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# 基于调和函数的复合材料弹性力学解析

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## 摘　要

**本文按照实变函数分析理念推导出复合材料弹性力学偏微分方程的通解，利用坐标变换法与调和函数求解各向异性板应力边值问题。通过典例阐明解决弹性力学应力边值问题的具体办法。为了满足给定应力边界条件要求，必须选择合适的调和函数，进而推导出各向异性材料确切的应力场表达式。**

## 关键词

**复合材料，偏微分方程，调和函数，应力场**

# Analysis of Elastic Mechanics for Composite Materials Based on the Harmonious Function

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## Abstract

**In this paper, based on the analytic conception of real variable function, the general solution of the partial derivative equation has been determined on the elastic mechanics for composite materials. By the method of the coordinate transition and the use of the harmonious functions, some stress boundary problems of the anisotropic plate are solved. The practical course to solve the boundary problem of elastic mechanics can be proved clearly from several typical examples. By way**

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**of selecting reasonable harmonious functions in order to satisfy the needs of the given stress boundary conditions, the specific formulae of the stress fields are derived for the anisotropic materials.**

## Keywords

**Composite Materials, Partial Derivative Equation, Harmonious Function, Stress Field**



## 1. 引言

复合材料已成为本世纪关键工程结构的主导材料，尤其在航空宇航领域具有广泛的应用潜力。伴随着尖端工程技术需求，先进复合材料结构力学领域的深入研究更加重要。鉴于先进复合材料的工程应用日益扩大，复合材料弹性力学的理论发展显得更加突出，常用复变函数法解决复合材料弹性力学平面问题已获得显著结果[1] [2] [3] [4] [5]。为了全面探讨复合材料弹性力学边值问题的求解方法和研究思路，采用实函数分析方法求解各向异性板受力边值问题作为充实弹性理论研究的必要途径。一般弹性力学书籍中都以实变函数为基础解决有关偏微分方程[6] [7]，主要是应用双调和函数求解典型边值问题。对于正交异性材料平面应力问题的实变函数解法已有研究论述[8]，其求解方法就是利用坐标变换将偏微分方程转化为调和方程与参数方程，再选择恰当的调和函数能够满足给定边界条件。本文针对各向异性材料采用实变函数分析法求解弹性力学应力边值问题，借鉴弹性力学书籍中有关直角坐标和极坐标的基础理论和求解方法，并举例说明不同类型边界条件下的典型调和函数选择及其详细解题过程，以便于推广应用。

## 2. 复合材料弹性力学的基本方程

为了叙述清楚，先将有关基础理论列举出来。众所周知，解决弹性力学问题基本方法主要从三个方面考虑：静力学、几何学和物理学。通常用应力分量作为基本变量求解平面应力边值问题，静力平衡的偏微分方程为(忽略体力)：

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0,\quad \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0 \tag{1}$$

这组偏微分方程的通解可用应力函数 $F(x,y)$ 表示，应力分量可表达为：

$$\sigma_x=\frac{\partial^2 F}{\partial y^2},\quad \sigma_y=\frac{\partial^2 F}{\partial x^2},\quad \tau_{xy}=-\frac{\partial^2 F}{\partial x\partial y} \tag{2}$$

物体内各点的相对位移产生应变，平面中一点的三个应变分量 $\varepsilon_x,\varepsilon_y,\gamma_{xy}$ 可由两个位移分量 $u(x,y)$ 和 $v(x,y)$ 而确定，即所谓的几何方程：

$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \gamma_{xy}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y} \tag{3}$$

由此推出三个应变必须满足相容条件，也就是应变协调方程：

$$\frac{\partial^2 \varepsilon_x}{\partial y^2}+\frac{\partial^2 \varepsilon_y}{\partial x^2}=\frac{\partial^2 \gamma_{xy}}{\partial x\partial y} \tag{4}$$

常见复合材料薄板的宏观力学性能可视为均匀各向异性材料特征，其板内的弹性力学分析可按照平面应力状态处理。为了便于复合材料力学分析，主要考虑面内形变的物理学性能。在平面应力状态下，各向异性材料的物理方程为：

$$\left.\begin{aligned}\varepsilon_x &= a_{11}\sigma_x + a_{12}\sigma_y + a_{16}\tau_{xy}\\ \varepsilon_y &= a_{12}\sigma_x + a_{22}\sigma_y + a_{26}\tau_{xy}\\ \gamma_{xy} &= a_{16}\sigma_x + a_{26}\sigma_y + a_{66}\tau_{xy}\end{aligned}\right\} \tag{5}$$

式中：各个常数 $a_{ij}$ 为参考坐标系 $(x, y)$ 下各向异性材料的柔度系数。

将应力分量表达式(2)代入物理方程(5)，再将所得的应变分量代入协调方程(4)，由此可导出求解一般复合材料平面应力问题的基本方程为：

$$\frac{\partial^4 F}{\partial y^4} + A\frac{\partial^4 F}{\partial x\partial y^3} + B\frac{\partial^4 F}{\partial x^2\partial y^2} + C\frac{\partial^4 F}{\partial x^3\partial y} + D\frac{\partial^4 F}{\partial x^4} = 0 \tag{6}$$

式中： $A = -\dfrac{2a_{16}}{a_{11}}$， $B = \dfrac{a_{66}+2a_{12}}{a_{11}}$， $C = -\dfrac{2a_{26}}{a_{11}}$， $D = \dfrac{a_{22}}{a_{11}}$。

基本方程(6)是一个常系数齐次线性偏微分方程，应力函数 $F(x, y)$ 为实函数，可根据给定边值问题选择合适的函数类型，以便求得具体问题的解答。根据几何方程与物理方程，可将平面变形利用应力函数表示为：

$$\left.\begin{aligned}\varepsilon_x &= \frac{\partial u}{\partial x} = a_{11}\frac{\partial^2 F}{\partial y^2} + a_{12}\frac{\partial^2 F}{\partial x^2} - a_{16}\frac{\partial^2 F}{\partial x\partial y}\\ \varepsilon_y &= \frac{\partial v}{\partial y} = a_{12}\frac{\partial^2 F}{\partial y^2} + a_{22}\frac{\partial^2 F}{\partial x^2} - a_{26}\frac{\partial^2 F}{\partial x\partial y}\\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = a_{16}\frac{\partial^2 F}{\partial y^2} + a_{26}\frac{\partial^2 F}{\partial x^2} - a_{66}\frac{\partial^2 F}{\partial x\partial y}\end{aligned}\right\} \tag{7}$$

通过上式就可利用设定的应力函数确定出平面变形及其位移，在此不做赘述。

根据应力边界条件确定应力函数过程中，通常要利用应力状态转换关系式以便简化应力分析。根据弹性力学中的应力状态分析，直角坐标 $x-y$ 与极坐标 $r-\theta$ 之间的应力分量转化关系为：

$$\left.\begin{aligned}\sigma_r &= \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + \tau_{xy}\sin 2\theta\\ \sigma_\theta &= \sigma_x\sin^2\theta + \sigma_y\cos^2\theta - \tau_{xy}\sin 2\theta\\ \tau_{r\theta} &= (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}\cos 2\theta\end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned}\sigma_x &= \sigma_r\cos^2\theta + \sigma_\theta\sin^2\theta - \tau_{r\theta}\sin 2\theta\\ \sigma_y &= \sigma_r\sin^2\theta + \sigma_\theta\cos^2\theta + \tau_{r\theta}\sin 2\theta\\ \tau_{xy} &= (\sigma_r - \sigma_\theta)\sin\theta\cos\theta + \tau_{r\theta}\cos 2\theta\end{aligned}\right\} \tag{8*}$$

下面利用以上基本公式进行各向异性材料应力边值问题的弹性力学解析，包括直角坐标和极坐标的变量转换与参数确定，求解基本方程并确定应力表达式。

## 3. 基于变量转换的应力分析

### 3.1. 含参数的坐标变换及偏导数

在解决弹性力学边值问题时要合理建立坐标系以便满足边界条件，一般按照直角坐标与极坐标的关

系建立坐标变换法。在讨论基本方程解答时，通常以原坐标系 $x-y$ 为基础，再构建新的坐标系 $X-Y$，如图 1 所示。引入两个待定参数 $g,h$，且令 $h>0$，新坐标与原坐标之间的变换关系确定为：

$$\left.\begin{aligned} X &= x+gy = r\cos\theta + gr\sin\theta = L\cos\beta \\ Y &= hy = hr\sin\theta = L\sin\beta \end{aligned}\right\} \tag{9}$$

由此便可导出以下结果：

$$\left.\begin{aligned} & X^2+Y^2=L^2=\left(x+gy\right)^2+h^2y^2=r^2\lambda^2,\ L=r\lambda \\ & \tan\beta=\frac{Y}{X}=\frac{h\tan\theta}{1+g\tan\theta},\ \lambda=\sqrt{\left(\cos\theta+g\sin\theta\right)^2+h^2\sin^2\theta} \end{aligned}\right\} \tag{10}$$

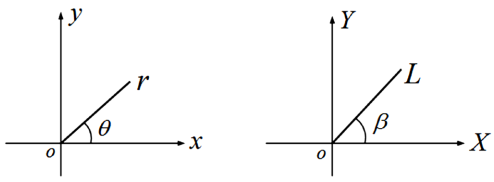


**Figure 1.** Sketch map showing the coordinate axes
**图 1.** 坐标轴示意图

新坐标下的变换关系式为：

$$L=\sqrt{X^2+Y^2},\quad \beta=\arctan\frac{Y}{X} \tag{11}$$

因此，可推导出新坐标中的偏微分表达式如下：

$$\frac{\partial L}{\partial X}=\frac{X}{L}=\cos\beta,\quad \frac{\partial L}{\partial Y}=\frac{Y}{L}=\sin\beta,\quad \frac{\partial\beta}{\partial X}=-\frac{\sin\beta}{L},\quad \frac{\partial\beta}{\partial Y}=\frac{\cos\beta}{L}$$

在新坐标系下，函数 $U\left(X,Y\right)$ 的偏导数公式为：

$$\left.\begin{aligned} \frac{\partial U}{\partial X} &= \frac{\partial U}{\partial L}\frac{\partial L}{\partial X}+\frac{\partial U}{\partial\beta}\frac{\partial\beta}{\partial X}=\cos\beta\frac{\partial U}{\partial L}-\frac{\sin\beta}{L}\frac{\partial U}{\partial\beta} \\ \frac{\partial U}{\partial Y} &= \frac{\partial U}{\partial L}\frac{\partial L}{\partial Y}+\frac{\partial U}{\partial\beta}\frac{\partial\beta}{\partial Y}=\sin\beta\frac{\partial U}{\partial L}+\frac{\cos\beta}{L}\frac{\partial U}{\partial\beta} \end{aligned}\right\} \tag{12}$$

再求出函数的二阶偏导数：

$$\begin{aligned} \frac{\partial^2 U}{\partial X^2} &= \left(\cos\beta\frac{\partial}{\partial L}-\frac{\sin\beta}{L}\frac{\partial}{\partial\beta}\right)\left(\cos\beta\frac{\partial U}{\partial L}-\frac{\sin\beta}{L}\frac{\partial U}{\partial\beta}\right) \\ &= \cos^2\beta\frac{\partial^2 U}{\partial L^2}+\sin^2\beta\left(\frac{1}{L}\frac{\partial U}{\partial L}+\frac{1}{L^2}\frac{\partial^2 U}{\partial\beta^2}\right)+\frac{\sin 2\beta}{L}\left(\frac{1}{L}\frac{\partial U}{\partial\beta}-\frac{\partial^2 U}{\partial L\partial\beta}\right) \\ \frac{\partial^2 U}{\partial Y^2} &= \left(\sin\beta\frac{\partial}{\partial L}+\frac{\cos\beta}{L}\frac{\partial}{\partial\beta}\right)\left(\sin\beta\frac{\partial U}{\partial L}+\frac{\cos\beta}{L}\frac{\partial U}{\partial\beta}\right) \\ &= \sin^2\beta\frac{\partial^2 U}{\partial L^2}+\cos^2\beta\left(\frac{1}{L}\frac{\partial U}{\partial L}+\frac{1}{L^2}\frac{\partial^2 U}{\partial\beta^2}\right)-\frac{\sin 2\beta}{L}\left(\frac{1}{L}\frac{\partial U}{\partial\beta}-\frac{\partial^2 U}{\partial L\partial\beta}\right) \\ \frac{\partial^2 U}{\partial X\partial Y} &= \left(\cos\beta\frac{\partial}{\partial L}-\frac{\sin\beta}{L}\frac{\partial}{\partial\beta}\right)\left(\sin\beta\frac{\partial U}{\partial L}+\frac{\cos\beta}{L}\frac{\partial U}{\partial\beta}\right) \\ &= \frac{\sin 2\beta}{2}\left(\frac{\partial^2 U}{\partial L^2}-\frac{1}{L}\frac{\partial U}{\partial L}-\frac{1}{L^2}\frac{\partial^2 U}{\partial\beta^2}\right)+\frac{\cos 2\beta}{L}\left(\frac{\partial^2 U}{\partial L\partial\beta}-\frac{1}{L}\frac{\partial U}{\partial\beta}\right) \end{aligned}$$

显然有：

$$\nabla^2 U = \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} = \frac{\partial^2 U}{\partial L^2} + \frac{1}{L}\frac{\partial U}{\partial L} + \frac{1}{L^2}\frac{\partial^2 U}{\partial \beta^2}$$

如果选取函数 $U(X,Y)$ 为调和函数，则满足下列调和方程：

$$\nabla^2 U = \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} = \frac{\partial^2 U}{\partial L^2} + \frac{1}{L}\frac{\partial U}{\partial L} + \frac{1}{L^2}\frac{\partial^2 U}{\partial \beta^2} = 0 \tag{13}$$

由此可将调和函数 $U(X,Y)$ 的二阶偏导数表达为：

$$\left.\begin{aligned} \frac{\partial^2 U}{\partial X^2} = -\frac{\partial^2 U}{\partial Y^2} = \cos 2\beta \frac{\partial^2 U}{\partial L^2} + \frac{\sin 2\beta}{L}\left(\frac{1}{L}\frac{\partial U}{\partial \beta} - \frac{\partial^2 U}{\partial L \partial \beta}\right) \\ \frac{\partial^2 U}{\partial X \partial Y} = \sin 2\beta \frac{\partial^2 U}{\partial L^2} + \frac{\cos 2\beta}{L}\left(\frac{\partial^2 U}{\partial L \partial \beta} - \frac{1}{L}\frac{\partial U}{\partial \beta}\right) \end{aligned}\right\} \tag{14}$$

### 3.2. 微分方程求解与应力函数

对于应力边值问题，通常利用应力函数 $F(x,y)$ 求解相关方程并确定出应力分量。按照图 1 所示的两个坐标系，可将原坐标系的函数 $F(x,y)$ 转换成新坐标系函数 $U(X,Y)$。先确定两类函数具有以下的对等关系：

$$U = U(X,Y) = U(x+gy, hy) = F(x,y) = F$$

为了便于求解偏微分方程(6)，选择新坐标 $X-Y$ 中的函数 $U$ 为调和函数，即令：

$$\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} = 0$$

按照求导法则推导出原坐标系应力函数 $F(x,y)$ 的偏导数变换关系式如下：

$$\frac{\partial F}{\partial x} = \frac{\partial U}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial U}{\partial Y}\frac{\partial Y}{\partial x} = \frac{\partial U}{\partial X}, \quad \frac{\partial F}{\partial y} = \frac{\partial U}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial U}{\partial Y}\frac{\partial Y}{\partial y} = g\frac{\partial U}{\partial X} + h\frac{\partial U}{\partial Y}$$

$$\left.\begin{aligned} \frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 U}{\partial X^2}, \quad \frac{\partial^2 F}{\partial x \partial y} = g\frac{\partial^2 U}{\partial X^2} + h\frac{\partial^2 U}{\partial X \partial Y} \\ \frac{\partial^2 F}{\partial y^2} = g^2\frac{\partial^2 U}{\partial X^2} + 2gh\frac{\partial^2 U}{\partial X \partial Y} + h^2\frac{\partial^2 U}{\partial Y^2} \end{aligned}\right\} \tag{15}$$

显然就会导出应力函数对应的调和方程具有以下形式：

$$\frac{\partial^2 F}{\partial y^2} - 2g\frac{\partial^2 F}{\partial x \partial y} + \left(g^2 + h^2\right)\frac{\partial^2 F}{\partial x^2} = h^2\left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right) = 0 \tag{16}$$

把二阶偏导数代入式(2)，可将原坐标平面内的应力用调和函数 $U$ 表示为：

$$\left.\begin{aligned} \sigma_x = \left(g^2 - h^2\right)\frac{\partial^2 U}{\partial X^2} + 2gh\frac{\partial^2 U}{\partial X \partial Y} \\ \sigma_y = \frac{\partial^2 U}{\partial X^2}, \quad \tau_{xy} = -g\frac{\partial^2 U}{\partial X^2} - h\frac{\partial^2 U}{\partial X \partial Y} \end{aligned}\right\} \tag{17}$$

利用上面的导数公式可求出四阶偏导数表达式为：

$$\frac{\partial^4 F}{\partial x^4}=\frac{\partial^4 U}{\partial X^4},\quad \frac{\partial^4 F}{\partial x^2\partial y^2}=\left(g^2-h^2\right)\frac{\partial^4 U}{\partial X^4}+2gh\frac{\partial^4 U}{\partial X^3\partial Y}$$

$$\frac{\partial^4 F}{\partial x^3\partial y}=g\frac{\partial^4 U}{\partial X^4}+h\frac{\partial^4 U}{\partial X^3\partial Y},\quad \frac{\partial^4 F}{\partial x\partial y^3}=\left(g^3-3gh^2\right)\frac{\partial^4 U}{\partial X^4}+\left(3g^2h-h^3\right)\frac{\partial^4 U}{\partial X^3\partial Y}$$

$$\frac{\partial^4 F}{\partial y^4}=\left(g^4-6g^2h^2+h^4\right)\frac{\partial^4 U}{\partial X^4}+4gh\left(g^2-h^2\right)\frac{\partial^4 U}{\partial X^3\partial Y}$$

将以上偏导数表达式代入基本方程(6)，可转变该方程为如下形式：

$$\begin{aligned}&\left[\left(g^4-6g^2h^2+h^4\right)+A\left(g^3-3gh^2\right)+B\left(g^2-h^2\right)+Cg+D\right]\frac{\partial^4 U}{\partial X^4}\\&+\left[4gh\left(g^2-h^2\right)+A\left(3g^2h-h^3\right)+2Bgh+Ch\right]\frac{\partial^4 U}{\partial X^3\partial Y}=0\end{aligned}\tag{18}$$

显而易见，上列四阶偏微分方程则可转化为求解两个参数$(g,h)$的特征方程组：

$$\left.\begin{aligned}&g^4-6g^2h^2+h^4+A\left(g^3-3gh^2\right)+B\left(g^2-h^2\right)+Cg+D=0\\&h\left[4g\left(g^2-h^2\right)+A\left(3g^2-h^2\right)+2Bg+C\right]=0\end{aligned}\right\}\tag{19}$$

两个方程联立求解就可确定出特征值$(g,h)$。也可利用复数运算法则$\left(i^2=-1\right)$，把两个实参数方程(15)合并为一个复代数方程：

$$\begin{aligned}&g^4-6g^2h^2+h^4+A\left(g^3-3gh^2\right)+B\left(g^2-h^2\right)+Cg+D\\&+ih\left[4g\left(g^2-h^2\right)+A\left(3g^2-h^2\right)+2Bg+C\right]=0\end{aligned}$$

该方程可转化为：

$$(g+ih)^4+A(g+ih)^3+B(g+ih)^2+C(g+ih)+D=0\tag{20}$$

显然，可用单一复数($z=g+ih$)表达出上列的特征方程，即为：

$$z^4+Az^3+Bz^2+Cz+D=0$$

因此，就需根据一元四次方程求根公式计算两个实参数$(g,h)$的确切值。

对于某些典型应力边值问题，采用极坐标函数求解更为简便。通常将调和函数 $U$ 用极坐标变量$(L,\beta)$给定，即为：$U=U(L,\beta)$。所以将表达式(14)代入式(17)，确定出应力分量，既用极坐标变量表示如下：

$$\begin{aligned}\sigma_x=&\left[\left(g^2-h^2\right)\cos 2\beta+2gh\sin 2\beta\right]\frac{\partial^2 U}{\partial L^2}\\&+\left[\left(g^2-h^2\right)\sin 2\beta-2gh\cos 2\beta\right]\left(\frac{1}{L^2}\frac{\partial U}{\partial \beta}-\frac{1}{L}\frac{\partial^2 U}{\partial L\partial\beta}\right)\end{aligned}\tag{21}$$

$$\left.\begin{aligned}&\sigma_y=\cos 2\beta\frac{\partial^2 U}{\partial L^2}+\sin 2\beta\left(\frac{1}{L^2}\frac{\partial U}{\partial \beta}-\frac{1}{L}\frac{\partial^2 U}{\partial L\partial\beta}\right)\\&\tau_{xy}=-\left(g\cos 2\beta+h\sin 2\beta\right)\frac{\partial^2 U}{\partial L^2}+\left(h\cos 2\beta-g\sin 2\beta\right)\left(\frac{1}{L^2}\frac{\partial U}{\partial \beta}-\frac{1}{L}\frac{\partial^2 U}{\partial L\partial\beta}\right)\end{aligned}\right\}\tag{21*}$$

### 3.3. 基本方程解析与参数计算

复合材料弹性力学平面问题可归结为求解偏微分方程(6)，并根据应力边界条件确定应力函数。基本

方程中的常系数 $A,B,C,D$ 取决于复合材料柔度参数，根据常用复合材料类型可简化求解偏微分方程以及确定特征参数。

#### 3.3.1. 正交异性材料

工程结构中复合材料通常为薄板形式，且呈现出正交异性材料力学特点。对于正交异性材料，平面应力状态下的柔度系数 $a_{ij}$ 与工程弹性常数的关系为：

$$a_{11}=\frac{1}{E_1},\quad a_{22}=\frac{1}{E_2},\quad a_{12}=-\frac{\mu_{12}}{E_1},\quad a_{66}=\frac{1}{G_{12}},\quad a_{16}=a_{26}=0$$

因而可求得常系数 $A,B,C,D$ 如下：

$$A=C=0,\quad B=\frac{a_{66}+2a_{12}}{a_{11}}=\frac{E_1}{G_{12}}-2\mu_{12},\quad D=\frac{a_{22}}{a_{11}}=\frac{E_1}{E_2}$$

代入方程(20)后就变成：

$$\left(g+ih\right)^4+B\left(g+ih\right)^2+D=0$$

则可利用代数方程求根公式，解得：

$$\left(g+hi\right)^2=g^2-h^2+2ghi=-\frac{1}{2}\left(B\pm\sqrt{B^2-4D}\right)\tag{22}$$

对于上式的解答可按两类情况分析。

**第 1 类情况**( $B^2>4D$ )，可取 $g=0,h>0$，由式(22)解出两个正根为：

$$h_1=\sqrt{\frac{B}{2}+\sqrt{\frac{B^2}{4}-D}},\quad h_2=\sqrt{\frac{B}{2}-\sqrt{\frac{B^2}{4}-D}}\tag{23}$$

因此可将调和函数 $U$ 表示为：

$$U=U_1+U_2=U_1\left(X_1,Y_1\right)+U_2\left(X_2,Y_2\right)$$

式中： $X_1=X_2=x$， $Y_1=h_1y$， $Y_2=h_2y$ 。

**第 2 类情况**( $B^2<4D$ )，由复数表达式(22)分解出实部和虚部，即有：

$$g^2-h^2=-\frac{B}{2},\quad 2gh=\pm\sqrt{D-\frac{B^2}{4}}$$

两式联合求解( $h>0$ )，可得：

$$g_{1,2}=\pm\sqrt{\frac{\sqrt{D}}{2}-\frac{B}{4}},\quad h=\sqrt{\frac{\sqrt{D}}{2}+\frac{B}{4}}$$

由此可确定出两组实根为：

$$\left.\begin{aligned}&g_1=g=\sqrt{\frac{\sqrt{D}}{2}-\frac{B}{4}},\quad h_1=h=\sqrt{\frac{\sqrt{D}}{2}+\frac{B}{4}}\\&g_2=-g,\quad h_2=h\end{aligned}\right\}\tag{24}$$

因此可将调和函数 $U$ 表示为：

$$U=U_1+U_2=U_1\left(X_1,Y_1\right)+U_2\left(X_2,Y_2\right)$$

式中： $X_1=x+gy$， $Y_1=hy$， $X_2=x-gy$， $Y_2=hy$ 。

### 3.3.2. 各向异性材料

通常复合材料在参考坐标下呈现出各向异性力学特性，因而弹性力学基本方程中的材料常数就会有任意性。对于不同性能的复合材料，需要按照特征方程(19)或公式(20)求解出实参数( $g,h$ )的相应数值。下面针对方程组(19)讨论特征值的求解方法。

假如令 $h=0$，则方程组(19)就变成单一参数方程：

$$g^4+Ag^3+Bg^2+Cg+D=0$$

可按一元四次方程的求根公式计算出实数 $g$，这里不必讨论。一般地按 $h\neq 0$ 进行讨论，由复数四次方程式(20)确定两个实参数。

**《计算举例》**选取常数为： $A=1,\quad B=3,\quad C=2,\quad D=8$

则式(20)变为：

$$(g+ih)^4+(g+ih)^3+3(g+ih)^2+2(g+ih)+8=0 \tag{25}$$

根据四次方程求解方法可确定出两个参数的真实值，计算结果为：

$$g_1=0.6055,\quad h_1=1.4924,\quad g_2=-1.1055,\quad h_2=1.3645$$

由此可确定出坐标转换关系为：

$$X_1=x+g_1y=x+0.6055y,\quad Y_1=h_1y=1.4924y$$
$$X_2=x+g_2y=x-1.1055y,\quad Y_2=h_2y=1.3645y$$

按极坐标确定的转换关系为：

$$\tan\beta_1=\frac{1.4924\tan\theta}{1+0.6055\tan\theta},\quad \lambda_1=\sqrt{(\cos\theta+0.6055\sin\theta)^2+2.2273\sin^2\theta}$$
$$\tan\beta_2=\frac{1.3645\tan\theta}{1-1.1055\tan\theta},\quad \lambda_2=\sqrt{(\cos\theta-1.1055\sin\theta)^2+1.8619\sin^2\theta}$$

以上说明了运用求解特征方程而得到特征参数 $(g,h)$ 的方法，且将 $h$ 取为正值，确定出两组实参数 $(g_1,h_1)$，$(g_2,h_2)$。因此，新坐标 $(X-Y)$ 与原坐标 $(x-y)$ 之间的转换关系就有两组可用于求解具体应力边值问题。两组坐标转换关系为：

$$\left.\begin{aligned}X_1=x+g_1y=r\cos\theta+g_1r\sin\theta,\quad Y_1=h_1y=h_1r\sin\theta\\ X_2=x+g_2y=r\cos\theta+g_2r\sin\theta,\quad Y_2=h_2y=h_2r\sin\theta\end{aligned}\right\} \tag{26}$$

因此可用两个调和函数 $U_1$ 与 $U_2$ 表示应力函数 $F(x,y)$ 如下：

$$F=U_1(X_1,Y_1)+U_2(X_2,Y_2)=U_1(x+g_1y,h_1y)+U_2(x+g_2y,h_2y)$$

## 4. 典型应力边值问题解析

前面叙述了求解复合材料弹性力学平面问题的基本理论，确定了基本方程的求解方法及其相关参数计算过程。下面就讨论复合材料应力边值问题的具体解析办法，通过构造恰当的调和函数解决几个典型的各向异性板应力边值问题。

### 4.1. 楔形体顶端承受集中力

对于各向异性材料弹性力学问题的解答归结为选择合理的调和函数以适应具体边界受力状态。为了说明具体问题求解过程，考虑一块单位厚度的楔形平板在顶端受到横向集中力 $Q$ 作用，如图 2 所示。本

例求解主要满足两条斜边面的自由条件，即考虑的应力边界条件为(两条斜边的张角为 $2\alpha$ )：

$$\sigma_\theta = 0, \quad \tau_{r\theta} = 0 \quad (\theta = \pm\alpha)$$

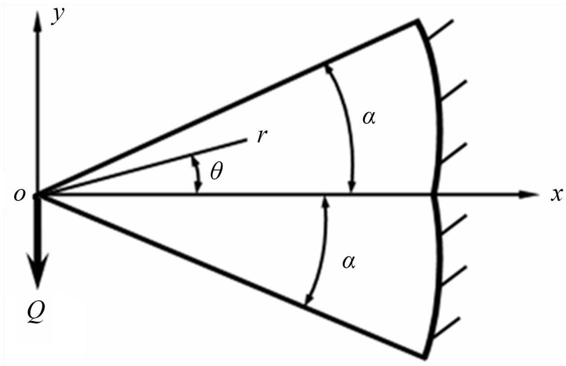


**Figure 2.** Wedge body subjected to a concentrated force and coordinate system
**图 2.** 楔形体承受集中力及坐标系

根据前面介绍的坐标变换求解方法，需要选择适合本例的调和函数 $U(X,Y)$ 以便满足应力边界条件。首先考虑的函数形式表示如下：

$$U_0 = A_0\left(X\ln\sqrt{X^2+Y^2} - Y\arctan\frac{Y}{X}\right) + B_0\left(Y\ln\sqrt{X^2+Y^2} + X\arctan\frac{Y}{X}\right) \tag{27}$$

求其偏导数并化简可得：

$$\frac{\partial U_0}{\partial X} = A_0\left(\ln\sqrt{X^2+Y^2}+1\right) + B_0\arctan\frac{Y}{X}, \quad \frac{\partial^2 U_0}{\partial X^2} = \frac{A_0X - B_0Y}{X^2+Y^2}$$

$$\frac{\partial U_0}{\partial Y} = -A_0\arctan\frac{Y}{X} + B_0\left(\ln\sqrt{X^2+Y^2}+1\right), \quad \frac{\partial^2 U_0}{\partial Y^2} = \frac{-A_0X + B_0Y}{X^2+Y^2}$$

显然容易获得下列的调和方程：

$$\frac{\partial^2 U_0}{\partial X^2} + \frac{\partial^2 U_0}{\partial Y^2} = \nabla^2 U_0 = 0$$

$$\nabla^2\left[A_0\left(X\ln\sqrt{X^2+Y^2} - Y\arctan\frac{Y}{X}\right) + B_0\left(Y\ln\sqrt{X^2+Y^2} + X\arctan\frac{Y}{X}\right)\right] = 0$$

由此说明式(27)表达的函数类型是 $X-Y$ 坐标面内的调和函数。由于单一函数难以满足调和方程，就利用不同类型基本函数进行适当组合，便于构造出新的调和函数。

根据式(26)给出的两组坐标转换关系和式(27)确定的调和函数类型，可将应力函数 $F(x,y)$ 表达成两组调和函数之和(叠加法)，即为：

$$F = U_1 + U_2 \tag{28}$$

$$\left.\begin{aligned} U_1 &= A_1\left(X_1\ln\sqrt{X_1^2+Y_1^2} - Y_1\arctan\frac{Y_1}{X_1}\right) + B_1\left(Y_1\ln\sqrt{X_1^2+Y_1^2} + X_1\arctan\frac{Y_1}{X_1}\right) \\ U_2 &= A_2\left(X_2\ln\sqrt{X_2^2+Y_2^2} - Y_2\arctan\frac{Y_2}{X_2}\right) + B_2\left(Y_2\ln\sqrt{X_2^2+Y_2^2} + X_2\arctan\frac{Y_2}{X_2}\right) \end{aligned}\right\} \tag{28*}$$

即满足调和方程：

$$\frac{\partial^2 U_1}{\partial X_1^2}+\frac{\partial^2 U_1}{\partial Y_1^2}=0,\quad \frac{\partial^2 U_2}{\partial X_2^2}+\frac{\partial^2 U_2}{\partial Y_2^2}=0$$

利用前面推导的应力表达式(17)，并按照叠加法可将应力表示为：

$$\left.\begin{aligned}
\sigma_x &= \left(g_1^2-h_1^2\right)\frac{\partial^2 U_1}{\partial X_1^2}+2g_1h_1\frac{\partial^2 U_1}{\partial X_1\partial Y_1}+\left(g_2^2-h_2^2\right)\frac{\partial^2 U_2}{\partial X_2^2}+2g_2h_2\frac{\partial^2 U_2}{\partial X_2\partial Y_2}\\
\sigma_y &= \frac{\partial^2 U_1}{\partial X_1^2}+\frac{\partial^2 U_2}{\partial X_2^2}\\
\tau_{xy} &= -g_1\frac{\partial^2 U_1}{\partial X_1^2}-h_1\frac{\partial^2 U_1}{\partial X_1\partial Y_1}-g_2\frac{\partial^2 U_2}{\partial X_2^2}-h_2\frac{\partial^2 U_2}{\partial X_2\partial Y_2}
\end{aligned}\right\}\tag{29}$$

把式(28)的函数求二阶偏导数后代入上式，再化简可得应力表达式为：

$$\begin{aligned}
\sigma_x &= \frac{\left(A_1g_1^2-A_1h_1^2+2B_1g_1h_1\right)X_1+\left(2A_1g_1h_1-B_1g_1^2+B_1h_1^2\right)Y_1}{X_1^2+Y_1^2}\\
&\quad+\frac{\left(A_2g_2^2-A_2h_2^2+2B_2g_2h_2\right)X_2+\left(2A_2g_2h_2-B_2g_2^2+B_2h_2^2\right)Y_2}{X_2^2+Y_2^2}
\end{aligned}$$

$$\sigma_y=\frac{A_1X_1-B_1Y_1}{X_1^2+Y_1^2}+\frac{A_2X_2-B_2Y_2}{X_2^2+Y_2^2}$$

$$\tau_{xy}=-\frac{\left(A_1g_1+B_1h_1\right)X_1+\left(A_1h_1-B_1g_1\right)Y_1}{X_1^2+Y_1^2}-\frac{\left(A_2g_2+B_2h_2\right)X_2+\left(A_2h_2-B_2g_2\right)Y_2}{X_2^2+Y_2^2}$$

再将式中新坐标变量代换成原坐标变量$(x,y)$，经组合后得到原坐标的应力表达式：

$$\left.\begin{aligned}
\sigma_x &= \frac{C_1x+C_2\left(g_1^2+h_1^2\right)y}{\left(x+g_1y\right)^2+\left(h_1y\right)^2}+\frac{C_3x+C_4\left(g_2^2+h_2^2\right)y}{\left(x+g_2y\right)^2+\left(h_2y\right)^2}\\
\sigma_y &= \frac{A_1x+\left(A_1g_1-B_1h_1\right)y}{\left(x+g_1y\right)^2+\left(h_1y\right)^2}+\frac{A_2x+\left(A_2g_2-B_2h_2\right)y}{\left(x+g_2y\right)^2+\left(h_2y\right)^2}\\
\tau_{xy} &= -\frac{C_2x+A_1\left(g_1^2+h_1^2\right)y}{\left(x+g_1y\right)^2+\left(h_1y\right)^2}-\frac{C_4x+A_2\left(g_2^2+h_2^2\right)y}{\left(x+g_2y\right)^2+\left(h_2y\right)^2}
\end{aligned}\right\}\tag{30}$$

式中：

$$\begin{aligned}
&C_1=A_1g_1^2-A_1h_1^2+2B_1g_1h_1,\quad C_2=A_1g_1+B_1h_1\\
&C_3=A_2g_2^2-A_2h_2^2+2B_2g_2h_2,\quad C_4=A_2g_2+B_2h_2
\end{aligned}\tag{30*}$$

亦可用极坐标变量$(r,\theta)$将应力表示为：

$$\left.\begin{aligned}
\sigma_x &= \frac{C_1\cos\theta+C_2\left(g_1^2+h_1^2\right)\sin\theta}{r\lambda_1^2}+\frac{C_3\cos\theta+C_4\left(g_2^2+h_2^2\right)\sin\theta}{r\lambda_2^2}\\
\sigma_y &= \frac{A_1\cos\theta+\left(A_1g_1-B_1h_1\right)\sin\theta}{r\lambda_1^2}+\frac{A_2\cos\theta+\left(A_2g_2-B_2h_2\right)\sin\theta}{r\lambda_2^2}\\
\tau_{xy} &= -\frac{C_2\cos\theta+A_1\left(g_1^2+h_1^2\right)\sin\theta}{r\lambda_1^2}-\frac{C_4\cos\theta+A_2\left(g_2^2+h_2^2\right)\sin\theta}{r\lambda_2^2}
\end{aligned}\right\}\tag{31}$$

式中：

$$\lambda_1^2=(\cos\theta+g_1\sin\theta)^2+(h_1\sin\theta)^2,\quad \lambda_2^2=(\cos\theta+g_2\sin\theta)^2+(h_2\sin\theta)^2$$

应力表达式中包含的待定常数取决于加载与边界条件，而应力边界条件按照极坐标更易表达清楚。因此需要利用直角坐标与极坐标之间的应力分量转化关系式(8)确定出极坐标下的应力分量。首先将环向正应力和切应力进行转化，可得：

$$\begin{aligned}\sigma_\theta&=\sigma_x\sin^2\theta+\sigma_y\cos^2\theta-\tau_{xy}\sin2\theta\\&=\frac{\sin^2\theta}{r\lambda_1^2}\left[C_1\cos\theta+C_2\left(g_1^2+h_1^2\right)\sin\theta\right]+\frac{\sin^2\theta}{r\lambda_2^2}\left[C_3\cos\theta+C_4\left(g_2^2+h_2^2\right)\sin\theta\right]\\&\quad+\frac{\cos^2\theta}{r\lambda_1^2}\left[A_1\cos\theta+\left(A_1g_1-B_1h_1\right)\sin\theta\right]+\frac{\cos^2\theta}{r\lambda_2^2}\left[A_2\cos\theta+\left(A_2g_2-B_2h_2\right)\sin\theta\right]\\&\quad+\frac{\sin2\theta}{r\lambda_1^2}\left[C_2\cos\theta+A_1\left(g_1^2+h_1^2\right)\sin\theta\right]+\frac{\sin2\theta}{r\lambda_2^2}\left[C_4\cos\theta+A_2\left(g_2^2+h_2^2\right)\sin\theta\right]\end{aligned}$$

$$\begin{aligned}\tau_{r\theta}&=\left(\sigma_y-\sigma_x\right)\sin\theta\cos\theta+\tau_{xy}\cos2\theta\\&=\frac{\sin\theta\cos\theta}{r\lambda_1^2}\left[\left(A_1-C_1\right)\cos\theta+\left(A_1g_1-B_1h_1-C_2g_1^2-C_2h_1^2\right)\sin\theta\right]\\&\quad+\frac{\sin\theta\cos\theta}{r\lambda_2^2}\left[\left(A_2-C_3\right)\cos\theta+\left(A_2g_2-B_2h_2-C_4g_2^2-C_4h_2^2\right)\sin\theta\right]\\&\quad-\frac{\cos2\theta}{r\lambda_1^2}\left[C_2\cos\theta+A_1\left(g_1^2+h_1^2\right)\sin\theta\right]-\frac{\cos2\theta}{r\lambda_2^2}\left[C_4\cos\theta+A_2\left(g_2^2+h_2^2\right)\sin\theta\right]\end{aligned}$$

对两个应力表达式进行合并同类项，再经化简后的结果如下：

$$\left.\begin{aligned}\sigma_\theta&=\frac{1}{r}\left[\left(A_1+A_2\right)\cos\theta+\left(A_1g_1+B_1h_1+A_2g_2+B_2h_2\right)\sin\theta\right]\\\tau_{r\theta}&=\frac{1}{r}\left[\left(A_1+A_2\right))\text{in}\,\theta-\left(A_1g_1+B_1h_1+A_2g_2+B_2h_2\right)\cos\theta\right]\end{aligned}\right\}\tag{32}$$

为了满足图 2 所示楔形体两条斜边无面力的自由条件，显然可令：$\sigma_\theta=0,\tau_{r\theta}=0$。因此由上式可确定出几个常数相适应的关系式为：

$$A_1+A_2=0,\quad A_1g_1+B_1h_1+A_2g_2+B_2h_2=C_2+C_4=0$$

即有：

$$A_2=-A_1,\quad C_4=-C_2,\quad A_1\left(g_2-g_1\right)=B_1h_1+B_2h_2\tag{33}$$

下面再考虑楔形体的静力平衡条件。按照图 2 所示的楔形体坐标与角度可知，斜边的方程为：$y=\pm x\tan\alpha$。沿 $x$ 轴向任取一个横截面 $\bar{x}$，该截面的两个端点为 $\bar{y}_d$ 与 $\bar{y}_u$，既有：$\bar{y}_d=-\bar{x}\tan\alpha$，$\bar{y}_u=\bar{x}\tan\alpha$。现对截面左侧部分列出静力平衡方程如下：

$$\left.\begin{aligned}\int_{\bar{y}_d}^{\bar{y}_u}\sigma_x\left(\bar{x},y\right)\mathrm{d}y=\int_{-\bar{x}\tan\alpha}^{\bar{x}\tan\alpha}\sigma_x\left(\bar{x},y\right)\mathrm{d}y=0\\\int_{\bar{y}_d}^{\bar{y}_u}\tau_{xy}\left(\bar{x},y\right)\mathrm{d}y=\int_{-\bar{x}\tan\alpha}^{\bar{x}\tan\alpha}\tau_{xy}\left(\bar{x},y\right)\mathrm{d}y=Q\end{aligned}\right\}\tag{34}$$

再将式(30)的应力表达式代入积分中，则有：

$$\int_{-\bar{x}\tan\alpha}^{\bar{x}\tan\alpha}\left[\frac{C_1\bar{x}+C_2\left(g_1^2+h_1^2\right)y}{\left(\bar{x}+g_1y\right)^2+\left(h_1y\right)^2}+\frac{C_3\bar{x}+C_4\left(g_2^2+h_2^2\right)y}{\left(\bar{x}+g_2y\right)^2+\left(h_2y\right)^2}\right]\mathrm{d}y=0$$

$$\int_{-\bar{x}\tan\alpha}^{\bar{x}\tan\alpha}\left[-\frac{C_2\bar{x}+A_1\left(g_1^2+h_1^2\right)y}{\left(\bar{x}+g_1y\right)^2+\left(h_1y\right)^2}-\frac{C_4\bar{x}+A_2\left(g_2^2+h_2^2\right)y}{\left(\bar{x}+g_2y\right)^2+\left(h_2y\right)^2}\right]\mathrm{d}y=Q$$

对各项进行积分，可得：

$$\left|\begin{aligned}&\frac{C_1-C_2g_1}{h_1}\arctan\frac{g_1\bar{x}+\left(g_1^2+h_1^2\right)y}{h_1\bar{x}}+\frac{C_3-C_4g_2}{h_2}\arctan\frac{g_2\bar{x}+\left(g_2^2+h_2^2\right)y}{h_2\bar{x}}\\&+C_2\ln\sqrt{\left(\bar{x}+g_1y\right)^2+\left(h_1y\right)^2}+C_4\ln\sqrt{\left(\bar{x}+g_2y\right)^2+\left(h_2y\right)^2}\end{aligned}\right|_{y=-\bar{x}\tan\alpha}^{y=\bar{x}\tan\alpha}=0$$

$$\left|\begin{aligned}&\frac{C_2-A_1g_1}{h_1}\arctan\frac{g_1\bar{x}+\left(g_1^2+h_1^2\right)y}{h_1\bar{x}}+\frac{C_4-A_2g_2}{h_2}\arctan\frac{g_2\bar{x}+\left(g_2^2+h_2^2\right)y}{h_2\bar{x}}\\&+A_1\ln\sqrt{\left(\bar{x}+g_1y\right)^2+\left(h_1y\right)^2}+A_2\ln\sqrt{\left(\bar{x}+g_2y\right)^2+\left(h_2y\right)^2}\end{aligned}\right|_{y=-\bar{x}\tan\alpha}^{y=\bar{x}\tan\alpha}=-Q$$

注意到几个常数之间具有下列关系式：

$$C_1-C_2g_1=\left(B_1g_1-A_1h_1\right)h_1,\quad C_3-C_4g_2=\left(B_2g_2-A_2h_2\right)h_2$$
$$C_2-A_1g_1=B_1h_1,\quad C_4-A_2g_2=B_2h_2,\quad C_4=-C_2,\quad A_2=-A_1$$

因而可将上列两个定积分的结果表示为：

$$\left.\begin{aligned}&\left(B_1g_1-A_1h_1\right)\varpi_1+\left(B_2g_2-A_2h_2\right)\varpi_2+\left(A_1g_1+B_1h_1\right)\varpi_3=0\\&B_1\varpi_1+B_2\varpi_2+A_1\varpi_3=-Q\end{aligned}\right\}\tag{35}$$

式中：

$$\left.\begin{aligned}&\varpi_1=\arctan\frac{g_1+\left(g_1^2+h_1^2\right)\tan\alpha}{h_1}-\arctan\frac{g_1-\left(g_1^2+h_1^2\right)\tan\alpha}{h_1}\\&\varpi_2=\arctan\frac{g_2+\left(g_2^2+h_2^2\right)\tan\alpha}{h_2}-\arctan\frac{g_2-\left(g_2^2+h_2^2\right)\tan\alpha}{h_2}\\&\varpi_3=\frac{1}{2}\ln\frac{\left(1+g_1\tan\alpha\right)^2+\left(h_1\tan\alpha\right)^2}{\left(1-g_1\tan\alpha\right)^2+\left(h_1\tan\alpha\right)^2}-\frac{1}{2}\ln\frac{\left(1+g_2\tan\alpha\right)^2+\left(h_2\tan\alpha\right)^2}{\left(1-g_2\tan\alpha\right)^2+\left(h_2\tan\alpha\right)^2}\end{aligned}\right\}\tag{35*}$$

再利用式(33)的常数关系，即：

$$A_2=-A_1=\frac{B_1h_1+B_2h_2}{g_1-g_2}$$

将其代入常数方程组(35)后，就可求解出 $B_1,B_2$，再返回上式求得 $A_1,A_2$，从而确定出四个常数表示如下：

$$A_1=-A_2=Q\eta_1,\quad B_1=Q\eta_2,\quad B_2=Q\eta_2\eta_3\tag{36}$$

式中：

$$\left.\begin{aligned}&\eta_1=\frac{h_1+h_2\eta_3}{\left(g_1-g_2\right)\left(\varpi_1+\varpi_2\eta_3\right)-\left(h_1+h_2\eta_3\right)\varpi_3}\\&\eta_2=\frac{g_2-g_1}{\left(g_1-g_2\right)\left(\varpi_1+\varpi_2\eta_3\right)-\left(h_1+h_2\eta_3\right)\varpi_3}\\&\eta_3=\frac{\left(g_1g_2-g_1^2-h_1^2\right)\varpi_1+h_1h_2\varpi_2+g_2h_1\varpi_3}{\left(g_1g_2-g_2^2-h_2^2\right)\varpi_2+h_1h_2\varpi_1-g_1h_2\varpi_3}\end{aligned}\right\}\tag{36*}$$

由此也可将常数 $C_1, C_2, C_3, C_4$ 表达为：

$$\left.\begin{aligned}&C_1 = Q\eta_4,\quad C_2 = -C_4 = Q\eta_5,\quad C_3 = Q\eta_6\\&\eta_4 = \left(g_1^2 - h_1^2\right)\eta_1 + 2g_1h_1\eta_2,\quad \eta_5 = g_1\eta_1 + h_1\eta_2 = g_2\eta_1 - h_2\eta_2\eta_3\\&\eta_6 = \left(h_2^2 - g_2^2\right)\eta_1 + 2g_2h_2\eta_2\eta_3\end{aligned}\right\}\tag{37}$$

把求出的常数代入到应力表达式(31)，则可得图 2 楔形体的应力表达式为：

$$\left.\begin{aligned}\sigma_x &= \frac{Q}{r}\left[\frac{\eta_4\cos\theta + \eta_5\left(g_1^2 + h_1^2\right)\sin\theta}{\left(\cos\theta + g_1\sin\theta\right)^2 + \left(h_1\sin\theta\right)^2} + \frac{\eta_6\cos\theta - \eta_5\left(g_2^2 + h_2^2\right)\sin\theta}{\left(\cos\theta + g_2\sin\theta\right)^2 + \left(h_2\sin\theta\right)^2}\right]\\ \sigma_y &= \frac{Q}{r}\left[\frac{\eta_1\cos\theta + \left(\eta_1 g_1 - \eta_2 h_1\right)\sin\theta}{\left(\cos\theta + g_1\sin\theta\right)^2 + \left(h_1\sin\theta\right)^2} - \frac{\eta_1\cos\theta + \left(\eta_1 g_2 + \eta_2\eta_3 h_2\right)\sin\theta}{\left(\cos\theta + g_2\sin\theta\right)^2 + \left(h_2\sin\theta\right)^2}\right]\\ \tau_{xy} &= \frac{Q}{r}\left[\frac{\eta_5\cos\theta + \eta_1\left(g_2^2 + h_2^2\right)\sin\theta}{\left(\cos\theta + g_2\sin\theta\right)^2 + \left(h_2\sin\theta\right)^2} - \frac{\eta_5\cos\theta + \eta_1\left(g_1^2 + h_1^2\right)\sin\theta}{\left(\cos\theta + g_1\sin\theta\right)^2 + \left(h_1\sin\theta\right)^2}\right]\end{aligned}\right\}\tag{38}$$

根据应力分量转化关系式(8)可知 $\sigma_r + \sigma_\theta = \sigma_x + \sigma_y$，因为已确定了 $\sigma_\theta = 0$，故可将径向应力 $\left(\sigma_r = \sigma_x + \sigma_y\right)$ 表示如下：

$$\sigma_r = \frac{Q}{r}\left[\frac{\left(\eta_1 + \eta_4\right)\cos\theta + \eta_7\sin\theta}{\left(\cos\theta + g_1\sin\theta\right)^2 + \left(h_1\sin\theta\right)^2} - \frac{\left(\eta_1 - \eta_6\right)\cos\theta + \eta_8\sin\theta}{\left(\cos\theta + g_2\sin\theta\right)^2 + \left(h_2\sin\theta\right)^2}\right]\tag{39}$$

式中：

$$\eta_7 = \eta_1 g_1 - \eta_2 h_1 + \eta_5\left(g_1^2 + h_1^2\right),\quad \eta_8 = \eta_1 g_2 + \eta_2\eta_3 h_2 + \eta_5\left(g_2^2 + h_2^2\right)$$

### 4.2. 平板边界面承受分布力

为了表明复合材料应力边值问题求解方法，再考虑各向异性板半边界表面受到分布剪切面力 $p$ 作用，如图 3 所示。本例求解主要满足一半直边面的自由状态与另一半直边面承受分布力作用。这个平板应该满足的应力边界条件为：

$$\begin{aligned}&\sigma_y = \sigma_\theta = 0,\quad \tau_{xy} = \tau_{r\theta} = 0\quad (\theta = 0)\\&\sigma_y = \sigma_\theta = 0,\quad \tau_{xy} = \tau_{r\theta} = p\quad (\theta = \pi)\end{aligned}$$

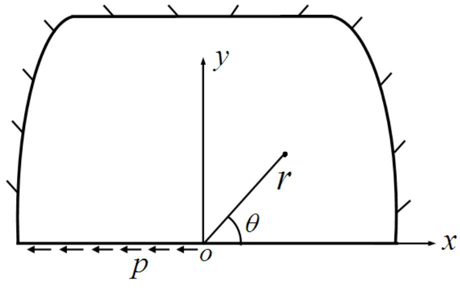


**Figure 3.** Half-edge surface subjected to distributed shear force
**图 3.** 半边界面承受分布剪切力

按照坐标变换求解办法，选择适合本问题的调和函数 $U\left(X, Y\right)$ 以便满足板边界的分布力条件。先考

查新坐标中的函数形式如下：

$$\begin{aligned}U_0 = A_0\left[\left(X^2-Y^2\right)\arctan\frac{Y}{X}+2XY\ln\sqrt{X^2+Y^2}\right] \\ +B_0\left[\left(X^2-Y^2\right)\ln\sqrt{X^2+Y^2}-2XY\arctan\frac{Y}{X}\right]\end{aligned} \tag{40}$$

求其偏导数并化简可得：

$$\begin{aligned}\frac{\partial U_0}{\partial X} = A_0\left[2X\arctan\frac{Y}{X}+2Y\ln\sqrt{X^2+Y^2}+Y\right] \\ +B_0\left[2X\ln\sqrt{X^2+Y^2}-2Y\arctan\frac{Y}{X}+X\right]\end{aligned}$$

$$\begin{aligned}\frac{\partial U_0}{\partial Y} = A_0\left[-2Y\arctan\frac{Y}{X}+2X\ln\sqrt{X^2+Y^2}+X\right] \\ -B_0\left[2Y\ln\sqrt{X^2+Y^2}+2X\arctan\frac{Y}{X}+Y\right]\end{aligned}$$

二阶偏导数确定为：

$$\left.\begin{aligned}\frac{\partial^2 U_0}{\partial X^2} = 2A_0\arctan\frac{Y}{X}+2B_0\ln\sqrt{X^2+Y^2}+3B_0 \\ \frac{\partial^2 U_0}{\partial Y^2} = -2A_0\arctan\frac{Y}{X}-2B_0\ln\sqrt{X^2+Y^2}-3B_0 \\ \frac{\partial^2 U_0}{\partial X\partial Y} = 2A_0\ln\sqrt{X^2+Y^2}+3A_0-2B_0\arctan\frac{Y}{X}\end{aligned}\right\} \tag{41}$$

显而易见，函数$U_0\left(X,Y\right)$是调和函数，既有：

$$\nabla^2 U_0 = \frac{\partial^2 U_0}{\partial X^2}+\frac{\partial^2 U_0}{\partial Y^2}=0$$

利用式(40)的调和函数形式构造出应力函数如下：

$$F = U_1+U_2 \tag{42}$$

$$\left.\begin{aligned}U_1 = A_1\left[\left(X_1^2-Y_1^2\right)\arctan\frac{Y_1}{X_1}+2X_1Y_1\ln\sqrt{X_1^2+Y_1^2}\right] \\ +B_1\left[\left(X_1^2-Y_1^2\right)\ln\sqrt{X_1^2-Y_1^2}-2X_1Y_1\arctan\frac{Y_1}{X_1}\right] \\ U_2 = A_2\left[\left(X_2^2-Y_2^2\right)\arctan\frac{Y_2}{X_2}+2X_2Y_2\ln\sqrt{X_2^2+Y_2^2}\right] \\ +B_2\left[\left(X_2^2-Y_2^2\right)\ln\sqrt{X_2^2-Y_2^2}-2X_2Y_2\arctan\frac{Y_2}{X_2}\right]\end{aligned}\right\} \tag{42*}$$

且满足调和方程：

$$\frac{\partial^2 U_1}{\partial X_1^2}+\frac{\partial^2 U_1}{\partial Y_1^2}=0,\quad \frac{\partial^2 U_2}{\partial X_2^2}+\frac{\partial^2 U_2}{\partial Y_2^2}=0$$

则可确定出二阶偏导数为：

$$\frac{\partial^2 U_1}{\partial X_1^2} = 2A_1 \arctan\frac{Y_1}{X_1} + 2B_1 \ln\sqrt{X_1^2 + Y_1^2} + 3B_1$$
$$\frac{\partial^2 U_1}{\partial X_1 \partial Y_1} = 2A_1 \ln\sqrt{X_1^2 + Y_1^2} + 3A_1 - 2B_1 \arctan\frac{Y_1}{X_1}$$
$$\frac{\partial^2 U_2}{\partial X_2^2} = 2A_2 \arctan\frac{Y_2}{X_2} + 2B_2 \ln\sqrt{X_2^2 + Y_2^2} + 3B_2$$
$$\frac{\partial^2 U_2}{\partial X_2 \partial Y_2} = 2A_2 \ln\sqrt{X_2^2 + Y_2^2} + 3A_2 - 2B_2 \arctan\frac{Y_2}{X_2}$$

参照应力表达式(29)与本处确定的调和函数及其二阶偏导数，可求得应力为：

$$\begin{aligned}\sigma_x &= \left(g_1^2 - h_1^2\right)\frac{\partial^2 U_1}{\partial X_1^2} + 2g_1 h_1 \frac{\partial^2 U_1}{\partial X_1 \partial Y_1} + \left(g_2^2 - h_2^2\right)\frac{\partial^2 U_2}{\partial X_2^2} + 2g_2 h_2 \frac{\partial^2 U_2}{\partial X_2 \partial Y_2} \\ &= 2\left[A_1\left(g_1^2 - h_1^2\right) - 2B_1 g_1 h_1\right]\arctan\frac{Y_1}{X_1} + 2\left[A_2\left(g_2^2 - h_2^2\right) - 2B_2 g_2 h_2\right]\arctan\frac{Y_2}{X_2} \\ &\quad + 2\left[2A_1 g_1 h_1 + B_1\left(g_1^2 - h_1^2\right)\right]\ln\sqrt{X_1^2 + Y_1^2} + 2\left[2A_2 g_2 h_2 + B_2\left(g_2^2 - h_2^2\right)\right]\ln\sqrt{X_2^2 + Y_2^2} \\ &\quad + 3\left[2A_1 g_1 h_1 + B_1\left(g_1^2 - h_1^2\right) + 2A_2 g_2 h_2 + B_2\left(g_2^2 - h_2^2\right)\right]\end{aligned}$$

$$\begin{aligned}\sigma_y &= \frac{\partial^2 U_1}{\partial X_1^2} + \frac{\partial^2 U_2}{\partial X_2^2} = 2A_1 \arctan\frac{Y_1}{X_1} + 2A_2 \arctan\frac{Y_2}{X_2} \\ &\quad + 2B_1 \ln\sqrt{X_1^2 + Y_1^2} + 2B_2 \ln\sqrt{X_2^2 + Y_2^2} + 3B_1 + 3B_2\end{aligned}$$

$$\begin{aligned}\tau_{xy} &= -g_1 \frac{\partial^2 U_1}{\partial X_1^2} - h_1 \frac{\partial^2 U_1}{\partial X_1 \partial Y_1} - g_2 \frac{\partial^2 U_2}{\partial X_2^2} - h_2 \frac{\partial^2 U_2}{\partial X_2 \partial Y_2} \\ &= -2\left(A_1 g_1 - B_1 h_1\right)\arctan\frac{Y_1}{X_1} - 2\left(A_2 g_2 - B_2 h_2\right)\arctan\frac{Y_2}{X_2} \\ &\quad - 2\left(A_1 h_1 + B_1 g_1\right)\ln\sqrt{X_1^2 + Y_1^2} - 2\left(A_2 h_2 + B_2 g_2\right)\ln\sqrt{X_2^2 + Y_2^2} \\ &\quad - 3\left(A_1 h_1 + B_1 g_1 + A_2 h_2 + B_2 g_2\right)\end{aligned}$$

按照极坐标变量转换上列应力表达式如下：

$$\left.\begin{aligned}\sigma_x &= 2\left[A_1\left(g_1^2 - h_1^2\right) - 2B_1 g_1 h_1\right]\beta_1 + 2\left[A_2\left(g_2^2 - h_2^2\right) - 2B_2 g_2 h_2\right]\beta_2 \\ &\quad + 2\left[2A_1 g_1 h_1 + B_1\left(g_1^2 - h_1^2\right)\right]\ln L_1 + 2\left[2A_2 g_2 h_2 + B_2\left(g_2^2 - h_2^2\right)\right]\ln L_2 \\ &\quad + 3\left[2A_1 g_1 h_1 + B_1\left(g_1^2 - h_1^2\right) + 2A_2 g_2 h_2 + B_2\left(g_2^2 - h_2^2\right)\right] \\ \sigma_y &= 2A_1\beta_1 + 2A_2\beta_2 + 2B_1 \ln L_1 + 2B_2 \ln L_2 + 3B_1 + 3B_2 \\ \tau_{xy} &= -2\left(A_1 g_1 - B_1 h_1\right)\beta_1 - 2\left(A_2 g_2 - B_2 h_2\right)\beta_2 - 2\left(A_1 h_1 + B_1 g_1\right)\ln L_1 \\ &\quad - 2\left(A_2 h_2 + B_2 g_2\right)\ln L_2 - 3\left(A_1 h_1 + B_1 g_1 + A_2 h_2 + B_2 g_2\right)\end{aligned}\right\} \tag{43}$$

式中：

$$\beta_1 = \arctan\frac{Y_1}{X_1},\quad \beta_2 = \arctan\frac{Y_2}{X_2},\quad L_1 = \sqrt{X_1^2 + Y_1^2},\quad L_2 = \sqrt{X_2^2 + Y_2^2}$$

为了确定上列应力表达式中的各个待定常数，必须考虑极坐标下的应力边界条件。根据新坐标与原

坐标之间的变换关系式(9)与(10)可得：

$$\tan\beta_1=\frac{Y_1}{X_1}=\frac{h_1\tan\theta}{1+g_1\tan\theta}$$
$$\tan\beta_2=\frac{Y_2}{X_2}=\frac{h_2\tan\theta}{1+g_2\tan\theta}$$
$$L_1=r\lambda_1=r\sqrt{\left(\cos\theta+g_1\sin\theta\right)^2+\left(h_1\sin\theta\right)^2}$$
$$L_2=r\lambda_2=r\sqrt{\left(\cos\theta+g_2\sin\theta\right)^2+\left(h_2\sin\theta\right)^2}$$

由图 3 的面力边界条件可知：

$$\left.\begin{array}{l}\theta=0,\ \ \beta_1=\beta_2=0,\ \ L_1=L_2=r,\quad \sigma_y=\tau_{xy}=0\\ \theta=\pi,\ \ \beta_1=\beta_2=\pi,\ \ L_1=L_2=r,\ \ \sigma_y=0,\ \ \tau_{xy}=p\end{array}\right\}\tag{44}$$

将式(44)给定的应力边值条件代入式(43)，可得：

$$\begin{aligned}&\left(B_1+B_2\right)\left(2\ln r+3\right)=0\\&2\pi\left(A_1+A_2\right)+\left(B_1+B_2\right)\left(2\ln r+3\right)=0\\&\left(A_1h_1+B_1g_1+A_2h_2+B_2g_2\right)\left(2\ln r+3\right)=0\\&2\pi\left(A_1g_1-B_1h_1+A_2g_2-B_2h_2\right)+\left(A_1h_1+B_1g_1+A_2h_2+B_2g_2\right)\left(2\ln r+3\right)=-p\end{aligned}$$

由此四式联合求解可确定出四个常数，表示为：

$$\left.\begin{array}{l}A_1=-A_2=\dfrac{p}{2\pi}\dfrac{g_2-g_1}{\left(g_1-g_2\right)^2+\left(h_1-h_2\right)^2}\\ B_1=-B_2=\dfrac{p}{2\pi}\dfrac{h_1-h_2}{\left(g_1-g_2\right)^2+\left(h_1-h_2\right)^2}\end{array}\right\}\tag{45}$$

将所得常数带回到应力表达式(43)，再进行同类函数合并化简后就得到图 3 边界承受剪切的应力场公式如下：

$$\left.\begin{array}{l}\sigma_x=-\dfrac{p}{\pi}\left[\dfrac{D_4}{D_{12}}\beta_1+\dfrac{D_5}{D_{12}}\beta_2+\dfrac{D_6}{D_{12}}\ln\left(r\lambda_1\right)+\dfrac{D_7}{D_{12}}\ln\left(r\lambda_2\right)+\dfrac{3\left(D_6+D_7\right)}{2D_{12}}\right]\\ \sigma_y=\dfrac{p}{\pi}\left[\dfrac{g_2-g_1}{D_{12}}\left(\beta_1-\beta_2\right)+\dfrac{h_1-h_2}{D_{12}}\ln\dfrac{\lambda_1}{\lambda_2}\right]\\ \tau_{xy}=\dfrac{p}{\pi}\left(\dfrac{D_1}{D_{12}}\beta_1+\dfrac{D_2}{D_{12}}\beta_2+\dfrac{D_3}{D_{12}}\ln\dfrac{\lambda_2}{\lambda_1}\right)\end{array}\right\}\tag{46}$$

式中坐标变量为：

$$\beta_1=\arctan\frac{h_1\tan\theta}{1+g_1\tan\theta},\qquad \beta_2=\arctan\frac{h_2\tan\theta}{1+g_2\tan\theta}$$
$$\lambda_1=\sqrt{\left(\cos\theta+g_1\sin\theta\right)^2+\left(h_1\sin\theta\right)^2},\qquad \lambda_2=\sqrt{\left(\cos\theta+g_2\sin\theta\right)^2+\left(h_2\sin\theta\right)^2}$$

几个常数计算关系式如下：

$$D_1=g_1\left(g_1-g_2\right)+h_1\left(h_1-h_2\right),\qquad D_2=-g_2\left(g_1-g_2\right)-h_2\left(h_1-h_2\right)$$
$$D_{12}=\left(g_1-g_2\right)^2+\left(h_1-h_2\right)^2=D_1+D_2,\qquad D_3=g_2h_1-g_1h_2$$

$$D_4 = g_1D_1 + h_1D_3，\quad D_5 = g_2D_2 - h_2D_3$$
$$D_6 = h_1D_1 - g_1D_3，\quad D_7 = h_2D_2 + g_2D_3$$

### 4.3. 各向异性板裂纹端部应力场

复合材料断裂力学基础就是裂纹端部应力场理论，因此对裂纹端部的应力分析十分重要。下面以各向异性材料含裂纹平板为例，并针对常见 I 型裂纹板进行裂纹尖端区域的应力场分析，张开型裂纹板及坐标系如图 4 所示。

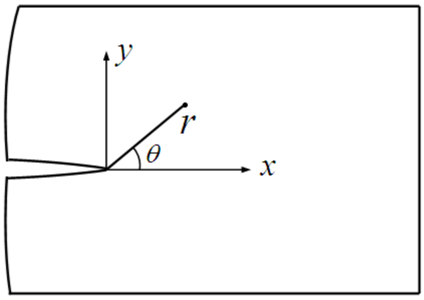


**Figure 4.** Anisotropic materials with a crack and the axes
**图 4.** 各向异性材料裂纹板及坐标轴

按照弹性力学理论和坐标变换求解办法，首先选择适合一般裂纹体边值问题的调和函数 $U(X,Y)$ 以便满足 I 型加载和裂纹面自由边界条件。对于裂纹端部，利用极坐标较为简便，因此在新坐标系中选取极坐标函数 $U_0(L,\beta)$ 表示如下：

$$U_0 = A_0\sqrt{L^3}\cos\frac{3\beta}{2} + B_0\sqrt{L^3}\sin\frac{3\beta}{2} \tag{47}$$

对其求偏导数可得：

$$\frac{\partial U_0}{\partial L} = \frac{3}{2}\sqrt{L}\left(A_0\cos\frac{3\beta}{2} + B_0\sin\frac{3\beta}{2}\right),\quad \frac{1}{L}\frac{\partial U_0}{\partial \beta} = \frac{3}{2}\sqrt{L}\left(B_0\cos\frac{3\beta}{2} - A_0\sin\frac{3\beta}{2}\right)$$
$$\frac{\partial^2 U_0}{\partial L^2} = \frac{3}{4\sqrt{L}}\left(A_0\cos\frac{3\beta}{2} + B_0\sin\frac{3\beta}{2}\right),\quad \frac{\partial^2 U_0}{\partial L\partial \beta} = \frac{9}{4}\sqrt{L}\left(B_0\cos\frac{3\beta}{2} - A_0\sin\frac{3\beta}{2}\right)$$
$$\frac{1}{L^2}\frac{\partial^2 U_0}{\partial \beta^2} = -\frac{9}{4\sqrt{L}}\left(A_0\cos\frac{3\beta}{2} + B_0\sin\frac{3\beta}{2}\right)$$

容易验证函数 $U_0(L,\beta)$ 是新坐标系的调和函数，即满足极坐标的调和方程：

$$\frac{\partial^2 U_0}{\partial L^2} + \frac{1}{L}\frac{\partial U_0}{\partial L} + \frac{1}{L^2}\frac{\partial^2 U_0}{\partial \beta^2} = 0$$

再将调和函数 $U_0$ 的偏导数代入式(14)，可得二阶偏导数的坐标变换表达式为：

$$\frac{\partial^2 U_0}{\partial X^2} = \frac{3}{4\sqrt{L}}\left[\cos 2\beta\left(A_0\cos\frac{3\beta}{2} + B_0\sin\frac{3\beta}{2}\right) + \sin 2\beta\left(A_0\sin\frac{3\beta}{2} - B_0\cos\frac{3\beta}{2}\right)\right]$$
$$\frac{\partial^2 U_0}{\partial X\partial Y} = \frac{3}{4\sqrt{L}}\left[\sin 2\beta\left(A_0\cos\frac{3\beta}{2} + B_0\sin\frac{3\beta}{2}\right) - \cos 2\beta\left(A_0\sin\frac{3\beta}{2} - B_0\cos\frac{3\beta}{2}\right)\right]$$

利用三角函数变换关系式简化上列表达式后，可得调和函数 $U_0$ 的偏导数为：

$$\left.\begin{aligned}\frac{\partial^2 U_0}{\partial X^2} &= -\frac{\partial^2 U_0}{\partial Y^2} = \frac{3}{4\sqrt{L}}\left(A_0 \cos\frac{\beta}{2} - B_0 \sin\frac{\beta}{2}\right) \\ \frac{\partial^2 U_0}{\partial X \partial Y} &= \frac{3}{4\sqrt{L}}\left(A_0 \sin\frac{\beta}{2} + B_0 \cos\frac{\beta}{2}\right)\end{aligned}\right\} \tag{48}$$

因此可利用式(47)之类的调和函数构造出应力函数 $F(x,y)$ 如下：

$$\left.\begin{aligned}F &= F_1 + F_2 = U_1 + U_2 \\ U_1 &= A_1\sqrt{L_1^3}\cos\frac{3\beta_1}{2} + B_1\sqrt{L_1^3}\sin\frac{3\beta_1}{2} \\ U_2 &= A_2\sqrt{L_2^3}\cos\frac{3\beta_2}{2} + B_2\sqrt{L_2^3}\sin\frac{3\beta_2}{2}\end{aligned}\right\} \tag{49}$$

按照式(48)可将 $U_1, U_2$ 的偏导数表示为：

$$\frac{\partial^2 U_1}{\partial X_1^2} = \frac{3A_1}{4\sqrt{L_1}}\cos\frac{\beta_1}{2} - \frac{3B_1}{4\sqrt{L_1}}\sin\frac{\beta_1}{2}, \quad \frac{\partial^2 U_1}{\partial X_1 \partial Y_1} = \frac{3A_1}{4\sqrt{L_1}}\sin\frac{\beta_1}{2} + \frac{3B_1}{4\sqrt{L_1}}\cos\frac{\beta_1}{2}$$

$$\frac{\partial^2 U_2}{\partial X_2^2} = \frac{3A_2}{4\sqrt{L_2}}\cos\frac{\beta_2}{2} - \frac{3B_2}{4\sqrt{L_2}}\sin\frac{\beta_2}{2}, \quad \frac{\partial^2 U_2}{\partial X_2 \partial Y_2} = \frac{3A_2}{4\sqrt{L_2}}\sin\frac{\beta_2}{2} + \frac{3B_2}{4\sqrt{L_2}}\cos\frac{\beta_2}{2}$$

根据式(29)的应力表达式与此处的调和函数偏导数，可求得应力分量为：

$$\begin{aligned}\sigma_x &= \left(g_1^2 - h_1^2\right)\frac{\partial^2 U_1}{\partial X_1^2} + 2g_1 h_1 \frac{\partial^2 U_1}{\partial X_1 \partial Y_1} + \left(g_2^2 - h_2^2\right)\frac{\partial^2 U_2}{\partial X_2^2} + 2g_2 h_2 \frac{\partial^2 U_2}{\partial X_2 \partial Y_2} \\ &= \frac{3}{4\sqrt{L_1}}\left\{\left[A_1\left(g_1^2 - h_1^2\right) + 2B_1 g_1 h_1\right]\cos\frac{\beta_1}{2} + \left[2A_1 g_1 h_1 - B_1\left(g_1^2 - h_1^2\right)\right]\sin\frac{\beta_1}{2}\right\} \\ &\quad + \frac{3}{4\sqrt{L_2}}\left\{\left[A_2\left(g_2^2 - h_2^2\right) + 2B_2 g_2 h_2\right]\cos\frac{\beta_2}{2} + \left[2A_2 g_2 h_2 - B_2\left(g_2^2 - h_2^2\right)\right]\sin\frac{\beta_2}{2}\right\}\end{aligned} \tag{50}$$

$$\sigma_y = \frac{\partial^2 U_1}{\partial X_1^2} + \frac{\partial^2 U_2}{\partial X_2^2} = \frac{3}{4\sqrt{L_1}}\left(A_1\cos\frac{\beta_1}{2} - B_1\sin\frac{\beta_1}{2}\right) + \frac{3}{4\sqrt{L_2}}\left(A_2\cos\frac{\beta_2}{2} - B_2\sin\frac{\beta_2}{2}\right) \tag{51}$$

$$\begin{aligned}\tau_{xy} &= -g_1\frac{\partial^2 U_1}{\partial X_1^2} - h_1\frac{\partial^2 U_1}{\partial X_1 \partial Y_1} - g_2\frac{\partial^2 U_2}{\partial X_2^2} - h_2\frac{\partial^2 U_2}{\partial X_2 \partial Y_2} \\ &= -\frac{3}{4\sqrt{L_1}}\left[\left(A_1 g_1 + B_1 h_1\right)\cos\frac{\beta_1}{2} + \left(A_1 h_1 - B_1 g_1\right)\sin\frac{\beta_1}{2}\right] \\ &\quad - \frac{3}{4\sqrt{L_2}}\left[\left(A_2 g_2 + B_2 h_2\right)\cos\frac{\beta_2}{2} + \left(A_2 h_2 - B_2 g_2\right)\sin\frac{\beta_2}{2}\right]\end{aligned} \tag{52}$$

根据图 4 所示的裂纹面自由边界条件，则应力分量需要满足的条件如下：

$$\sigma_y = 0, \quad \tau_{xy} = 0 \quad \left(\theta = \pm\pi, \ \beta_1 = \beta_2 = \pm\pi\right)$$

再按应力表达式(51)与式(52)可确定出常数需满足的关系式，即为：

$$B_1 + B_2 = 0, \quad A_1 h_1 - B_1 g_1 + A_2 h_2 - B_2 g_2 = 0$$

由此可解得：

$$B_1 = -B_2 = \frac{A_1 h_1 + A_2 h_2}{g_1 - g_2} \tag{53}$$

对于张开型裂纹(纯 I 型)，裂纹前沿( $\theta = 0$ )的切应力为零，按应力表达式(52)可确定出以下结果：

$$\tau_{xy}\Big|_{\theta=0} = -\frac{3}{4\sqrt{r}}\left(A_1 g_1 + B_1 h_1 + A_2 g_2 + B_2 h_2\right) = 0$$

即得常数关系式：

$$A_1 g_1 + B_1 h_1 + A_2 g_2 + B_2 h_2 = 0 \tag{54}$$

通常用应力强度因子 $K_I$ 作为衡量裂纹尖端应力场大小的重要参数，因此由相关正应力表达式(51)确定出 $K_I$ 与待定常数之间的关系如下：

$$K_I = \lim_{r\to 0}\sqrt{2\pi r}\,\sigma_y\Big|_{\theta=0} = \sqrt{2\pi r}\left(\frac{3}{4\sqrt{r}}A_1 + \frac{3}{4\sqrt{r}}A_2\right) = \frac{3\sqrt{2\pi}}{4}\left(A_1 + A_2\right) \tag{55}$$

联合上列几个常数关系式求解出四个常数，可表示为：

$$\left.\begin{aligned} A_1 &= \frac{K_I}{\sqrt{2\pi}}\frac{4D_2}{3D_{12}}, \quad A_2 = \frac{K_I}{\sqrt{2\pi}}\frac{4D_1}{3D_{12}} \\ B_1 &= -\frac{K_I}{\sqrt{2\pi}}\frac{4D_3}{3D_{12}}, \quad B_2 = \frac{K_I}{\sqrt{2\pi}}\frac{4D_3}{3D_{12}} \end{aligned}\right\} \tag{56}$$

式中：

$$D_1 = g_1\left(g_1 - g_2\right) + h_1\left(h_1 - h_2\right), \quad D_2 = -g_2\left(g_1 - g_2\right) - h_2\left(h_1 - h_2\right)$$
$$D_{12} = \left(g_1 - g_2\right)^2 + \left(h_1 - h_2\right)^2 = D_1 + D_2, \quad D_3 = g_2 h_1 - g_1 h_2$$

把以上常数带回到式(50)~(52)，就推导出 I 型裂纹端部的应力场为：

$$\left.\begin{aligned} \sigma_x &= \frac{K_I}{\sqrt{2\pi r}D_{12}}\left(\frac{C_1}{\sqrt{\lambda_1}}\cos\frac{\beta_1}{2} + \frac{C_2}{\sqrt{\lambda_2}}\cos\frac{\beta_2}{2} + \frac{C_3}{\sqrt{\lambda_1}}\sin\frac{\beta_1}{2} + \frac{C_4}{\sqrt{\lambda_2}}\sin\frac{\beta_2}{2}\right) \\ \sigma_y &= \frac{K_I}{\sqrt{2\pi r}D_{12}}\left(\frac{D_2}{\sqrt{\lambda_1}}\cos\frac{\beta_1}{2} + \frac{D_1}{\sqrt{\lambda_2}}\cos\frac{\beta_2}{2} + \frac{D_3}{\sqrt{\lambda_1}}\sin\frac{\beta_1}{2} - \frac{D_3}{\sqrt{\lambda_2}}\sin\frac{\beta_2}{2}\right) \\ \tau_{xy} &= \frac{K_I}{\sqrt{2\pi r}D_{12}}\left(\frac{D_4}{\sqrt{\lambda_1}}\cos\frac{\beta_1}{2} - \frac{D_4}{\sqrt{\lambda_2}}\cos\frac{\beta_2}{2} + \frac{D_5}{\sqrt{\lambda_1}}\sin\frac{\beta_1}{2} - \frac{D_5}{\sqrt{\lambda_2}}\sin\frac{\beta_2}{2}\right) \end{aligned}\right\} \tag{57}$$

式中：

$$\lambda_1 = \sqrt{\left(\cos\theta + g_1\sin\theta\right)^2 + \left(h_1\sin\theta\right)^2}, \quad \lambda_2 = \sqrt{\left(\cos\theta + g_2\sin\theta\right)^2 + \left(h_2\sin\theta\right)^2}$$
$$\beta_1 = \arctan\frac{h_1\tan\theta}{1 + g_1\tan\theta}, \quad \beta_2 = \arctan\frac{h_2\tan\theta}{1 + g_2\tan\theta}$$

几个常数由下列关系式确定：

$$C_1 = \left(g_1^2 - h_1^2\right)D_2 - 2g_1 h_1 D_3, \quad C_2 = \left(g_2^2 - h_2^2\right)D_1 + 2g_2 h_2 D_3$$
$$C_3 = 2g_1 h_1 D_2 + \left(g_1^2 - h_1^2\right)D_3, \quad C_4 = 2g_2 h_2 D_1 - \left(g_2^2 - h_2^2\right)D_3$$
$$D_4 = g_1 g_2\left(g_1 - g_2\right) + g_2 h_1^2 - g_1 h_2^2, \quad D_5 = h_1 h_2\left(h_1 - h_2\right) + g_1^2 h_2 - g_2^2 h_1$$

这就是各向异性板 I 型裂纹尖端奇异应力场的解析结果。

上面通过三个典型例题阐述了复合材料应力边值问题的具体解析办法，求得的应力场以满足局部区

边界为主，目的在于说明选用调和函数及其求解步骤。通过这些实例分析可以看出，解决各向异性板的弹性力学问题固然复杂，但其解析方法易于精通。

## 5. 结论

本文针对复合材料弹性力学平面问题，建立各向异性板的基本方程，利用坐标变换法与调和函数求解偏微分方程，推导出一般应力场，并以直角坐标和极坐标形式给出应力分量的简明表达式。举例说明求解各向异性板平面应力边值问题的具体办法，强调了选择不同类型基本函数构造出合适的调和函数，以便求得的应力可满足边界条件。本文注重基础理论推导完整，阐明主要公式应用与求解思路，包括材料参数与各个待定常数关系的确切算法，有助于掌握求解复合材料弹性力学边值问题的详细过程。

## 参考文献


[1] 杨维阳, 李俊林, 张雪霞. 复合材料断裂复变方法[M]. 北京: 科学出版社, 2005.

[2] 张行. 断裂与损伤力学[M]. 北京: 北京航空航天大学出版社, 2009.

[3] 李群, 欧卓成, 陈宜亨. 高等断裂力学[M]. 北京: 科学出版社, 2017.

[4] 贾普荣. 基于多复函的各向异性板弹性力学解法[J]. 力学研究, 2021, 10(3): 194-203. https://doi.org/10.12677/IJM.2021.103019

[5] Zhang, H. and Qiao, P.Z. (2019) A State-Based Peridynamic Model for Quantitative Elastic and Fracture Analysis of Orthotropic Materials. *Engineering Fracture Mechanics*, **206**, 147-171. https://doi.org/10.1016/j.engfracmech.2018.10.003

[6] 吴家龙. 弹性力学[M]. 北京: 高等教育出版社, 2001.

[7] 徐芝纶. 弹性力学简明教程(第三版) [M]. 北京: 高等教育出版社, 2002.

[8] 贾普荣. 正交异性材料弹性力学通解[J]. 力学研究, 2022, 11(3): 54-62. https://doi.org/10.12677/IJM.2022.113007