Pure Mathematics 理论数学, 2017, 7(1), 20-29
Published Online January 2017 in Hans. http://www.hanspub.org/journal/pm
http://dx.doi.org/10.12677/pm.2017.71004

The Automorphism Group of A Class of Abelian *p*-Group

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Received: Dec. 25th, 2016; accepted: Jan. 10th, 2017; published: Jan. 13th, 2017

Abstract

Let $C_{p^{\beta_i}}$ be the cyclic group of order p^{β_i} where p is a prime number, and $1 \le i \le 4$. Using matrix representation, J. N. S. Bidwell got the automorphisms of direct products $C_{p^m} \times C_{p^n}$, and $C_{p^{\beta_1}} \times C_{p^{\beta_2}} \times C_{p^{\beta_3}}$ which have no common direct factor. In this paper, the generators of the automorphism of direct product $C_{p^{\beta_1}} \times C_{p^{\beta_2}} \times C_{p^{\beta_3}} \times C_{p^{\beta_4}}$ and the relations between them are obtained.

Keywords

Automorphism, Cyclic Group; Matrix Representation; Direct Product

一类交换*p*-群的自同构群

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收稿日期：2016年12月25日；录用日期：2017年1月10日；发布日期：2017年1月13日

文章引用: 方树珍, 周芳. 一类交换 *p*-群的自同构群[J]. 理论数学, 2017, 7(1): 20-29.
http://dx.doi.org/10.12677/pm.2017.71004

摘　要

设$C_{p^{\beta_i}}$是p^{β_i}阶的循环群，其中p为素数，$1\le i\le 4$。J. N. S. Bidwell利用矩阵表示方法得到了没有共同直因子的直积$C_{p^m}\times C_{p^n}$和$C_{p^{\beta_1}}\times C_{p^{\beta_2}}\times C_{p^{\beta_3}}$的自同构群。本文采用同样的方法得到了直积$C_{p^{\beta_1}}\times C_{p^{\beta_2}}\times C_{p^{\beta_3}}\times C_{p^{\beta_4}}$的自同构群的生成元和生成关系。

关键词

自同构群，循环群，矩阵表示，直积

1. 引言

设群$G=C_{p^{\beta_1}}\times C_{p^{\beta_2}}\times C_{p^{\beta_3}}\times C_{p^{\beta_4}}$，其中$p$为素数，$\beta_i\left(i=1,2,3,4\right)$互不相同，$C_{p^{\beta_i}}$为$p^{\beta_i}$阶的循环群。J. N. S. Bidwell在文献[1] [2] [3]给出了没有共同直因子直积的自同构群的结构，并通过矩阵表示方法得到了直积$C_{p^m}\times C_{p^n}$，$C_{p^{\beta_1}}\times C_{p^{\beta_2}}\times C_{p^{\beta_3}}$的自同构群，其中$m,n$互不相同，并通过GAP软件计算了直积$C_{p^{\beta_1}}\times\cdots\times C_{p^{\beta_n}}$的自同构群，周芳等人在文献[4]中利用该方法得到了给定半直积的稳定自同构群可分解为若干结构简单的特殊子群乘积的充要条件。本文中我们用矩阵表示的方法计算了$C_{p^{\beta_1}}\times C_{p^{\beta_2}}\times C_{p^{\beta_3}}\times C_{p^{\beta_4}}$的自同构群，得到了该自同构群的生成元和生成关系。本文的符号是标准的，见[1]。

2. 主要内容

设$G=C_{p^{\beta_1}}\times C_{p^{\beta_2}}\times C_{p^{\beta_3}}\times C_{p^{\beta_4}}$，其中$p$为素数，$\beta_1,\beta_2,\beta_3,\beta_4$互不相同，不失一般性，设$\beta_1>\beta_2>\beta_3>\beta_4$，由[1]可得$\mathrm{Aut}\left(G\right)\cong\mathcal{A}=A_{11}A_{12}A_{13}A_{14}A_{21}A_{22}A_{23}A_{24}A_{31}A_{32}A_{33}A_{34}A_{41}A_{42}A_{43}A_{44}$，其中

$$A_{11}=\left\{\begin{pmatrix}\alpha_{11}&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}:\alpha_{11}\in\mathrm{Aut}\left(H_1\right)\right\},$$

$$A_{12}=\left\{\begin{pmatrix}1&\alpha_{12}&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}:\alpha_{12}\in\mathrm{Hom}\left(H_2,H_1\right)\right\},$$

$$\cdots$$

$$A_{44}=\left\{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&\alpha_{44}\end{pmatrix}:\alpha_{44}\in\mathrm{Aut}\left(H_4\right)\right\}$$

且$C_{p^{\beta_i}}\cong H_i=\left\langle x_i\right\rangle$，$s_i$分别为模$p^{\beta_i}$的原根，即有$\alpha_{ii}\left(x_i\right)=x_i^{s_i}$，其中$1\le i\le 4$，于是$\alpha_{ii}$生成$\mathrm{Aut}\left(H_i\right)$，进而$A_{ii}$是阶为$\varphi\left(p^{\beta_i}\right)$的循环群；进一步，$t_i=s_i^{-1}\left(\bmod p^{\beta_i}\right)$，于是$\alpha_{ii}^{-1}\left(x_i\right)=x_i^{t_i}$。

另若$i<j$，有$\alpha_{ij}\left(x_j\right)=x_i^{p^{\beta_i-\beta_j}}$；若$i>j$，有$\alpha_{ij}\left(x_{ij}\right)=x_i$，则$\alpha_{ij}$生成$\mathrm{Hom}\left(H_j,H_i\right)$，进而$A_{ij}\left(i\ne j\right)$是

阶为 $p^{\beta_{\max(i,j)}}$ 的循环群，其中 $1\le i,j\le 4$。

$$令\ a_{11}=\begin{pmatrix}\alpha_{11}&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},a_{12}=\begin{pmatrix}1&\alpha_{12}&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\cdots,a_{44}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&\alpha_{44}\end{pmatrix}$$

有 $o(a_{ij})=\begin{cases}\varphi\left(p^{\beta_i}\right), & i=j\\ p^{\beta_{\max(i,j)}}, & i\ne j\end{cases}$。

于是 $\mathcal{A}=\langle a_{11},a_{12},a_{13},a_{14},a_{21},a_{22},a_{23},a_{24},a_{31},a_{32},a_{33},a_{34},a_{41},a_{42},a_{43},a_{44}\rangle$。

通过对 $a_{ij}^{a_{kl}}$ 计算，当 $i=j,k=l$，有 $a_{ii}^{a_{kk}}=a_{ii}$；当 $i\ne l,j\ne k,l$，有 $a_{ij}^{a_{kl}}=a_{ij}$。

根据矩阵主对角线元素的顺序，首先计算 a_{11} 的共轭关系，

$$a_{11}^{a_{12}}=\begin{pmatrix}1&-\alpha_{12}&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}\alpha_{11}&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&\alpha_{12}&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}\alpha_{11}&\alpha_{11}\alpha_{12}-\alpha_{12}&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}\alpha_{11}&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&\alpha_{12}-\alpha_{11}^{-1}\alpha_{12}&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

由于 $\alpha_{12}-\alpha_{11}^{-1}\alpha_{12}=(1-t_1)\alpha_{12}$，于是 $a_{11}^{a_{12}}=a_{11}a_{12}^{1-t_1}$。

类似计算结果有 $a_{11}^{a_{13}}=a_{11}a_{13}^{1-t_1}$， $a_{11}^{a_{14}}=a_{11}a_{14}^{1-t_1}$。

$$a_{11}^{a_{21}}=\begin{pmatrix}1&0&0&0\\-\alpha_{21}&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}\alpha_{11}&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\\alpha_{21}&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}\alpha_{11}&0&0&0\\-\alpha_{21}\alpha_{11}+\alpha_{21}&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}1&0&0&0\\-\alpha_{21}+\alpha_{21}\alpha_{11}^{-1}&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}\alpha_{11}&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

由于 $-\alpha_{21}+\alpha_{21}\alpha_{11}^{-1}=(t_1-1)\alpha_{21}$，于是 $a_{11}^{a_{21}}=a_{21}^{t_1-1}a_{11}$。

类似计算结果有 $a_{11}^{a_{31}}=a_{31}^{t_1-1}a_{11}$， $a_{11}^{a_{41}}=a_{41}^{t_1-1}a_{11}$。

我们得到 a_{11} 的共轭关系如下：

$$a_{11}^{a_{12}}=a_{11}a_{12}^{1-t_1},\quad a_{11}^{a_{13}}=a_{11}a_{13}^{1-t_1},\quad a_{11}^{a_{14}}=a_{11}a_{14}^{1-t_1},$$

$$a_{11}^{a_{21}}=a_{21}^{t_1-1}a_{11},\quad a_{11}^{a_{22}}=a_{11},\quad a_{11}^{a_{23}}=a_{11},\quad a_{11}^{a_{24}}=a_{11},$$

$$a_{11}^{a_{31}}=a_{31}^{t_1-1}a_{11},\quad a_{11}^{a_{32}}=a_{11},\quad a_{11}^{a_{33}}=a_{11},\quad a_{11}^{a_{34}}=a_{11},$$

$$a_{11}^{a_{41}}=a_{41}^{t_1-1}a_{11},\quad a_{11}^{a_{42}}=a_{11},\quad a_{11}^{a_{43}}=a_{11},\quad a_{11}^{a_{44}}=a_{11}。$$

接下来计算 a_{22} 的共轭关系，

$$a_{22}^{\alpha_{23}}=\begin{pmatrix}1&0&0&0\\0&1&-\alpha_{23}&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&\alpha_{22}&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&\alpha_{23}&0\\0&0&1&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&\alpha_{22}&\alpha_{22}\alpha_{23}-\alpha_{23}&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}1&0&0&0\\0&\alpha_{22}&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&\alpha_{23}-\alpha_{22}^{-1}\alpha_{23}&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

由于 $\alpha_{23}-\alpha_{22}^{-1}\alpha_{23}=(1-t_2)\alpha_{23}$，于是 $a_{22}^{a_{23}}=a_{22}a_{23}^{1-t_2}$ 。

类似计算结果有 $a_{22}^{a_{24}}=a_{22}a_{24}^{1-t_2}$ 。

$$a_{22}^{\alpha_{32}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&-\alpha_{32}&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&\alpha_{22}&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&\alpha_{32}&1&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&\alpha_{22}&0&0\\0&-\alpha_{32}\alpha_{22}+\alpha_{32}&1&0\\0&0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&-\alpha_{32}+\alpha_{32}\alpha_{22}^{-1}&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&\alpha_{22}&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

由于 $-\alpha_{32}+\alpha_{32}\alpha_{22}^{-1}=(t_2-1)\alpha_{32}$，于是 $a_{22}^{a_{32}}=a_{32}^{t_2-1}a_{22}$ 。

类似计算结果有 $a_{22}^{a_{42}}=a_{42}^{t_2-1}a_{22}$ 。

我们得到 a_{22} 的共轭关系如下：

$$a_{22}^{a_{23}}=a_{22}a_{23}^{1-t_2}，\quad a_{22}^{a_{24}}=a_{22}a_{24}^{1-t_2}，$$

$$a_{22}^{a_{31}}=a_{22}，\quad a_{22}^{a_{32}}=a_{32}^{t_2-1}a_{22}，\quad a_{22}^{a_{33}}=a_{22}，\quad a_{22}^{a_{34}}=a_{22}，$$

$$a_{22}^{a_{41}}=a_{22}，\quad a_{22}^{a_{42}}=a_{42}^{t_2-1}a_{22}，\quad a_{22}^{a_{43}}=a_{22}，\quad a_{22}^{a_{44}}=a_{22}。$$

重复类似的计算过程，我们得到 a_{33} 的共轭关系：

$$a_{33}^{a_{34}}=a_{33}a_{34}^{1-t_3}，\quad a_{33}^{a_{41}}=a_{33}，$$

$$a_{33}^{a_{42}}=a_{33}，\quad a_{33}^{a_{43}}=a_{43}^{t_3-1}a_{33}，\quad a_{33}^{a_{44}}=a_{33}。$$

对于形如 $a_{ij}^{a_{kl}}\ (i\neq j)$ 较复杂的共轭关系，计算如下：

$$a_{12}^{a_{22}}=\begin{pmatrix}1&0&0&0\\0&-\alpha_{22}&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&\alpha_{12}&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&\alpha_{22}&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}1&\alpha_{12}\alpha_{22}&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

由于 $\alpha_{12}\alpha_{22}=s_2\alpha_{12}$，于是 $a_{12}^{a_{22}}=a_{12}^{s_2}$ 。

在接下来的计算中为方便书写，需定义参数 $\mu_{ij}=p^{\max(\beta_i-\beta_j,0)}$，其中 $i\neq j$，此时 $\alpha_{ij}\left(x_j\right)=x_i^{\mu_{ij}}$ 。

对于 a_{12} 的共轭关系，

$$a_{12}^{a_{23}}=\begin{pmatrix}1&0&0&0\\0&1&-\alpha_{23}&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&\alpha_{12}&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&\alpha_{23}&0\\0&0&1&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}1&\alpha_{12}&\alpha_{12}\alpha_{23}&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}1&\alpha_{12}&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&\alpha_{12}\alpha_{23}&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

由于 $\alpha_{12}\alpha_{23}=(\mu_{12}\mu_{23}/\mu_{13})\alpha_{13}$，于是 $a_{12}^{a_{23}}=a_{12}a_{13}^{\mu_{12}\mu_{23}/\mu_{13}}$。

类似计算结果有 $a_{12}^{a_{24}}=a_{12}a_{14}^{\mu_{12}\mu_{24}/\mu_{14}}$。

$$a_{12}^{a_{31}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\-\alpha_{31}&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&\alpha_{12}&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\\alpha_{31}&0&1&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}1&\alpha_{12}&0&0\\0&1&0&0\\0&-\alpha_{31}\alpha_{12}&1&0\\0&0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}1&\alpha_{12}&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&-\alpha_{31}\alpha_{12}&1&0\\0&0&0&1\end{pmatrix}$$

由于 $-\alpha_{31}\alpha_{12}=(-\mu_{31}\mu_{12}/\mu_{32})\alpha_{32}$，于是 $a_{12}^{a_{31}}=a_{12}a_{32}^{-\mu_{31}\mu_{12}/\mu_{32}}$。

类似计算结果有 $a_{12}^{a_{41}}=a_{12}a_{42}^{-\mu_{41}\mu_{12}/\mu_{42}}$。

我们得到 a_{12} 的共轭关系如下：

$$a_{12}^{a_{13}}=a_{12}，\quad a_{12}^{a_{14}}=a_{12}，$$

$$a_{12}^{a_{22}}=a_{12}^{s_2}，\quad a_{12}^{a_{23}}=a_{12}a_{13}^{\mu_{12}\mu_{23}/\mu_{13}}，\quad a_{12}^{a_{24}}=a_{12}a_{14}^{\mu_{12}\mu_{24}/\mu_{14}}，$$

$$a_{12}^{a_{31}}=a_{12}a_{32}^{-\mu_{31}\mu_{12}/\mu_{32}}，\quad a_{12}^{a_{32}}=a_{12}，\quad a_{12}^{a_{33}}=a_{12}，\quad a_{12}^{a_{34}}=a_{12}，$$

$$a_{12}^{a_{41}}=a_{12}a_{42}^{-\mu_{41}\mu_{12}/\mu_{42}}，\quad a_{12}^{a_{42}}=a_{12}，\quad a_{12}^{a_{43}}=a_{12}，\quad a_{12}^{a_{44}}=a_{12}。$$

对于 a_{13} 的共轭关系，

$$a_{13}^{a_{21}}=\begin{pmatrix}1&0&0&0\\-\alpha_{21}&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&\alpha_{13}&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\\alpha_{21}&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}1&0&\alpha_{13}&0\\0&1&-\alpha_{21}\alpha_{13}&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}1&0&\alpha_{13}&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&-\alpha_{21}\alpha_{13}&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

由于 $-\alpha_{21}\alpha_{13}=(-\mu_{21}\mu_{13}/\mu_{23})\alpha_{23}$，于是 $a_{13}^{a_{21}}=a_{13}a_{23}^{-\mu_{21}\mu_{13}/\mu_{23}}$。

类似计算结果有 $a_{13}^{a_{41}}=a_{13}a_{43}^{-\mu_{41}\mu_{13}/\mu_{43}}$。

$$a_{13}^{\alpha_{32}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&-\alpha_{32}&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&\alpha_{13}&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&\alpha_{32}&1&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}1&\alpha_{13}\alpha_{32}&\alpha_{13}&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$
$$=\begin{pmatrix}1&0&\alpha_{13}&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&\alpha_{13}\alpha_{32}&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

由于 $\alpha_{13}\alpha_{32}=(\mu_{13}\mu_{32}/\mu_{12})\alpha_{12}$，于是 $a_{13}^{\alpha_{32}}=a_{13}a_{12}^{\mu_{13}\mu_{32}/\mu_{12}}$。
类似计算结果有 $a_{13}^{\alpha_{34}}=a_{13}a_{14}^{\mu_{13}\mu_{34}/\mu_{14}}$。

$$a_{13}^{\alpha_{33}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-\alpha_{33}&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&\alpha_{13}&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\alpha_{33}&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}1&0&\alpha_{13}\alpha_{33}&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

由于 $\alpha_{13}\alpha_{33}=s_3\alpha_{13}$，于是 $a_{13}^{\alpha_{33}}=a_{13}^{s_3}$。
我们得到 a_{13} 的共轭关系如下：

$$a_{13}^{\alpha_{14}}=a_{13}，\quad a_{13}^{\alpha_{21}}=a_{13}a_{23}^{-\mu_{21}\mu_{13}/\mu_{23}}，\quad a_{13}^{\alpha_{22}}=a_{13}，\quad a_{13}^{\alpha_{23}}=a_{13}，\quad a_{13}^{\alpha_{24}}=a_{13}，\quad a_{13}^{\alpha_{32}}=a_{13}a_{12}^{\mu_{13}\mu_{32}/\mu_{12}}，$$
$$a_{13}^{\alpha_{33}}=a_{13}^{s_3}，\quad a_{13}^{\alpha_{34}}=a_{13}a_{14}^{\mu_{13}\mu_{34}/\mu_{14}}，\quad a_{13}^{\alpha_{41}}=a_{13}a_{43}^{-\mu_{41}\mu_{13}/\mu_{43}}，\quad a_{13}^{\alpha_{42}}=a_{13}，\quad a_{13}^{\alpha_{43}}=a_{13}，\quad a_{13}^{\alpha_{44}}=a_{13}。$$

接下来计算 a_{14} 的共轭关系，

$$a_{14}^{\alpha_{21}}=\begin{pmatrix}1&0&0&0\\-\alpha_{21}&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&\alpha_{14}\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\\alpha_{21}&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0&\alpha_{14}\\0&1&0&-\alpha_{21}\alpha_{14}\\0&0&1&0\\0&0&0&1\end{pmatrix}$$
$$=\begin{pmatrix}1&0&0&\alpha_{14}\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&-\alpha_{21}\alpha_{14}\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

由于 $-\alpha_{21}\alpha_{14}=(-\mu_{21}\mu_{14}/\mu_{24})\alpha_{24}$，于是 $a_{14}^{\alpha_{21}}=a_{14}a_{24}^{-\mu_{21}\mu_{14}/\mu_{24}}$。
类似计算结果有 $a_{14}^{\alpha_{31}}=a_{14}a_{34}^{-\mu_{31}\mu_{14}/\mu_{34}}$。

$$a_{14}^{\alpha_{42}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&-\alpha_{42}&0&1\end{pmatrix}\begin{pmatrix}1&0&0&\alpha_{14}\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&\alpha_{42}&0&1\end{pmatrix}=\begin{pmatrix}1&\alpha_{14}\alpha_{42}&0&\alpha_{14}\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$
$$=\begin{pmatrix}1&0&0&\alpha_{14}\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&\alpha_{14}\alpha_{42}&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

由于 $\alpha_{14}\alpha_{42}=(\mu_{14}\mu_{42}/\mu_{12})\alpha_{12}$，于是 $a_{14}^{a_{42}}=a_{14}a_{12}^{\mu_{14}\mu_{42}/\mu_{12}}$。

类似计算结果有 $a_{14}^{a_{43}}=a_{14}a_{13}^{\mu_{14}\mu_{43}/\mu_{13}}$。

$$a_{14}^{a_{44}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&-\alpha_{44}\end{pmatrix}\begin{pmatrix}1&0&0&\alpha_{14}\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&\alpha_{44}\end{pmatrix}=\begin{pmatrix}1&0&\alpha_{14}\alpha_{44}&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

由于 $\alpha_{14}\alpha_{44}=s_4\alpha_{14}$，于是 $a_{14}^{a_{44}}=a_{14}^{s_4}$。

我们得到 a_{14} 的共轭关系如下：

$$a_{14}^{a_{21}}=a_{14}a_{24}^{-\mu_{21}\mu_{14}/\mu_{24}}，\quad a_{14}^{a_{22}}=a_{14}，\quad a_{14}^{a_{23}}=a_{14}，\quad a_{14}^{a_{24}}=a_{14}，\quad a_{14}^{a_{31}}=a_{14}a_{34}^{-\mu_{31}\mu_{14}/\mu_{34}}，$$

$$a_{14}^{a_{32}}=a_{14}，\quad a_{14}^{a_{33}}=a_{14}，\quad a_{14}^{a_{34}}=a_{14}，\quad a_{14}^{a_{42}}=a_{14}a_{12}^{\mu_{14}\mu_{42}/\mu_{12}}，\quad a_{14}^{a_{43}}=a_{14}a_{13}^{\mu_{14}\mu_{43}/\mu_{13}}，\quad a_{14}^{a_{44}}=a_{14}^{s_4}。$$

对于 a_{21} 的共轭关系，

$$a_{21}^{a_{22}}=\begin{pmatrix}1&0&0&0\\0&-\alpha_{22}&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\\alpha_{21}&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&\alpha_{22}&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0&0\\-\alpha_{22}\alpha_{21}&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

由于 $-\alpha_{22}\alpha_{21}=-s_2\alpha_{21}$，于是 $a_{21}^{a_{22}}=a_{21}^{-s_2}=a_{21}^{t_2}$。

$$a_{21}^{a_{32}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&-\alpha_{32}&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\\alpha_{21}&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&\alpha_{32}&1&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0&0\\\alpha_{21}&1&0&0\\-\alpha_{32}\alpha_{21}&0&1&0\\0&0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}1&0&0&0\\\alpha_{21}&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\-\alpha_{32}\alpha_{21}&0&1&0\\0&0&0&1\end{pmatrix}$$

由于 $-\alpha_{32}\alpha_{21}=(-\mu_{32}\mu_{21}/\mu_{31})\alpha_{31}$，于是 $a_{21}^{a_{32}}=a_{21}a_{31}^{-\mu_{32}\mu_{21}/\mu_{31}}$。

类似计算结果有 $a_{21}^{a_{42}}=a_{21}a_{41}^{-\mu_{42}\mu_{21}/\mu_{41}}$。

我们得到的 a_{21} 共轭关系如下：

$$a_{21}^{a_{22}}=a_{21}^{t_2}，\quad a_{21}^{a_{23}}=a_{21}，\quad a_{21}^{a_{24}}=a_{21}，\quad a_{21}^{a_{31}}=a_{21}，\quad a_{21}^{a_{32}}=a_{21}a_{31}^{-\mu_{32}\mu_{21}/\mu_{31}}，\quad a_{21}^{a_{33}}=a_{21}，$$

$$a_{21}^{a_{34}}=a_{21}，\quad a_{21}^{a_{41}}=a_{21}，\quad a_{21}^{a_{42}}=a_{21}a_{41}^{-\mu_{42}\mu_{21}/\mu_{41}}，\quad a_{21}^{a_{43}}=a_{21}，\quad a_{21}^{a_{44}}=a_{21}。$$

重复类似的计算过程，我们得到生成元 $a_{23},a_{24},a_{31},a_{32},a_{34},a_{41},a_{42},a_{43}$ 的共轭关系，整理如下，

$$a_{23}^{a_{24}}=a_{23}，\quad a_{23}^{a_{31}}=a_{23}a_{21}^{\mu_{23}\mu_{31}/\mu_{21}}，\quad a_{23}^{a_{33}}=a_{23}^{s_3}，\quad a_{23}^{a_{34}}=a_{23}a_{24}^{\mu_{23}\mu_{34}/\mu_{24}}，$$

$$a_{23}^{a_{41}}=a_{23}，\quad a_{23}^{a_{42}}=a_{23}a_{43}^{-\mu_{42}\mu_{23}/\mu_{43}}，\quad a_{23}^{a_{43}}=a_{23}，\quad a_{23}^{a_{44}}=a_{23}；$$

$$a_{24}^{a_{31}}=a_{24}，\quad a_{24}^{a_{32}}=a_{24}a_{34}^{-\mu_{32}\mu_{24}/\mu_{34}}，\quad a_{24}^{a_{33}}=a_{24}，\quad a_{24}^{a_{34}}=a_{24}，\quad a_{24}^{a_{41}}=a_{24}a_{21}^{\mu_{24}\mu_{41}/\mu_{21}}，\quad a_{24}^{a_{43}}=a_{24}a_{23}^{\mu_{24}\mu_{43}/\mu_{23}}，\quad a_{24}^{a_{44}}=a_{24}^{s_4}；$$

$$a_{31}^{a_{32}}=a_{31}，\quad a_{31}^{a_{33}}=a_{31}^{t_3}，\quad a_{31}^{a_{34}}=a_{31}，\quad a_{31}^{a_{41}}=a_{31}，\quad a_{31}^{a_{42}}=a_{31}，\quad a_{31}^{a_{43}}=a_{31}a_{41}^{-\mu_{43}\mu_{31}/\mu_{41}}，\quad a_{31}^{a_{44}}=a_{31}；$$

$$a_{32}^{a_{33}}=a_{32}^{t_3}，\quad a_{32}^{a_{34}}=a_{32}，\quad a_{32}^{a_{41}}=a_{32},\quad a_{32}^{a_{42}}=a_{32}，\quad a_{32}^{a_{43}}=a_{31}a_{42}^{-\mu_{43}\mu_{32}/\mu_{42}}，\quad a_{32}^{a_{44}}=a_{32}；$$

$$a_{34}^{a_{41}}=a_{34}a_{31}^{\mu_{34}\mu_{41}/\mu_{31}}，\quad a_{34}^{a_{42}}=a_{34}a_{32}^{\mu_{34}\mu_{42}/\mu_{32}}，\quad a_{34}^{a_{44}}=a_{34}^{s_4}；$$

$$a_{41}^{a_{42}}=a_{41},\quad a_{41}^{a_{43}}=a_{41}，\quad a_{41}^{a_{44}}=a_{41}^{t_4}；\quad a_{42}^{a_{43}}=a_{42}，\quad a_{42}^{a_{44}}=a_{42}^{t_4}，\quad a_{43}^{a_{44}}=a_{43}^{t_4}。$$

最后计算更为复杂的转置共轭 $a_{ij}^{a_{ji}}$，其中 $i\neq j$，

$$a_{12}^{a_{21}}=\begin{pmatrix}1&0&0&0\\-\alpha_{21}&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&\alpha_{12}&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\\alpha_{21}&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}1+\alpha_{12}\alpha_{21}&\alpha_{12}&0&0\\-\alpha_{21}\alpha_{12}\alpha_{21}&-\alpha_{21}\alpha_{12}+1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}1+\alpha_{12}\alpha_{21}&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&(1+\alpha_{12}\alpha_{21})^{-1}\alpha_{12}(1-\alpha_{21}\alpha_{12})^{-1}&0&0\\-\alpha_{21}\alpha_{12}\alpha_{21}&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$\begin{pmatrix}1&0&0&0\\0&1-\alpha_{21}\alpha_{12}&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

此时，第一个矩阵和最后一个矩阵可分别记作 $a_{11}^{r_{12}},a_{22}^{k_{12}}$，这是由于 $1+\alpha_{12}\alpha_{21}=(1+\mu_{12})\alpha_{11}$，有 $s_1^{r_{12}}\equiv 1+\mu_{12}\left(\bmod p^{\beta_1}\right)$，$r_{12}$ 为对应的结果；同样由于 $1-\alpha_{21}\alpha_{12}=(1-\mu_{12})\alpha_{22}$，有 $s_2^{k_{12}}\equiv 1-\mu_{12}\left(\bmod p^{\beta_2}\right)$，$k_{12}$ 为对应的结果。而第二个矩阵可分解为

$$\begin{pmatrix}\alpha^*&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&\alpha^{*-1}(1+\alpha_{12}\alpha_{21})^{-1}\alpha_{12}(1-\alpha_{21}\alpha_{12})^{-1}&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\-\alpha_{21}\alpha_{12}\alpha_{21}&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

其中，

$$\begin{aligned}\alpha^*&=1-(1+\alpha_{12}\alpha_{21})^{-1}\alpha_{12}(1-\alpha_{21}\alpha_{12})^{-1}(-\alpha_{21}\alpha_{12}\alpha_{21})\\&=1+\left(\alpha_{11}^{r_{12}}\right)^{-1}\alpha_{12}\left(\alpha_{22}^{k_{12}}\right)^{-1}(\alpha_{21}\alpha_{12}\alpha_{21})\\&=1+\mu_{21}\mu_{12}\mu_{21}\left(\alpha_{11}^{r_{12}}\right)^{-1}\alpha_{12}\left(\alpha_{22}^{k_{12}}\right)^{-1}\\&=1+t_1^{r_{12}}\mu_{12}t_2^{k_{12}}\mu_{21}\mu_{12}\mu_{21}\alpha_{11}=1+\mu_{12}^2t_1^{r_{12}}t_2^{k_{12}}\alpha_{11}\end{aligned}。$$

将 α^* 记作 $\alpha_{11}^{h_{12}}$，其中 h_{12} 是 $s_1^{h_{12}}\equiv 1+\mu_{12}^2t_1^{r_{12}}t_2^{k_{12}}\left(\bmod p^{\beta_1}\right)$ 对应的结果，而 $-\alpha_{21}\alpha_{12}\alpha_{21}=-\mu_{12}\alpha_{21}$，$\alpha^{*-1}(1+\alpha_{12}\alpha_{21})^{-1}\alpha_{12}(1-\alpha_{21}\alpha_{12})^{-1}=t_1^{h_{12}+r_{12}}t_2^{k_{12}}\alpha_{12}$，于是得到 $a_{12}^{a_{21}}=a_{11}^{r_{12}+h_{12}}a_{12}^{t_1^{h_{12}+r_{12}}t_2^{k_{12}}}a_{21}^{-\mu_{12}}a_{22}^{k_{12}}$。

通过计算我们得到其余生成元的转置共轭：

$$a_{13}^{a_{31}}=a_{11}^{r_{13}+h_{13}}a_{13}^{t_1^{h_{13}+r_{13}}t_3^{k_{13}}}a_{31}^{-\mu_{13}}a_{33}^{k_{13}}$$

其中 r_{13},k_{13},h_{13} 分别为 $s_1^{r_{13}}\equiv 1+\mu_{13}\left(\bmod p^{\beta_1}\right)$， $s_3^{k_{13}}\equiv 1-\mu_{13}\left(\bmod p^{\beta_3}\right)$， $s_1^{h_{13}}\equiv 1+\mu_{13}^2 t_1^{r_{13}} t_3^{k_{13}}\left(\bmod p^{\beta_1}\right)$ 对应的结果；

$$a_{14}^{a_{41}}=a_{11}^{r_{14}+h_{14}}a_{14}^{t_1^{h_{14}+r_{14}}t_4^{k_{14}}}a_{41}^{-\mu_{14}}a_{44}^{k_{14}}$$

其中 r_{14},k_{14},h_{14} 分别为 $s_1^{r_{14}}\equiv 1+\mu_{14}\left(\bmod p^{\beta_1}\right)$， $s_4^{k_{14}}\equiv 1-\mu_{14}\left(\bmod p^{\beta_4}\right)$， $s_1^{h_{14}}\equiv 1+\mu_{14}^2 t_1^{r_{14}} t_4^{k_{14}}\left(\bmod p^{\beta_1}\right)$ 对应的结果；

$$a_{23}^{a_{32}}=a_{22}^{r_{23}+h_{23}}a_{23}^{t_2^{h_{23}+r_{23}}t_3^{k_{23}}}a_{32}^{-\mu_{23}}a_{33}^{k_{23}}$$

其中 r_{23},k_{23},h_{23} 分别为 $s_2^{r_{23}}\equiv 1+\mu_{23}\left(\bmod p^{\beta_2}\right)$， $s_3^{k_{23}}\equiv 1-\mu_{23}\left(\bmod p^{\beta_3}\right)$， $s_2^{h_{23}}\equiv 1+\mu_{23}^2 t_2^{r_{23}} t_3^{k_{23}}\left(\bmod p^{\beta_2}\right)$ 对应的结果；

$$a_{24}^{a_{42}}=a_{22}^{r_{24}+h_{24}}a_{24}^{t_2^{h_{24}+r_{24}}t_4^{k_{24}}}a_{42}^{-\mu_{24}}a_{44}^{k_{24}}$$

其中 r_{24},k_{24},h_{24} 分别为 $s_2^{r_{24}}\equiv 1+\mu_{24}\left(\bmod p^{\beta_2}\right)$， $s_4^{k_{24}}\equiv 1-\mu_{24}\left(\bmod p^{\beta_4}\right)$， $s_2^{h_{24}}\equiv 1+\mu_{24}^2 t_2^{r_{24}} t_4^{k_{24}}\left(\bmod p^{\beta_2}\right)$ 对应的结果；

$$a_{34}^{a_{43}}=a_{33}^{r_{34}+h_{34}}a_{34}^{t_3^{h_{34}+r_{34}}t_4^{k_{34}}}a_{43}^{-\mu_{34}}a_{44}^{k_{34}}$$

其中 r_{34},k_{34},h_{34} 分别为 $s_3^{r_{34}}\equiv 1+\mu_{34}\left(\bmod p^{\beta_3}\right)$， $s_4^{k_{34}}\equiv 1-\mu_{34}\left(\bmod p^{\beta_4}\right)$， $s_3^{h_{34}}\equiv 1+\mu_{34}^2 t_3^{r_{34}} t_4^{k_{34}}\left(\bmod p^{\beta_3}\right)$ 对应的结果。

至此，我们得到 16 个生成元的 136 种生成关系，

$$\mathrm{Aut}C_{p^{\beta_1}}\times C_{p^{\beta_2}}\times C_{p^{\beta_3}}\times C_{p^{\beta_4}}\cong\langle a_{11},a_{12},a_{13},a_{14},a_{21},a_{22},a_{23},a_{24},a_{31},a_{32},a_{33},a_{34},a_{41},a_{42},a_{43},a_{44}:$$

$$\begin{aligned}
&a_{11}^{\varphi\left(p^{\beta_1}\right)}=a_{12}^{p^{\beta_2}}=a_{13}^{p^{\beta_3}}=a_{14}^{p^{\beta_4}}=\\
&a_{21}^{p^{\beta_2}}=a_{22}^{\varphi\left(p^{\beta_2}\right)}=a_{23}^{p^{\beta_3}}=a_{24}^{p^{\beta_4}}=\\
&a_{31}^{p^{\beta_3}}=a_{32}^{p^{\beta_3}}=a_{33}^{\varphi\left(p^{\beta_3}\right)}=a_{34}^{p^{\beta_4}}=\\
&a_{41}^{p^{\beta_4}}=a_{42}^{p^{\beta_4}}=a_{43}^{p^{\beta_4}}=a_{44}^{\varphi\left(p^{\beta_4}\right)}=1,
\end{aligned}$$

$$\begin{aligned}
&a_{11}^{a_{12}}=a_{11}a_{12}^{1-t_1},a_{11}^{a_{13}}=a_{11}a_{13}^{1-t_1},a_{11}^{a_{14}}=a_{11}a_{14}^{1-t_1},a_{11}^{a_{21}}=a_{21}^{t_1-1}a_{11},a_{11}^{a_{22}}=a_{11},a_{11}^{a_{23}}=a_{11},\\
&a_{11}^{a_{24}}=a_{11},a_{11}^{a_{31}}=a_{31}^{t_1-1}a_{11},a_{11}^{a_{32}}=a_{11},a_{11}^{a_{33}}=a_{11},a_{11}^{a_{34}}=a_{11},a_{11}^{a_{41}}=a_{41}^{t_1-1}a_{11},\\
&a_{11}^{a_{42}}=a_{11},a_{11}^{a_{43}}=a_{11},a_{11}^{a_{44}}=a_{11},\\
&a_{22}^{a_{23}}=a_{22}a_{23}^{1-t_2},a_{22}^{a_{24}}=a_{22}a_{24}^{1-t_2},a_{22}^{a_{31}}=a_{22},a_{22}^{a_{32}}=a_{32}^{t_2-1}a_{22},a_{22}^{a_{33}}=a_{22},a_{22}^{a_{34}}=a_{22},\\
&a_{22}^{a_{41}}=a_{22},a_{22}^{a_{42}}=a_{42}^{t_2-1}a_{22},a_{22}^{a_{43}}=a_{22},a_{22}^{a_{44}}=a_{22},\\
&a_{33}^{a_{34}}=a_{33}a_{34}^{1-t_3},a_{33}^{a_{41}}=a_{33},a_{33}^{a_{42}}=a_{33},a_{33}^{a_{43}}=a_{43}^{t_3-1}a_{33},a_{33}^{a_{44}}=a_{33},\\
&a_{12}^{a_{13}}=a_{12},a_{12}^{a_{14}}=a_{12},a_{12}^{a_{22}}=a_{12}^{s_2},a_{12}^{a_{23}}=a_{12}a_{13}^{\mu_{12}\mu_{23}/\mu_{13}},a_{12}^{a_{24}}=a_{12}a_{14}^{\mu_{12}\mu_{24}/\mu_{14}},a_{12}^{a_{31}}=a_{12}a_{32}^{-\mu_{31}\mu_{12}/\mu_{32}},\\
&a_{12}^{a_{32}}=a_{12},a_{12}^{a_{33}}=a_{12},a_{12}^{a_{34}}=a_{12},a_{12}^{a_{41}}=a_{12}a_{42}^{-\mu_{41}\mu_{12}/\mu_{42}},a_{12}^{a_{42}}=a_{12},a_{12}^{a_{43}}=a_{12},a_{12}^{a_{44}}=a_{12},\\
&a_{13}^{a_{14}}=a_{13},a_{13}^{a_{21}}=a_{13}a_{23}^{-\mu_{21}\mu_{13}/\mu_{23}},a_{13}^{a_{22}}=a_{13},a_{13}^{a_{23}}=a_{13},a_{13}^{a_{24}}=a_{13},a_{13}^{a_{32}}=a_{13}a_{12}^{\mu_{13}\mu_{32}/\mu_{12}},\\
&a_{13}^{a_{33}}=a_{13}^{s_3},a_{13}^{a_{34}}=a_{13}a_{14}^{\mu_{13}\mu_{34}/\mu_{14}},a_{13}^{a_{41}}=a_{13}a_{43}^{-\mu_{41}\mu_{13}/\mu_{43}},a_{13}^{a_{42}}=a_{13},a_{13}^{a_{43}}=a_{13},a_{13}^{a_{44}}=a_{13},\\
&a_{14}^{a_{21}}=a_{14}a_{24}^{-\mu_{21}\mu_{14}/\mu_{24}},a_{14}^{a_{22}}=a_{14},a_{14}^{a_{23}}=a_{14},a_{14}^{a_{24}}=a_{14},a_{14}^{a_{31}}=a_{14}a_{34}^{-\mu_{31}\mu_{14}/\mu_{34}},a_{14}^{a_{32}}=a_{14},\\
&a_{14}^{a_{33}}=a_{14},a_{14}^{a_{34}}=a_{14},a_{14}^{a_{42}}=a_{14}a_{12}^{\mu_{14}\mu_{42}/\mu_{12}},a_{14}^{a_{43}}=a_{14}a_{13}^{\mu_{14}\mu_{43}/\mu_{13}},a_{14}^{a_{44}}=a_{14}^{s_4},\\
&a_{21}^{a_{22}}=a_{21}^{t_2},a_{21}^{a_{23}}=a_{21},a_{21}^{a_{24}}=a_{21},a_{21}^{a_{31}}=a_{21},a_{21}^{a_{32}}=a_{21}a_{31}^{-\mu_{32}\mu_{21}/\mu_{31}},a_{21}^{a_{33}}=a_{21},a_{21}^{a_{34}}=a_{21},
\end{aligned}$$

$$
\begin{gathered}
a_{21}^{a_{41}}=a_{21}, a_{21}^{a_{42}}=a_{21} a_{41}^{-\mu_{42} \mu_{21} / \mu_{41}}, a_{21}^{a_{43}}=a_{21}, a_{21}^{a_{44}}=a_{21}, \\
a_{23}^{a_{24}}=a_{23}, a_{23}^{a_{31}}=a_{23} a_{21}^{\mu_{23} \mu_{31} / \mu_{21}}, a_{23}^{a_{33}}=a_{23}^{s_3}, a_{23}^{a_{34}}=a_{23} a_{24}^{\mu_{23} \mu_{34} / \mu_{24}}, \\
a_{23}^{a_{41}}=a_{23}, a_{23}^{a_{42}}=a_{23} a_{43}^{-\mu_{42} \mu_{23} / \mu_{43}}, a_{23}^{a_{43}}=a_{23}, a_{23}^{a_{44}}=a_{23}, \\
a_{24}^{a_{31}}=a_{24}, a_{24}^{a_{32}}=a_{24} a_{34}^{-\mu_{32} \mu_{24} / \mu_{34}}, a_{24}^{a_{33}}=a_{24}, a_{24}^{a_{34}}=a_{24}, \\
a_{24}^{a_{41}}=a_{24} a_{21}^{\mu_{24} \mu_{41} / \mu_{21}}, a_{24}^{a_{43}}=a_{24} a_{23}^{\mu_{24} \mu_{43} / \mu_{23}}, a_{24}^{a_{44}}=a_{24}^{s_4}, \\
a_{31}^{a_{32}}=a_{31}, a_{31}^{a_{33}}=a_{31}^{t_3}, a_{31}^{a_{34}}=a_{31}, a_{31}^{a_{41}}=a_{31}, a_{31}^{a_{42}}=a_{31}, a_{31}^{a_{43}}=a_{31} a_{41}^{-\mu_{43} \mu_{31} / \mu_{41}}, a_{31}^{a_{44}}=a_{31}, \\
a_{32}^{a_{33}}=a_{32}^{t_3}, a_{32}^{a_{34}}=a_{32}, a_{32}^{a_{41}}=a_{32}, a_{32}^{a_{42}}=a_{32}, a_{32}^{a_{43}}=a_{32} a_{42}^{-\mu_{43} \mu_{32} / \mu_{42}}, a_{32}^{a_{44}}=a_{32}, \\
a_{34}^{a_{41}}=a_{34} a_{31}^{\mu_{34} \mu_{41} / \mu_{31}}, a_{34}^{a_{42}}=a_{34} a_{32}^{\mu_{34} \mu_{42} / \mu_{32}}, a_{34}^{a_{44}}=a_{34}^{s_4}, \\
a_{41}^{a_{42}}=a_{41}, a_{41}^{a_{43}}=a_{41}, a_{41}^{a_{44}}=a_{41}^{t_4}, a_{42}^{a_{43}}=a_{42}, a_{42}^{a_{44}}=a_{42}^{t_4}, a_{43}^{a_{44}}=a_{43}^{t_4}, \\
a_{12}^{a_{21}}=a_{11}^{r_{12}+h_{12}} a_{12}^{t_1^{h_{12}+r_{12}} t_2^{k_{12}}} a_{21}^{-\mu_{12}} a_{22}^{k_{12}}, a_{13}^{a_{31}}=a_{11}^{r_{13}+h_{13}} a_{13}^{t_1^{h_{13}+r_{13}} t_3^{k_{13}}} a_{31}^{-\mu_{13}} a_{33}^{k_{13}}, \\
a_{14}^{a_{41}}=a_{11}^{r_{14}+h_{14}} a_{14}^{t_1^{h_{14}+r_{14}} t_4^{k_{14}}} a_{41}^{-\mu_{14}} a_{44}^{k_{14}}, a_{23}^{a_{32}}=a_{22}^{r_{23}+h_{23}} a_{23}^{t_2^{h_{23}+r_{23}} t_3^{k_{23}}} a_{32}^{-\mu_{23}} a_{33}^{k_{23}}, \\
a_{24}^{a_{42}}=a_{22}^{r_{24}+h_{24}} a_{24}^{t_2^{h_{24}+r_{24}} t_4^{k_{24}}} a_{42}^{-\mu_{24}} a_{44}^{k_{24}}, a_{34}^{a_{43}}=a_{33}^{r_{34}+h_{34}} a_{34}^{t_3^{h_{34}+r_{34}} t_4^{k_{34}}} a_{43}^{-\mu_{34}} a_{44}^{k_{34}} \Big\rangle
\end{gathered}
$$

基金项目

国家自然科学基金(11401424)和山西省自然科学基金(2013011001-3)资助项目资助。

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