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The Diversity of Solutions Satisfied Partial Boundary Conditions for a Partial Differential Equation

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Abstract

For (initial) boundary value problems of partial differential equations, most of current methods are based on their partial (initial) boundary conditions. Then the solution obtained in this way could satisfy all the boundary conditions? For this reason, we based on Adomian decomposition method to solve the Dirichlet boundary value problem of groundwater recharge effect model on triangular area. We find that: 1) the solution satisfies all boundary conditions sometimes, sometimes not satisfies; 2) the solution satisfied partial boundary conditions is not unique; 3) the solution obtained by the Adomian decomposition method is a particular solution that satisfies partial boundary conditions.

Keywords

Partial Differential Equations, Dirichlet Boundary Value Problem, Adomian Decomposition Method

偏微分方程的满足部分边界条件的解的多样性

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摘　要

对偏微分方程的(初)边值问题，目前的大多数求解方法基于其部分(初)边界条件。那么这样得到的解能否满足所有边界条件呢？为此，我们基于Adomian分解法，求解三角形地下水域上补给效应模型的Dirichlet边值问题。我们发现：1) 求出的解有时满足所有边界条件，有时不满足；2) 满足部分边界条件的解不是唯一；3) Adomian分解法得到的解是满足部分边界条件的某一个特解。

关键词

偏微分方程，Dirichlet边值问题，Adomian分解法

1. 引言

求解偏微分方程精确解的方法有很多，如双线性导数法[1] [2]，齐次平衡法[3]，辅助方程法[4] [5] [6]，Lie 对称[7] [8] [9]，达布变换[10]，tanh 函数法[11]，G 展开法[12]等等。但这些方法很难应用于偏微分方程的(初)边值问题。对偏微分方程(初)边值问题，近期人们提出并发展了很多方法，如摄动法[13]，Adomian 分解法[14]、同伦摄动法[15] [16]、同伦分析法[17]等等。而这些求解偏微分方程的边值问题的方法，求解过程中主要依赖部分边界条件，不是所有边界条件。那么，基于部分边界条件能否得到满足所有边界条件的解呢？即满足部分边界条件的解是否唯一呢？针对该问题，我们基于 Adomian 分解考虑三角形地下水域上的补给效应模型。

Adomian 分解法是美国数学物理学家 Georgie Adomian [18] [19]提出的一种分解法。该方法克服了传统摄动方法对小参数的依赖性，并且具有很好的收敛性和易计算性。对偏微分方程的(初)边值问题，传统的 Adomian 分解法只基于部分边界条件，不是所有边界条件(见文[20] [21] [22] [23] [24])。Serrano [24] [25]等人研究地下水流补给效应模型。地下水流模型控制微分方程为：

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=-\frac{Rg}{T},\ 0\le x\le 600, x\le y\le 600-x \tag{1.1}$$

其中 $T=100\ \mathrm{m}^2/\mathrm{month}, Rg=10\ \mathrm{mm/month}$，假设附加的边界条件为

$$h(x,x)=f_1(x), \tag{1.2}$$

$$h(x,600-x)=f_2(x), \tag{1.3}$$

$$h(x,0)=f_3(x). \tag{1.4}$$

其中

$$f_1(x)=100+\frac{2x}{125}-\frac{x^2}{50000}, \tag{1.5}$$

$$f_2(x)=\frac{448}{5}+\frac{103x}{1500}-\frac{x^2}{12500}, \tag{1.6}$$

$$f_3(x)=100+\frac{17x}{3750}-\frac{x^2}{500000} \tag{1.7}$$

为了方便令其中 $L_x=\frac{\partial^2}{\partial x^2},L_y=\frac{\partial^2}{\partial y^2}$ 则方程(1.1)可改写为

$$L_x h(x,y)+L_y h(x,y)=-\frac{Rg}{T} \tag{1.8}$$

2. 基于边界条件(1.2)和(1.3)的 Adomian 近似解

根据 Adomian 算法[23]，把逆算子

$$L_1^{-1}=\int_y^x\int_y^x\cdot\mathrm{d}x\mathrm{d}x-\frac{x-y}{600-2y}\int_y^{600-y}\int_y^x\cdot\mathrm{d}x\mathrm{d}x$$

作用于方程(1.8)两边，并考虑边界条件(1.2)，(1.3)可得

$$\sum_{n=0}^{\infty}h_n=\left(1-\frac{x-y}{600-2y}\right)f_1(y)+\frac{x-y}{600-2y}f_2(600-y)-L_1^{-1}\frac{Rg}{T}-L_1^{-1}L_y\sum_{n=0}^{\infty}h_n \tag{2.1}$$

由上式构造迭代公式

$$h_0(x,y)=\left(1-\frac{x-y}{600-2y}\right)f_1(y)+\frac{x-y}{600-2y}f_2(600-y)-L_x^{-1}\frac{rg}{t},$$

$$h_n(x,y)=-L_x^{-1}L_y h_{n-1} \tag{2.2}$$

从迭代公式(2.2)，我们得到

$$h_0(x,y)=100-\frac{x^2}{20000}+x\left(\frac{1}{30}+\frac{3y}{100000}\right)-\frac{13y}{750},h_n=0,(n\ge 1)$$

由此得到方程(1.1)的满足边界条件(1.2)和(1.3)的精确解：

$$H_1(x,y)=100-\frac{x^2}{20000}+x\left(\frac{1}{30}+\frac{3y}{100000}\right)-\frac{13y}{750} \tag{2.3}$$

经验证 $H_1(x,y)$ 不满足边界条件(1.4)。

3. 基于(1.3)和(1.4)的 Adomian 近似解

将逆算子

$$L_2^{-1}=\int_0^y\int_0^y\cdot\mathrm{d}y\mathrm{d}y-\frac{y}{600-x}\int_0^{600-x}\int_0^y\cdot\mathrm{d}y\mathrm{d}y$$

作用于方程(1.8)两侧，考虑边界条件(1.3)和(1.4)，可得

$$\sum_{n=0}^{\infty}h_n=\left(1-\frac{y}{600-x}\right)f_3(x)+\frac{y}{600-x}f_2(x)-L_2^{-1}\frac{Rg}{T}-L_2^{-1}L_x\sum_{n=0}^{\infty}h_n, \tag{3.1}$$

从上式，我们构造迭代公式：

$$h_0(x,y)=\left(1-\frac{y}{600-x}\right)f_3(x)+\frac{y}{600-x}f_2(x)-L_y^{-1}\frac{rg}{t},$$

$$h_n(x,y)=-L_y^{-1}L_x h_{n-1} \tag{3.2}$$

从(3.2)，我们得到

$$h_0(x,y)=\frac{-3x^2+x(6800+42y)+25(6000000+760y-3y^2)}{1500000} \tag{3.3}$$

$$h_1(x,y)=\frac{y(-600+x+y)}{500000} \tag{3.4}$$

$$h_n(x,y)=0,(n\geq 2) \tag{3.5}$$

由此得到方程满足边界条件(1.3)和(1.4)的精确解:

$$H_2(x,y)=\frac{-3x^2+5x(1360+9y)+8(18750000+2150y-9y^2)}{1500000} \tag{3.6}$$

通过验证发现 $H_2(x,y)$ 满足所有边界条件。

4. 总结

从上述求解结果可以发现：基于(1.2)，(1.3)用 Adomian 算法求出的解(2.3)只满足边界条件(1.2)，(1.3)，不满足(1.4)；而基于(1.3)，(1.4)的 Adomian 解满足所有边界条件(1.2)~(1.3)。这说明，利用部分边界条件求出的解有时满足所有的边界条件，有时只满足部分边界条件，不是所有边界条件。Adomian 解(2.3)和(3.6)都满足边界条件(1.2)与(1.3)，从而得知满足偏微分方程部分边界条件的解不是唯一，是多样的。并且说明了有 Adomian 分解法得到的解是满足部分边界条件的某一个特解。

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参考文献

[1] Hirota, R. and Satsuma, J. (1976) N-Soliton Solutions of Model Equations for Shallow Water Waves. *Journal of the Physical Society of Japan*, **40**, 611-612. https://doi.org/10.1143/JPSJ.40.611

[2] Hirota, R. (2004) The Direct Method in Soliton Theory. Cambridge University Press, Cambridge. https://doi.org/10.1017/CBO9780511543043

[3] Wang, M.L. (1996) Exact Solutions for a Compound KdV-Burgers Equation. *Physics Letters A*, **213**, 279-287. https://doi.org/10.1016/0375-9601(96)00103-X

[4] Chen, Y., Li, B. and Zhang, H.Q. (2003) Generalized Riccati Equation Expansion Method and Its Application to the Bogoyavlenskiis Generalized Breaking Soliton Equation. *Chinese Physics*, **12**, 940-945. https://doi.org/10.1088/1009-1963/12/9/303

[5] 伊丽娜，套格图桑. 非线性耦合 KdV 方程组的一种新求解法[J]. 数学杂志, 2017, 37(4): 823-832.

[6] 李宁，套格图桑. 几种广义非线性发展方程的新解[J]. 数学杂志, 2016, 36(5): 1103-1110.

[7] Bluman, G.W. and Kumei, S. (1989) Symmetries and Di Erential Equations. Springer-Verlag, New York. https://doi.org/10.1007/978-1-4757-4307-4

[8] Yun, Y.S. and Chaolu, T. (2015) Classical and Nonclassical Symmetry Classifications of Nonlinear Wave Equation with Dissipation. *Applied Mathematics and Mechanics* (*English Edition*), **36**, 365-378. https://doi.org/10.1007/s10483-015-1910-6

[9] 白月星，苏道毕力格. Poisson 方程的一维最优系统和不变解[J]. 数学杂志, 2018, 38(4): 706-712.

[10] 谷超豪，胡和生，周子翔. 孤立子理论中的达布变换及其几何应用[M]. 上海：科学技术出版社, 2005.

[11] Fan, E.G. (2000) Extended Tanh-Function Method and Its Applications to Nonlinear Equations. *Physics Letters A*, **277**,

212-218. https://doi.org/10.1016/S0375-9601(00)00725-8
[12] 王鑫, 邢文雅, 李胜军. 一类推广的 KdV 方程的新行波解[J]. 数学杂志, 2017, 37(4): 859-864.
[13] Nayfeh, A.H. (1973) Perturbation Methods. John Wiley and Sons, New York.
[14] Adomian, G. (1988) A Review of the Decomposition Method in Applied Mathematics. *Journal of Mathematical Analysis and Applications*, **135**, 501-544. https://doi.org/10.1016/0022-247X(88)90170-9
[15] He, J.H. (2000) A Coupling Method of a Homotopy Technique and a Perturbation Technique for Non-Linear Problems. *International Journal of Non-Linear Mechanics*, **35**, 37-43. https://doi.org/10.1016/S0020-7462(98)00085-7
[16] Yun, Y. and Temuer, C. (2015) Application of the Homotopy Perturbation Method for the Large Deflection Problem of a Circular Plate. *Applied Mathematical Modelling*, **39**, 1308-1316. https://doi.org/10.1016/j.apm.2014.09.001
[17] Liao, S. (1992) The Proposed Homotopy Analysis Technique for the Solution of Nonlinear Problems. PhD Thesis, Shanghai Jiao Tong University, Shanghai.
[18] Adomian, G. (1998) Solutions of Nonlinear P.D.E. *Applied Mathematics Letters*, **11**, 121-123. https://doi.org/10.1016/S0893-9659(98)00043-3
[19] Adomian, G. (1983) Stochastic Systems. Academic Press, Pittsburgh.
[20] Benneouala, T., Cherruault, Y. and Abbaoui, K. (2005) New Methods for Applying the Adomian Method to Partial Differential Equations with Boundary Conditions. *Kybernetes*, **34**, 924-933. https://doi.org/10.1108/03684920510605740
[21] Patel, A. and Serrano, S.E. (2011) Decomposition Solution of Multidimensional Groundwater Equations. *Journal of Hydrology*, **397**, 202-209. https://doi.org/10.1016/j.jhydrol.2010.11.032
[22] Shidfar, G. (2009) A Weighted Algorithm Based on Adomian Decomposition Method for Solving an Special Class of Evolution Equations. *Communications in Nonlinear Science and Numerical Simulation*, **14**, 1146-1151. https://doi.org/10.1016/j.cnsns.2008.04.004
[23] Yun, Y., Temuer, C. and Duan, J. (2014) A Segmented and Weighted Adomian Decomposition Algorithm for Boundary Value Problem of Nonlinear Groundwater Equation. *Mathematical Methods in the Applied Sciences*, **37**, 2406-2418. https://doi.org/10.1002/mma.2986
[24] Syafrin, T. and Serrano, S.E. (2015) Regional Groundwater Flow in the Louisville Aquifer. *Ground Water*, **53**, 550-557. https://doi.org/10.1111/gwat.12242
[25] Serrano, S. (2013) A Simple Approach to Groundwater Modelling with Decomposition. *International Association of Scientific Hydrology Bulletin*, **58**, 177-187. https://doi.org/10.1080/02626667.2012.745938