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A New Half-Discrete Hilbert's Inequality

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Abstract: In this paper, by introducing some parameters and estimating the weight function, we give a new half-discrete Hilbert-type inequality with a best constant factor. The equivalent inequality forms is considered.

Keywords: Half-Discrete; Hilbert's Inequality; Hölder's Inequality

一个新的半离散 Hilbert 型不等式

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摘　要：应用权函数，给出一个新的有最佳常数因子的半离散 Hilbert 型不等式。同时给出他的等价式。

关键词：半离散；Hilbert 不等式；Hölder 不等式

1. 引言

设 $p>1,\dfrac{1}{p}+\dfrac{1}{q}=1,\ a_n,b_n>0,$ 且$0<\sum\limits_{n=1}^{\infty}a_n^p<\infty$，及 $0<\sum\limits_{n=1}^{\infty}b_n^q<\infty$，则有如下含最佳常数因子的 Hardy-Hilbert 积分不等式[1]：

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_n b_m}{m+n}\right)<\frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=1}^{\infty}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{1/q} \tag{1}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_n b_m}{m+n}\right)^p<\left(\frac{\pi}{\sin(\pi/p)}\right)^p\left\{\sum_{n=1}^{\infty}a_n^p\right\} \tag{2}$$

近年来，人们陆续对不等式(1)(2)作了大量推广[2-16]。2011 年杨必成教授给出以下半离散 Hilbert 型不等式[2]：

设 $p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1,\ p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1,\ \lambda_1>0,\ 1\ge\lambda_2>0,\ p>1,\ \lambda_1+\lambda_2=\lambda,$ 且 $0<\int_0^{\infty}x^{p(1-\lambda_1)-1}f^p(x)\mathrm{d}x<\infty$，

$0<\sum\limits_{n=1}^{\infty}n^{q(1-\lambda_2)-1}<\infty$，则

$$\sum_{n=1}^{\infty}a_n\int_0^{\infty}\frac{f(x)}{(x+n)^{\lambda}}\mathrm{d}x=\int_0^{\infty}f(x)\sum_{n=1}^{\infty}\frac{a_n}{(x+n)^{\lambda}}\mathrm{d}x<B(\lambda_1,\lambda_2)\left\{\int_0^{\infty}x^{p(1-\lambda_1)-1}f^p(x)\mathrm{d}x\right\}^{1/p}\cdot\left\{\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}\right\}^{1/q} \tag{3}$$

其中 $B(\lambda_1,\lambda_2)$ 为 β 函数。

我们应用权函数，将给出一个 -3μ 齐次的有最佳常数因子的半离散 Hilbert 型不等式。同时给出他的等价式。以下我们总假设 $p>1,\frac{1}{p}+\frac{1}{q}=1,\ 0\le\mu\le\frac{2}{3}$。

2. 一些引理

引理 1 定义权系数及权函数 $W(n)$ 和 $\tilde{W}(x)$ 如下

$$W(n)=\int_0^{\infty}\frac{1}{\max\{x^{\mu},n^{\mu}\}(x^{\mu}+a^2n^{\mu})(x^{\mu}+b^2n^{\mu})}\frac{n^{\frac{3\mu}{2}}}{x^{1-\frac{3\mu}{2}}}\mathrm{d}x;$$

$$\tilde{W}(x)=\sum_{n=1}^{\infty}\frac{1}{\max\{x^{\mu},n^{\mu}\}(x^{\mu}+a^2n^{\mu})(x^{\mu}+b^2n^{\mu})}\cdot\frac{x^{\frac{3\mu}{2}}}{n^{1-\frac{3\mu}{2}}}$$

则有

$$W(n)=K=\frac{h}{\mu};\ \tilde{W}(x)<K;$$

其中

$$h=\int_0^{\infty}\frac{u^{1/2}\mathrm{d}u}{\max\{u,1\}(u+a^2)(u+b^2)}=\int_0^{\infty}\frac{u^{1/2}\mathrm{d}u}{\max\{1,u\}(1+a^2u)(1+b^2u)}$$
$$=\begin{cases}\frac{\pi}{ab(a+b)}+\frac{2}{a^2-b^2}\left(\frac{a^2+1}{a}\arctan\frac{1}{a}-\frac{b^2+1}{b}\arctan\frac{1}{b}\right), & 当a\ne b时,\\ \frac{\pi}{2a^3}-\frac{1}{a^2}+\frac{1}{a}\left(1-\frac{1}{a^2}\right)\arctan\frac{1}{a}, & 当a=b时,\end{cases}$$

证明：首先我们易有

$$\int_0^{\infty}\frac{u^{1/2}\mathrm{d}u}{\max\{u,1\}(u+a^2)(u+b^2)}=\int_0^{\infty}\frac{u^{1/2}\mathrm{d}u}{\max\{1,u\}(1+a^2u)(1+b^2u)}$$

设 $x=nu^{1/\mu}$，则

$$W(n)=\int_0^{1}\frac{u^{1/2}\mathrm{d}u}{\max\{u,1\}(u+a^2)(u+b^2)}+\int_1^{\infty}\frac{u^{1/2}\mathrm{d}u}{\max\{u,1\}(u+a^2)(u+b^2)}$$
$$=\int_0^{1}\frac{u^{1/2}\mathrm{d}u}{(u+a^2)(u+b^2)}+\int_1^{\infty}\frac{u^{1/2}\mathrm{d}u}{u(u+a^2)(u+b^2)}=\int_0^{1}\frac{2t^2\mathrm{d}t}{(t^2+a^2)(t^2+b^2)}+\int_1^{\infty}\frac{2\mathrm{d}t}{(t^2+a^2)(t^2+b^2)}$$
$$=\begin{cases}\frac{\pi}{ab(a+b)}+\frac{2}{a^2-b^2}\left(\frac{a^2+1}{a}\arctan\frac{1}{a}-\frac{b^2+1}{b}\arctan\frac{1}{b}\right), & 当a\ne b时,\\ \frac{\pi}{2a^3}-\frac{1}{a^2}+\frac{1}{a}\left(1-\frac{1}{a^2}\right)\arctan\frac{1}{a}, & 当a=b时,\end{cases}$$

又 $\frac{1}{\max\left\{x^{\mu},n^{\mu}\right\}\left(x^{\mu}+a^{2}n^{\mu}\right)\left(x^{\mu}+b^{2}n^{\mu}\right)}\cdot\frac{x^{\frac{3\mu}{2}}}{n^{1-\frac{3\mu}{2}}}$ 关于 n 严格单调下降，于是

$$\tilde{W}(x)<\int_{0}^{\infty}\frac{1}{\max\left\{x^{\mu},y^{\mu}\right\}\left(x^{\mu}+a^{2}y^{\mu}\right)\left(x^{\mu}+b^{2}y^{\mu}\right)}\frac{x^{\frac{3\mu}{2}}}{y^{1-\frac{3\mu}{2}}}\mathrm{d}y=\frac{1}{\mu}\int_{0}^{\infty}\frac{u^{1/2}\mathrm{d}u}{\max\{1,u\}(1+a^{2}u)(1+b^{2}u)}=K$$

引理获证。

引理 2 设 $p>1,\ a_n\geq 0$，$f(x)$ 在 $(0,\infty)$ 非负可测，且

$$0<\int_{0}^{\infty}x^{p\left(1-\frac{3\mu}{2}\right)-1}f^{p}(x)\mathrm{d}x<\infty\text{，}\quad 0<\sum_{n=1}^{\infty}n^{q\left(1-\frac{3\mu}{2}\right)-1}a_{n}^{q}<\infty\text{，}$$

则有如下不等式：

$$J_{1}:=\sum_{n=1}^{\infty}n^{\frac{3p\mu}{2}-1}\left(\int_{0}^{\infty}\frac{f(x)}{\max\left\{x^{\mu},n^{\mu}\right\}\left(x^{\mu}+a^{2}n^{\mu}\right)\left(x^{\mu}+b^{2}n^{\mu}\right)}\mathrm{d}x\right)^{p}\leq K^{p}\int_{0}^{\infty}x^{p\left(1-\frac{3\mu}{2}\right)-1}f^{p}(x)\mathrm{d}x \tag{4}$$

$$J_{2}:=\int_{0}^{\infty}x^{\frac{3q\mu}{2}-1}\left(\sum_{n=1}^{\infty}\frac{a_{n}}{\max\left\{x^{\mu},n^{\mu}\right\}\left(x^{\mu}+a^{2}n^{\mu}\right)\left(x^{\mu}+b^{2}n^{\mu}\right)}\right)^{q}\mathrm{d}x\leq K^{q}\sum_{n=1}^{\infty}n^{q\left(1-\frac{3\mu}{2}\right)-1}a_{n}^{q} \tag{5}$$

证明：由带权 Hölder 不等式及引理 1，有

$$\left(\int_{0}^{\infty}\frac{f(x)}{\max\left\{x^{\mu},n^{\mu}\right\}\left(x^{\mu}+a^{2}n^{\mu}\right)\left(x^{\mu}+b^{2}n^{\mu}\right)}\mathrm{d}x\right)^{p}$$

$$=\left\{\int_{0}^{\infty}\frac{1}{\max\left\{x^{\mu},n^{\mu}\right\}\left(x^{\mu}+a^{2}n^{\mu}\right)\left(x^{\mu}+b^{2}n^{\mu}\right)}\left[\frac{x^{\left(1-\frac{3\mu}{2}\right)/q}}{n^{\left(1-\frac{3\mu}{2}\right)/p}}f(x)\right]\cdot\left[\frac{n^{\left(1-\frac{3\mu}{2}\right)/p}}{x^{\left(1-\frac{3\mu}{2}\right)/q}}\right]\mathrm{d}x\right\}^{p}$$

$$\leq\int_{0}^{\infty}\frac{1}{\max\left\{x^{\mu},n^{\mu}\right\}(x^{\mu}+a^{2}n^{\mu})(x^{\mu}+b^{2}n^{\mu})}\frac{x^{\left(1-\frac{3\mu}{2}\right)(p-1)}}{n^{1-\frac{3\mu}{2}}}f^{p}(x)\mathrm{d}x\cdot\left\{\int_{0}^{\infty}\frac{1}{\max\left\{x^{\mu},n^{\mu}\right\}(x^{\mu}+a^{2}n^{\mu})\left(x^{\mu}+b^{2}n^{\mu}\right)}\frac{n^{\left(1-\frac{3\mu}{2}\right)(q-1)}}{x^{1-\frac{3\mu}{2}}}\mathrm{d}x\right\}^{p-1}$$

$$=\left[W(n)\right]^{p-1}n^{1-\frac{3p\mu}{2}}\int_{0}^{\infty}\frac{1}{\max\left\{x^{\mu},n^{\mu}\right\}\left(x^{\mu}+a^{2}n^{\mu}\right)\left(x^{\mu}+b^{2}n^{\mu}\right)}\frac{x^{\left(1-\frac{3\mu}{2}\right)(p-1)}}{n^{1-\frac{3\mu}{2}}}f^{p}(x)\mathrm{d}x$$

$$=K^{p-1}n^{1-\frac{3p\mu}{2}}\int_{0}^{\infty}\frac{1}{\max\left\{x^{\mu},n^{\mu}\right\}\left(x^{\mu}+a^{2}n^{\mu}\right)\left(x^{\mu}+b^{2}n^{\mu}\right)}\frac{x^{\left(1-\frac{3\mu}{2}\right)(p-1)}}{n^{1-\frac{3\mu}{2}}}f^{p}(x)\mathrm{d}x$$

$$J_{1}\leq K^{p-1}\sum_{n=1}^{\infty}\left[\int_{0}^{\infty}\frac{1}{\max\left\{x^{\mu},n^{\mu}\right\}\left(x^{\mu}+a^{2}n^{\mu}\right)\left(x^{\mu}+b^{2}n^{\mu}\right)}\frac{x^{\left(1-\frac{3\mu}{2}\right)(p-1)}}{n^{1-\frac{3\mu}{2}}}f^{p}(x)\mathrm{d}x\right]$$

$$=K^{p-1}\int_{0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\max\left\{x^{\mu},n^{\mu}\right\}\left(x^{\mu}+a^{2}n^{\mu}\right)\left(x^{\mu}+b^{2}n^{\mu}\right)}\frac{x^{\left(1-\frac{3\mu}{2}\right)(p-1)}}{n^{1-\frac{3\mu}{2}}}f^{p}(x)\mathrm{d}x\leq K^{p}\int_{0}^{\infty}x^{p\left(1-\frac{3\mu}{2}\right)-1}f^{p}(x)\mathrm{d}x$$

故(4)成立。类似地有，

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty}\frac{a_n}{\max\left\{x^{\mu},n^{\mu}\right\}\left(x^{\mu}+a^2n^{\mu}\right)\left(x^{\mu}+b^2n^{\mu}\right)}\right)^q\\
&=\left\{\sum_{n=1}^{\infty}\frac{1}{\max\left\{x^{\mu},n^{\mu}\right\}\left(x^{\mu}+a^2n^{\mu}\right)\left(x^{\mu}+b^2n^{\mu}\right)}\left[\frac{x^{\left(1-\frac{3\mu}{2}\right)/q}}{n^{\left(1-\frac{3\mu}{2}\right)/p}}\right]\cdot\left[\frac{n^{\left(1-\frac{3\mu}{2}\right)/p}}{x^{\left(1-\frac{3\mu}{2}\right)/q}}a_n\right]\right\}^q\\
&\le\left[\sum_{n=1}^{\infty}\frac{1}{\max\left\{x^{\mu},n^{\mu}\right\}\left(x^{\mu}+a^2n^{\mu}\right)\left(x^{\mu}+b^2n^{\mu}\right)}\frac{x^{\left(1-\frac{3\mu}{2}\right)(p-1)}}{n^{1-\frac{3\mu}{2}}}\right]^{q-1}\\
&\quad\cdot\sum_{n=1}^{\infty}\frac{1}{\max\left\{x^{\mu},n^{\mu}\right\}\left(x^{\mu}+a^2n^{\mu}\right)\left(x^{\mu}+b^2n^{\mu}\right)}\frac{x^{\left(1-\frac{3\mu}{2}\right)(q-1)}}{n^{1-\frac{3\mu}{2}}}a_n^q\\
&=\left[\tilde{W}(x)\right]^{q-1}x^{1-\frac{3q\mu}{2}}\sum_{n=1}^{\infty}\frac{1}{\max\left\{x^{\mu},n^{\mu}\right\}\left(x^{\mu}+a^2n^{\mu}\right)\left(x^{\mu}+b^2n^{\mu}\right)}\cdot\frac{n^{\left(1-\frac{3\mu}{2}\right)(q-1)}}{x^{1-\frac{3\mu}{2}}}a_n^q\\
&\le K^{q-1}x^{1-\frac{3q\mu}{2}}\sum_{n=1}^{\infty}\frac{1}{\max\left\{x^{\mu},n^{\mu}\right\}\left(x^{\mu}+a^2n^{\mu}\right)\left(x^{\mu}+b^2n^{\mu}\right)}\cdot\frac{n^{\left(1-\frac{3\mu}{2}\right)(q-1)}}{x^{1-\frac{3\mu}{2}}}a_n^q
\end{aligned}
$$

及类似地,

$$
J_2=\int_0^{\infty}x^{\frac{3q\mu}{2}-1}\left(\sum_{n=1}^{\infty}\frac{a_n}{\max\left\{x^{\mu},n^{\mu}\right\}\left(x^{\mu}+a^2n^{\mu}\right)\left(x^{\mu}+b^2n^{\mu}\right)}\right)^q\mathrm{d}x\le K^q\sum_{n=1}^{\infty}n^{q\left(1-\frac{3\mu}{2}\right)-1}a_n^q
$$

有(5)成立。

引理 3 设 $p>\varepsilon>0,\ \varepsilon$ 充分小，定义 $\tilde{f}(x)=0$ ，$x\in(0,1)$ ，$\tilde{f}(x)=x^{\frac{3\mu}{2}-1-\frac{\varepsilon}{p}}$ ，$x\in(1,\infty)$ ；及 $\tilde{a}_n=n^{\frac{3\mu}{2}-1-\frac{\varepsilon}{q}}$ ，$n\in\mathbb{N}$ 。则

$$
I(\varepsilon):=\varepsilon\left\{\int_1^{\infty}x^{p\left(1-\frac{3\mu}{2}\right)-1}\tilde{f}^p(x)\mathrm{d}x\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q\left(1-\frac{3\mu}{2}\right)-1}\tilde{a}_n^q\right\}^{\frac{1}{q}}=1+\mathrm{o}(1)\ \left(\varepsilon\to0^+\right)\tag{6}
$$

$$
\tilde{I}(\varepsilon):=\varepsilon\int_1^{\infty}x^{\frac{3\mu}{2}-1-\frac{\varepsilon}{p}}\left[\sum_{n=1}^{\infty}\frac{n^{\frac{3\mu}{2}-1-\frac{\varepsilon}{q}}}{\max\left\{x^{\mu},n^{\mu}\right\}\left(x^{\mu}+a^2n^{\mu}\right)\left(x^{\mu}+b^2n^{\mu}\right)}\right]\mathrm{d}x>K+\tilde{\mathrm{o}}(1)\ \left(\varepsilon\to0^+\right)\tag{7}
$$

证明 易有，

$$
I(\varepsilon)=\varepsilon\left\{\int_1^{\infty}x^{-1-\varepsilon}\mathrm{d}x\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{-1-\varepsilon}\right\}^{\frac{1}{q}}
$$

注意及右边最后一项有以下双边不等式

$$
\frac{1}{\varepsilon}=\int_1^{\infty}x^{-1-\varepsilon}\mathrm{d}x<\sum_{n=1}^{\infty}n^{-1-\varepsilon}=1+\sum_{n=2}^{\infty}n^{-1-\varepsilon}<1+\int_1^{\infty}x^{-1-\varepsilon}\mathrm{d}x=1+\frac{1}{\varepsilon}
$$

故有式(6)。又设 $y = xt^{1/\mu}$

$$\tilde{I}(\varepsilon) := \varepsilon\int_1^\infty x^{\frac{3\mu}{2}-1-\frac{\varepsilon}{p}}\left[\sum_{n=1}^{\infty}\frac{n^{\frac{3\mu}{2}-1-\frac{\varepsilon}{q}}}{\max\left\{x^\mu,n^\mu\right\}\left(x^\mu+a^2n^\mu\right)\left(x^\mu+b^2n^\mu\right)}\right]\mathrm{d}x$$

$$> \varepsilon\int_1^\infty x^{\frac{3\mu}{2}-1-\frac{\varepsilon}{p}}\left[\int_1^\infty\frac{y^{\frac{3\mu}{2}-1-\frac{\varepsilon}{q}}}{\max\left\{x^\mu,y^\mu\right\}\left(x^\mu+a^2y^\mu\right)\left(x^\mu+b^2y^\mu\right)}\mathrm{d}y\right]\mathrm{d}x$$

$$= \frac{\varepsilon}{\mu}\int_1^\infty x^{-1-\varepsilon}\left[\int_{\frac{1}{x^\mu}}^\infty\frac{t^{\frac{1}{2}-\frac{\varepsilon}{q\mu}}}{\max\{1,t\}\left(1+a^2t\right)\left(1+b^2t\right)}\mathrm{d}t\right]\mathrm{d}x$$

$$= \frac{\varepsilon}{\mu}\left\{\int_1^\infty x^{-1-\varepsilon}\left[\int_0^\infty\frac{t^{\frac{1}{2}-\frac{\varepsilon}{q\mu}}}{\max\{1,t\}\left(1+a^2t\right)\left(1+b^2t\right)}\mathrm{d}t\right]\mathrm{d}x - \int_1^\infty x^{-1-\varepsilon}\left[\int_0^{\frac{1}{x^\mu}}\frac{t^{\frac{1}{2}-\frac{\varepsilon}{q\mu}}}{\max\{1,t\}\left(1+a^2t\right)\left(1+b^2t\right)}\mathrm{d}t\right]\mathrm{d}x\right\}$$

$$= \frac{1}{\mu}\int_0^\infty\frac{t^{1/2}\mathrm{d}t}{\max\{1,t\}\left(1+a^2t\right)\left(1+b^2t\right)} - \frac{\varepsilon}{\mu}\int_0^1\frac{t^{\frac{1}{2}-\frac{\varepsilon}{q\mu}}\mathrm{d}t}{\max\{1,t\}\left(1+a^2t\right)\left(1+b^2t\right)}\left[\int_1^{\frac{1}{t^{1/\mu}}} x^{-1-\varepsilon}\mathrm{d}x\right]$$

$$= K + \frac{1}{\mu}\int_0^\infty\frac{t^{\frac{1}{2}}\left(t^{-\frac{\varepsilon}{q\mu}}-1\right)}{\max\{1,t\}\left(1+a^2t\right)\left(1+b^2t\right)}\mathrm{d}t + \frac{1}{\mu}\int_0^1\frac{t^{\frac{1}{2}}\left(t^{-\frac{\varepsilon}{q\mu}}-t^{\frac{\varepsilon}{p\mu}}\right)}{\max\{1,t\}\left(1+a^2t\right)\left(1+b^2t\right)}\mathrm{d}t = K+\eta(\varepsilon)$$

其中 $\lim\limits_{\varepsilon\to 0^+}\eta(\varepsilon)=0$ 知(7)成立，引理得证。

3. 主要结果

定理：设 $p>1,\ a_n\ge 0$，$f(x)$ 在 $(0,\infty)$ 非负可测，且 $0<\int_0^\infty x^{p\left(1-\frac{3\mu}{2}\right)-1}f^p(x)\mathrm{d}x<\infty$， $0<\sum_{n=1}^{\infty}n^{q\left(1-\frac{3\mu}{2}\right)-1}a_n^q<\infty$，则有如下等价不等式：

$$\begin{aligned} I &:= \sum_{n=1}^{\infty}a_n\int_0^\infty\frac{f(x)}{\max\left\{x^\mu,n^\mu\right\}\left(x^\mu+a^2n^\mu\right)\left(x^\mu+b^2n^\mu\right)}\mathrm{d}x \\ &= \int_0^\infty f(x)\sum_{n=1}^{\infty}\frac{a_n}{\max\left\{x^\mu,n^\mu\right\}\left(x^\mu+a^2n^\mu\right)\left(x^\mu+b^2n^\mu\right)}\mathrm{d}x \\ &< K\left\{\int_1^\infty x^{p\left(1-\frac{3\mu}{2}\right)-1}f^p(x)\mathrm{d}x\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q\left(1-\frac{3\mu}{2}\right)-1}a_n^q\right\}^{\frac{1}{q}} \end{aligned} \tag{8}$$

$$J_1=\sum_{n=1}^{\infty}n^{\frac{3p\mu}{2}-1}\cdot\left(\int_0^\infty\frac{f(x)}{\max\left\{x^\mu,n^\mu\right\}(x^\mu+a^2n^\mu)(x^\mu+b^2n^\mu)}\mathrm{d}x\right)^p < K^p\int_0^\infty x^{p\left(1-\frac{3\mu}{2}\right)-1}f^p(x)\mathrm{d}x \tag{9}$$

$$J_2=\int_0^\infty x^{\frac{3q\mu}{2}-1}\left(\sum_{n=1}^{\infty}\frac{a_n}{\max\left\{x^\mu,n^\mu\right\}\left(x^\mu+a^2n^\mu\right)\left(x^\mu+b^2n^\mu\right)}\right)^q\mathrm{d}x < K^q\sum_{n=1}^{\infty}n^{q\left(1-\frac{3\mu}{2}\right)-1}a_n^q \tag{10}$$

这里常数因子 K 由引理 1 定义，且 K， K^p 及 K^q 均为最佳值。

证明 由逐项积分定理，I 有两种表示，由 Hölder 不等式，有

$$I := \sum_{n=1}^{\infty}\left[n^{\frac{3\mu}{2}-\frac{1}{p}}\int_0^{\infty}\frac{f(x)}{\max\{x^{\mu},n^{\mu}\}(x^{\mu}+a^2n^{\mu})(x^{\mu}+b^2n^{\mu})}\mathrm{d}x\right]\left[n^{\frac{1}{p}-\frac{3\mu}{2}}a_n\right] \le J_1^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q\left(1-\frac{3\mu}{2}\right)-1}a_n^q\right]^{\frac{1}{q}}$$

由式(9)得式(8)。反之，设(8)成立，取

$$a_n = n^{\frac{3p\mu}{2}-1}\left(\int_0^{\infty}\frac{f(x)}{\max\{x^{\mu},n^{\mu}\}(x^{\mu}+a^2n^{\mu})(x^{\mu}+b^2n^{\mu})}\mathrm{d}x\right)^{p-1}$$

则由式(8)，有

$$\sum_{n=1}^{\infty}n^{q\left(1-\frac{3\mu}{2}\right)-1}a_n^q = J_1 = I < K\left\{\int_1^{\infty}x^{p\left(1-\frac{3\mu}{2}\right)-1}f^p(x)\mathrm{d}x\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q\left(1-\frac{3\mu}{2}\right)-1}a_n^q\right\}^{\frac{1}{q}}$$

易由条件知 $J_1<\infty$，如 $J_1=0$，则式(9)自然成立；如 $J_1>0$ 则式(8)条件都具备，上式取严格不等号，且有(9)成立。可知(8)与(9)两式等价。由条件，上式取严格不等号。类似不难证明，(8)与(10)两式等价。

类似地，设(8)成立，取

$$f(x) = x^{\frac{3q\mu}{2}-1}\cdot\left(\sum_{n=1}^{\infty}\frac{a_n}{\max\{x^{\mu},n^{\mu}\}(x^{\mu}+a^2n^{\mu})(x^{\mu}+b^2n^{\mu})}\right)^{q-1}$$

由式(8)有

$$\int_1^{\infty}x^{p\left(1-\frac{3\mu}{2}\right)-1}f^p(x)\mathrm{d}x = J_2 = I < K\left\{\int_1^{\infty}x^{p\left(1-\frac{3\mu}{2}\right)-1}f^p(x)\mathrm{d}x\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q\left(1-\frac{3\mu}{2}\right)-1}a_n^q\right\}^{\frac{1}{q}}$$

易由条件知 $J_2<\infty$，如 $J_2=0$，则式(10)自然成立；如 $J_2>0$ 则式(8)条件都具备，上式取严格不等号，且有(10)成立。(8)与(10)两式等价。故式(8)，式(9)与式(10)等价。

设有常数 $\tilde{K}\ (0<\tilde{K}<K)$，使 $\tilde{K}$ 代替式(8)中的常数因子 K 后不等式(8)仍然成立。取 $\tilde{f}(x)$ 和 $\tilde{a}_n$ 代入，($\tilde{f}(x)$ 和 $\tilde{a}_n$ 如引理 3 所定义)，有

$$\varepsilon\int_0^{\infty}\tilde{f}(x)\sum_{n=1}^{\infty}\frac{\tilde{a}_n}{\max\{x^{\mu},n^{\mu}\}(x^{\mu}+a^2n^{\mu})(x^{\mu}+b^2n^{\mu})}\mathrm{d}x < \varepsilon\tilde{K}\left\{\int_1^{\infty}x^{p\left(1-\frac{3\mu}{2}\right)-1}\tilde{f}^p(x)\mathrm{d}x\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q\left(1-\frac{3\mu}{2}\right)-1}\tilde{a}_n^q\right\}^{\frac{1}{q}}$$

由引理 3 并令 ε 充分小，得 $K(1+o(1))<\tilde{K}(1+\tilde{o}(1))$，再令 $\varepsilon\to 0^+$，有 $K\le\tilde{K}$，与 $\tilde{K}<K$ 矛盾，即式(8)中的常数因子 K 为最佳的。又由等价性易知(9)和式(10)中的 K^p 和 K^q 也为最佳值。证毕。

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