(1-|a|^2)}{\bar{a}^n(1-\bar{a}z)^{\alpha+1}}\right).$$

则$f_a, k_a \in \mathcal{B}^\alpha$且

$$f_a^{(n)}(z) = \frac{1-|a|^2}{(1-\bar{a}z)^{\alpha+n}},$$

$$k_a^{(n)}(z) = \frac{(a-z)(1-|a|^2)}{(1-\bar{a}z)^{\alpha+n+1}}.$$

3. 主要结论

为了方便, 记

$$\mathfrak{D}_{\varphi,g}(z) := \frac{(1-|z|^2)^\beta g(z)}{(1-|\varphi(z)|^2)^{\alpha+n-1}}.$$

并令

$$I_1(z) = |\mathfrak{D}_{\varphi_1,g_1}(z)|\rho(\varphi_1(z),\varphi_2(z)),$$

$$I_2(z) = |\mathfrak{D}_{\varphi_2,g_2}(z)|\rho(\varphi_1(z),\varphi_2(z)),$$

$$I_3(z) = |\mathfrak{D}_{\varphi_1,g_1}(z) - \mathfrak{D}_{\varphi_2,g_2}(z)|.$$

定理1 设$\varphi_1, \varphi_2 \in S(\mathbb{D})$, $g_1, g_2 \in H(\mathbb{D})$且$n \in \mathbb{N}$. 则下述条件等价:

(i) $I_{g_1,\varphi_1}^{(n)} - I_{g_2,\varphi_2}^{(n)} : \mathcal{B}^\alpha \to \mathcal{B}^\beta$有界;

(ii) $\sup_{z\in\mathbb{D}} I_1(z) < \infty$ and $\sup_{z\in\mathbb{D}} I_3(z) < \infty$;

(iii) $\sup_{z\in\mathbb{D}} I_2(z) < \infty$ and $\sup_{z\in\mathbb{D}} I_3(z) < \infty$.

证明 $(i) \Rightarrow (ii)$. 假设$I_{g_1,\varphi_1}^{(n)} - I_{g_2,\varphi_2}^{(n)} : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ 有界. 对$a \in \mathbb{D}$且$a \neq 0$, 令

$$f_a(z) = \frac{1-|a|^2}{\alpha(\alpha+1)\cdots(\alpha+n-1)\bar{a}^n(1-\bar{a}z)^\alpha},$$

$$k_a(z) = \frac{1}{\alpha(\alpha+1)\cdots(\alpha+n)}\left(\frac{n(1-|a|^2)}{\bar{a}^{n+1}(1-\bar{a}z)^\alpha} + \alpha\cdot\frac{(a-z)(1-|a|^2)}{\bar{a}^n(1-\bar{a}z)^{\alpha+1}}\right).$$

则由引理4可知, $f_a, k_a \in \mathcal{B}^\alpha$. 于是对固定的$w \in \mathbb{D}$且满足$\varphi_1(w) \neq 0$, 有

$$\begin{aligned} \infty &> \|(I_{g_1,\varphi_1}^{(n)} - I_{g_2,\varphi_2}^{(n)})f_{\varphi_1(w)}\|_{\mathcal{B}^\beta} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|((I_{g_1,\varphi_1}^{(n)} - I_{g_2,\varphi_2}^{(n)})f_{\varphi_1(w)})'(z)| \\ &\geq \left|\frac{(1-|w|^2)^\beta g_1(w)(1-|\varphi_1(w)|^2)}{(1-|\varphi_1(w)|^2)^{\alpha+n}} - \frac{(1-|w|^2)^\beta g_2(w)(1-|\varphi_1(w)|^2)}{(1-\overline{\varphi_1(w)}\varphi_2(w))^{\alpha+n}}\right| \\ &\geq |\mathfrak{D}_{\varphi_1,g_1}(w)| - \left|\mathfrak{D}_{\varphi_2,g_2}(w)\frac{(1-|\varphi_1(w)|^2)(1-|\varphi_2(w)|^2)^{\alpha+n-1}}{(1-\overline{\varphi_1(w)}\varphi_2(w))^{\alpha+n}}\right|, \end{aligned} \tag{1}$$

和

$$\begin{aligned} \infty &> \|(I_{g_1,\varphi_1}^{(n)} - I_{g_2,\varphi_2}^{(n)})k_{\varphi_1(w)}\|_{\mathcal{B}^\beta} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|((I_{g_1,\varphi_1}^{(n)} - I_{g_2,\varphi_2}^{(n)})k_{\varphi_1(w)})'(z)| \\ &\geq \left|\frac{(1-|w|^2)^\beta g_2(w)(\varphi_1(w)-\varphi_2(w))(1-|\varphi_1(w)|^2)}{(1-\overline{\varphi_1(w)}\varphi_2(w))^{\alpha+n+1}}\right| \\ &= \left|\mathfrak{D}_{\varphi_2,g_2}(w)\frac{(1-|\varphi_1(w)|^2)(1-|\varphi_2(w)|^2)^{\alpha+n-1}}{(1-\overline{\varphi_1(w)}\varphi_2(w))^{\alpha+n}}\right| \cdot \rho(\varphi_1(w),\varphi_2(w)). \end{aligned} \tag{2}$$

(1)式两边同时乘以$\rho(\varphi_1(w), \varphi_2(w))$, 结合(2)可得

$$\sup_{w\in\mathbb{D}\setminus\mathbb{D}_1}|\mathfrak{D}_{\varphi_1,g_1}(w)|\rho(\varphi_1(w),\varphi_2(w)) < \infty, \tag{3}$$

其中$\mathbb{D}_1 = \{w \in \mathbb{D} : \varphi_1(w) = 0\}$.

同理可得

$$\sup_{w\in\mathbb{D}\setminus\mathbb{D}_2}|\mathfrak{D}_{\varphi_2,g_2}(w)|\rho(\varphi_1(w),\varphi_2(w)) < \infty, \tag{4}$$

其中$\mathbb{D}_2 = \{w \in \mathbb{D} : \varphi_2(w) = 0\}$.

另一方面, 由(1)可以得到

$$\begin{aligned} \infty &> \|(I_{g_1,\varphi_1}^{(n)} - I_{g_2,\varphi_2}^{(n)})f_{\varphi_1(w)}\|_{\mathcal{B}^\beta} \\ &\geq \left|\frac{(1-|w|^2)^\beta g_1(w)(1-|\varphi_1(w)|^2)}{(1-|\varphi_1(w)|^2)^{\alpha+n}} - \frac{(1-|w|^2)^\beta g_2(w)(1-|\varphi_1(w)|^2)}{(1-\overline{\varphi_1(w)}\varphi_2(w))^{\alpha+n}}\right| \\ &\geq |\mathfrak{D}_{\varphi_1,g_1}(w) - \mathfrak{D}_{\varphi_2,g_2}(w)| - |\mathfrak{D}_{\varphi_2,g_2}(w)| \cdot \left|1 - \frac{(1-|\varphi_1(w)|^2)(1-|\varphi_2(w)|^2)^{\alpha+n-1}}{(1-\overline{\varphi_1(w)}\varphi_2(w))^{\alpha+n}}\right| \\ &\geq C\left(|\mathfrak{D}_{\varphi_1,g_1}(w) - \mathfrak{D}_{\varphi_2,g_2}(w)| - |\mathfrak{D}_{\varphi_2,g_2}(w)|\rho(\varphi_1(w),\varphi_2(w))\right), \end{aligned} \tag{5}$$

从而

$$\sup_{w\in\mathbb{D}\setminus\{\mathbb{D}_1\cup\mathbb{D}_2\}}|\mathfrak{D}_{\varphi_1,g_1}(w) - \mathfrak{D}_{\varphi_2,g_2}(w)| < \infty. \tag{6}$$

当$\varphi_1(w)=\varphi_2(w)=0$时, 令$f_0(z)=\frac{z^n}{n!}$, 则由$I^{(n)}_{g_1,\varphi_1}-I^{(n)}_{g_2,\varphi_2}:\mathcal{B}^\alpha\to\mathcal{B}^\beta$ 的有界性可得

$$\begin{aligned}\sup_{w\in\mathbb{D}_1\cap\mathbb{D}_2}|\mathfrak{D}_{\varphi_1,g_1}(w)-\mathfrak{D}_{\varphi_2,g_2}(w)| &= \sup_{w\in\mathbb{D}_1\cap\mathbb{D}_2}\mu(|w|)|g_1(w)-g_2(w)|\\ &\le \|(I^{(n)}_{g_1,\varphi_1}-I^{(n)}_{g_2,\varphi_2})f_0\|_{\mathcal{B}^\beta}<\infty, \end{aligned}\tag{7}$$

$$\sup_{w\in\mathbb{D}_1\cap\mathbb{D}_2}|\mathfrak{D}_{\varphi_1,g_1}(w)|\rho(\varphi_1(w),\varphi_2(w))=0,\tag{8}$$

且

$$\sup_{w\in\mathbb{D}_1\cap\mathbb{D}_2}|\mathfrak{D}_{\varphi_2,g_2}(w)|\rho(\varphi_1(w),\varphi_2(w))=0.\tag{9}$$

当$\varphi_2(w)=0,\varphi(w)_1\neq 0$时, 令

$$P_{\varphi_1(w)}(z)=\frac{1}{\alpha\cdots(\alpha+n)}\left(\frac{n}{\overline{\varphi_1(w)}^{n+1}(1-\overline{\varphi_1(w)}z)^\alpha}+\frac{\alpha(\varphi_1(w)-z)}{\overline{\varphi_1(w)}^{n}(1-\overline{\varphi_1(w)}z)^{\alpha+1}}\right).$$

则

$$\begin{aligned}\infty &> \|(I^{(n)}_{g_1,\varphi_1}-I^{(n)}_{g_2,\varphi_2})P_{\varphi_1(w)}\|_{\mathcal{B}^\beta}=\sup_{z\in\mathbb{D}}|((I^{(n)}_{g_1,\varphi_1}-I^{(n)}_{g_2,\varphi_2})P_{\varphi_1(w)})'(z))|\\ &\ge \left|\frac{(1-|w|^2)^\beta g_2(w)(\varphi_1(w)-\varphi_2(w))}{(1-\overline{\varphi_1(w)}\varphi_2(w))^{\alpha+n+1}}\right|\\ &= (1-|w|^2)^\beta|\varphi_1(w)g_2(w)|\\ &= |\mathfrak{D}_{\varphi_2,g_2}(w)|\rho(\varphi_1(w),\varphi_2(w)),\end{aligned}$$

于是

$$\sup_{w\in\mathbb{D}_2\backslash\mathbb{D}_1}|\mathfrak{D}_{\varphi_2,g_2}(w)|\rho(\varphi_1(w),\varphi_2(w))<\infty.\tag{10}$$

因此, 结合(5)和(10)可以得到

$$\sup_{w\in\mathbb{D}_2\backslash\mathbb{D}_1}|\mathfrak{D}_{\varphi_1,g_1}(w)-\mathfrak{D}_{\varphi_2,g_2}(w)|<\infty.\tag{11}$$

同理可得, 当$\varphi_1(w)=0,\varphi(w)_2\neq 0$时, 有

$$\sup_{w\in\mathbb{D}_1\backslash\mathbb{D}_2}|\mathfrak{D}_{\varphi_1,g_1}(w)-\mathfrak{D}_{\varphi_2,g_2}(w)|<\infty\tag{12}$$

$$\sup_{w\in\mathbb{D}_1\backslash\mathbb{D}_2}|\mathfrak{D}_{\varphi_2,g_2}(w)|\rho(\varphi_1(w),\varphi_2(w))<\infty.\tag{13}$$

故而, 由(3), (8)和(13)可得$\sup_{z\in\mathbb{D}} I_1(z) < \infty$; 由(6), (7), (11)和(12)可得$\sup_{z\in\mathbb{D}} I_3(z) < \infty$.

$(ii) \Rightarrow (iii)$. 若(ii)成立, 则

$$\begin{aligned}\sup_{z\in\mathbb{D}} I_2(z) &= \sup_{z\in\mathbb{D}} |\mathfrak{D}_{\varphi_2,g_2}(z)|\rho(\varphi_1(z),\varphi_2(z)) \\ &\leq \sup_{z\in\mathbb{D}} |\mathfrak{D}_{\varphi_1,g_1}(z)|\rho(\varphi_1(z),\varphi_2(z)) \\ &\quad + \sup_{z\in\mathbb{D}} |\mathfrak{D}_{\varphi_1,g_1}(z) - \mathfrak{D}_{\varphi_2,g_2}(z)|\rho(\varphi_1(z),\varphi_2(z)) \\ &\leq \sup_{z\in\mathbb{D}} I_1(z) + \sup_{z\in\mathbb{D}} I_3(z) < \infty,\end{aligned}$$

从而(iii)也成立.

$(iii) \Rightarrow (i)$. 若(iii)成立, 则由引理1可得, 对$f\in\mathcal{B}^\alpha$且$\|f\|_{\mathcal{B}^\alpha}\leq 1$, 有

$$\begin{aligned}&\|(I^{(n)}_{g_1,\varphi_1} - I^{(n)}_{g_2,\varphi_2})f\|_{\mathcal{B}^\beta} \\ =\ & \sup_{z\in\mathbb{D}} |(1-|z|^2)^\beta f^{(n)}(\varphi_1(z))g_1(z) - (1-|z|^2)^\beta f^{(n)}(\varphi_2(z))g_2(z)| \\ =\ & \sup_{z\in\mathbb{D}} |\mathfrak{D}_{\varphi_1,g_1}(z)(1-|\varphi_1(z)|^2)^{\alpha+n-1} f^{(n)}(\varphi_1(z)) \\ & -\mathfrak{D}_{\varphi_2,g_2}(z)(1-|\varphi_2(z)|^2)^{\alpha+n-1} f^{(n)}(\varphi_2(z))| \\ \leq\ & \sup_{z\in\mathbb{D}} |\mathfrak{D}_{\varphi_1,g_1}(z) - \mathfrak{D}_{\varphi_2,g_2}(z)| \\ & + C\sup_{z\in\mathbb{D}} |\mathfrak{D}_{\varphi_2,g_2}(z)|\rho(\varphi_1(z),\varphi_2(z)) \\ <\ & \infty,\end{aligned}$$

所以$I^{(n)}_{g_1,\varphi_1} - I^{(n)}_{g_2,\varphi_2} : \mathcal{B}^\alpha \to \mathcal{B}^\beta$有界.

在讨论$I^{(n)}_{g_1,\varphi_1} - I^{(n)}_{g_2,\varphi_2} : F(p,q,s) \to \mathcal{B}_\mu$的紧性之前, 我们引入下列记号:

$$\Gamma(\varphi) = \{\{z_k\}\subset\mathbb{D} : |\varphi(z_k)| \to 1\},$$

$$D(g,\varphi) := \{\{z_k\}\subset\mathbb{D} : |\varphi(z_k)| \to 1, |\mathfrak{D}_{\varphi,g}(z_k)| \nrightarrow 0\}.$$

定理2 设φ_1, $\varphi_2\in S(\mathbb{D})$, $g_1, g_2\in H(\mathbb{D})$. 若$I^{(n)}_{g_1,\varphi_1} - I^{(n)}_{g_2,\varphi_2} : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ 有界, 但$I^{(n)}_{g_1,\varphi_1}$和$I^{(n)}_{g_2,\varphi_2}$都不是紧算子, 则$I^{(n)}_{g_1,\varphi_1} - I^{(n)}_{g_2,\varphi_2} : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ 是紧算子当且仅当

(i) $D(g_1,\varphi_1) = D(g_2,\varphi_2) \neq 0$, $\quad D(g_1,\varphi_1)\subset\Gamma(\varphi_2)$,

(ii) 对$z_k\in\Gamma(\varphi_1)\cap\Gamma(\varphi_2)$,

$$\lim_{k\to\infty} I_1(z_k) = \lim_{k\to\infty} I_2(z_k) = \lim_{k\to\infty} I_3(z_k) = 0.$$

证明 充分性. 设$\{f_k\}$是$\mathcal{B}^\alpha$中的序列, 其在$\mathbb{D}$上的任意紧子集上都一致收敛于0, 且$\|f_k\|_{\mathcal{B}^\alpha} \leq 1$. 若$\|(I^{(n)}_{g_1,\varphi_1} - I^{(n)}_{g_2,\varphi_2})f_k\|_{\mathcal{B}^\beta} \nrightarrow 0$, 则存在$\varepsilon > 0$, 使得对任意k有$\|(I^{(n)}_{g_1,\varphi_1} - I^{(n)}_{g_2,\varphi_2})f_k\|_{\mathcal{B}^\beta} > \varepsilon$. 于是对每个$k$, 存在$z_k \in \mathbb{D}$使得

$$|\mathfrak{D}_{\varphi_1,g_1}(z_k)(1-|\varphi_1(z_k)|^2)^{\alpha+n-1} f_k^{(n)}(\varphi_1(z_k)) - \mathfrak{D}_{\varphi_2,g_2}(z_k)(1-|\varphi_2(z_k)|^2)^{\alpha+n-1} f_k^{(n)}(\varphi_2(z_k))| > \varepsilon. \tag{14}$$

从而$|\varphi_1(z_k)| \to 1$或者$|\varphi_2(z_k)| \to 1$. 若$|\varphi_1(z_k)| \to 1$, 设$w \in \overline{\mathbb{D}}$为$\{\varphi_2(z_k)\}$的极限. 则存在子列收敛于$w$, 不妨就设$\varphi_2(z_k) \to w$. 如果$|w| < 1$, 那么$z_k \notin \Gamma(\varphi_1) \cap \Gamma(\varphi_2)$. 于是由

$$D(g_1,\varphi_1) \subset \Gamma(\varphi_1) \cap \Gamma(\varphi_2)$$

可得$\mathfrak{D}_{\varphi_1,g_1}(z_k) \to 0$. 另一方面, 由于$I^{(n)}_{g_2,\varphi_2}$有界, 于是

$$|\mathfrak{D}_{\varphi_2,g_2}(z_k)|(1-|\varphi_2(z_k)|^2)^{\alpha+n-1} = (1-|z_k|^2)^\beta |g_2(z_k)| < \infty,$$

从而$|w| < 1$推出$f_k^{(n)}(\varphi_2(z_k)) \to 0$, 与(14)矛盾, 故$|w| = 1$. 因此, $|\varphi_1(z_k)| \to 1$ 且$|\varphi_2(z_k)| \to 1$. 由假设可以得到

$$\begin{aligned} & |\mathfrak{D}_{\varphi_1,g_1}(z_k)(1-|\varphi_1(z_k)|^2)^{\alpha+n-1} f_k^{(n)}(\varphi_1(z_k)) - \mathfrak{D}_{\varphi_2,g_2}(z_k)(1-|\varphi_2(z_k)|^2)^{\alpha+n-1} f_k^{(n)}(\varphi_2(z_k))| \\ \leq\ & |\mathfrak{D}_{\varphi_1,g_1}(z_k) - \mathfrak{D}_{\varphi_2,g_2}(z_k)| + \sup_{z\in\mathbb{D}} |\mathfrak{D}_{\varphi_2,g_2}(z_k)| \rho(\varphi_1(z_k),\varphi_2(z_k)) \to 0, \quad k \to \infty. \end{aligned}$$

与(14)矛盾.

必要性. 由假设可知, 若$I^{(n)}_{g_1,\varphi_1}$ 非紧, 则存在序列$\{z_k\} \subset D(g_1,\varphi_1)$ 使得当$|\varphi_1(z_k)| \to 1$时$|\mathfrak{D}_{\varphi_1,g_1}(z_k)| \nrightarrow 0$. 对$w_k = \varphi_1(z_k)$, 与定理1 类似地定义$f_{w_k}$和$k_{w_k}$, 则$\{f_{w_k}\}$和$\{k_{w_k}\}$是$\mathcal{B}^\alpha$中的有界序列, 且在$\mathbb{D}$ 的每个紧子集上都一致收敛于0. 因此, 由引理2可得, 当$k \to \infty$时

$$\begin{aligned} 0 \leftarrow\ & \|(I^{(n)}_{g_1,\varphi_1} - I^{(n)}_{g_2,\varphi_2})f_{\varphi_1(z_k)}\|_{\mathcal{B}^\beta} \\ \geq\ & \left(|\mathfrak{D}_{\varphi_1,g_1}(z_k)| - \left| \mathfrak{D}_{\varphi_2,g_2}(z_k) \frac{(1-|\varphi_1(z_k)|^2)(1-|\varphi_2(z_k)|^2)^{\alpha+n-1}}{(1-\overline{\varphi_1(z_k)}\varphi_2(z_k))^{\alpha+n}} \right| \right), \end{aligned} \tag{15}$$

和

$$\begin{aligned} 0 \leftarrow\ & \|(I^{(n)}_{g_1,\varphi_1} - I^{(n)}_{g_2,\varphi_2})k_{\varphi_1(z_k)}\|_{\mathcal{B}^\beta} \\ \geq\ & \left| \mathfrak{D}_{\varphi_2,g_2}(z_k) \frac{(1-|\varphi_1(z_k)|^2)(1-|\varphi_2(z_k)|^2)^{\alpha+n-1}}{(1-\overline{\varphi_1(z_k)}\varphi_2(z_k))^{\alpha+n}} \right| \cdot \rho(\varphi_1(z_k),\varphi_2(z_k)). \end{aligned} \tag{16}$$

于是结合(15)和(16)得到

$$\lim_{k\to\infty} I_1(z_k) = \lim_{k\to\infty} |\mathfrak{D}_{\varphi_1,g_1}(z_k)| \rho(\varphi_1(z_k),\varphi_2(z_k)) = 0. \tag{17}$$

由$|\mathfrak{D}_{\varphi_1,g_1}(z_k)| \nrightarrow 0$和(17)可得$\lim_{k\to\infty}\rho(\varphi_1(z_k),\varphi_2(z_k))=0$. 从而

$$\lim_{k\to\infty} I_2(z_k)=\lim_{k\to\infty}|\mathfrak{D}_{\varphi_2,g_2}(z_k)|\rho(\varphi_1(z_k),\varphi_2(z_k))=0. \tag{18}$$

进一步, 对任意$\{z_k\}\subset D(g_1,\varphi_1)$, 有$\lim_{k\to\infty}|\varphi_1(z_k)-\varphi_2(z_k)|=0$. 因此

$$D(g_1,\varphi_1)\subset\Gamma(\varphi_2). \tag{19}$$

此外, 当$k\to\infty$时, 有

$$|\mathfrak{D}_{\varphi_1,g_1}(z_k)-\mathfrak{D}_{\varphi_2,g_2}(z_k)|-|\mathfrak{D}_{\varphi_2,g_2}(z_k)|\rho(\varphi_1(z_k),\varphi_2(z_k))\to 0.$$

于是由(18)可以得到

$$\lim_{k\to\infty} I_3(z_k)=\lim_{k\to\infty}|\mathfrak{D}_{\varphi_1,g_1}(z_k)-\mathfrak{D}_{\varphi_2,g_2}(z_k)|=0. \tag{20}$$

因此, 由(19)和(20)可得$D(g_1,\varphi_1)\subset D(g_2,\varphi_2)$. 同理可得$D(g_2,\varphi_2)\subset D(g_1,\varphi_1)$. 故而, $D(g_1,\varphi_1)=D(g_2,\varphi_2)$.

对任意序列$\{z_k\}$, 当$|\varphi_1(z_k)|\to 1, |\varphi_2(z_k)|\to 1$且$|\mathfrak{D}_{\varphi_1,g_1}(z_k)|\to 0$时, 有

$$\lim_{k\to\infty} I_1(z_k)=\lim_{k\to\infty}|\mathfrak{D}_{\varphi_1,g_1}(z_k)|\rho(\varphi_1(z_k),\varphi_2(z_k))=0. \tag{21}$$

另一方面, 当$k\to\infty$时, 有

$$\begin{aligned} 0 \;&\leftarrow\; \|(I^{(n)}_{g_1,\varphi_1}-I^{(n)}_{g_2,\varphi_2})k_{\varphi_2(z_k)}\|_{\mathcal{B}^\beta} \\ &\geq\; C\left(|\mathfrak{D}_{\varphi_1,g_1}(w)-\mathfrak{D}_{\varphi_2,g_2}(w)|-|\mathfrak{D}_{\varphi_2,g_2}(w)|\rho(\varphi_1(w),\varphi_2(w))\right). \end{aligned}$$

于是, 可以得到

$$\lim_{k\to\infty} I_3(z_k)=\lim_{k\to\infty}|\mathfrak{D}_{\varphi_1,g_1}(z_k)-\mathfrak{D}_{\varphi_2,g_2}(z_k)|=0. \tag{22}$$

从而有

$$\lim_{k\to\infty}|\mathfrak{D}_{\varphi_1,g_1}(z_k)|=\lim_{k\to\infty}|\mathfrak{D}_{\varphi_2,g_2}(z_k)|=0.$$

因此, 有

$$\lim_{k\to\infty} I_2(z_k)=\lim_{k\to\infty}|\mathfrak{D}_{\varphi_2,g_2}(z_k)|\rho(\varphi_1(z_k),\varphi_2(z_k))=0.$$

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