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双线性分数次积分算子在极大变指标 Herz 空间上的有界性

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摘　要

借助双线性分数次积分算子在变指标 **Lebesgue** 空间上的有界性，利用函数分层分解和调和分析实方法，得到了双线性分数次积分算子在极大变指标 **Herz** 空间上的有界性。

关键词

双线性分数次积分算子，极大变指标 **Herz** 空间，有界性

Boundedness of Bilinear Fractional Integral Operators on Grand Variable Herz Spaces

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Abstract

Based on the boundedness of bilinear fractional integration operators on Lebesgue

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spaces with variable exponent, by using hierarchical decomposition of function and real methods in harmonic analysis, the boundedness of bilinear fractional integral operators is obtained on grand variable Herz spaces.

Keywords

Bilinear Fractional Integral Operator, Grand Variable Herz Space, Boundedness

1. 引言及主要结果

在偏微分方程中, 为了更好地研究 Possion 方程 $\Delta u = f$ 的解, Sobolev 在 1938 年引入了经典的分数次积分算子, 并证明了该算子从 $L^p(\mathbb{R}^n)$ 到 $L^q(\mathbb{R}^n)$ 的有界性 [1]. 此后, 分数次积分算子在各类函数空间上的有界性得到了国内外学者的广泛研究 [2–6]. 1999 年, Keng [4]把分数次积分算子推广到多线性的情形, 并研究了多线性分数次积分算子在 $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ 到 $L^p(\mathbb{R}^n)$ 的有界性, 其中 $\frac{1}{p(x)} = \frac{1}{p_1(x)} + \cdots + \frac{1}{p_m(x)} - \frac{\beta}{n}$. 2008 年, Shi 和 Tao [5]证明了多线性分数次积分算子在 Herz 空间上的有界性.

另一方面, 变指标函数空间不仅具有重要的理论意义, 而且在非线性弹性力学、具有非标准增长条件的偏微分方程和图像恢复等领域的应用十分广泛 [7,8]. 2010 年, Izuki [9]引入了变指标 Herz 空间, 得到了一类次线性算子的有界性. 2020 年, Nafis 等 [10]定义了极大变指标 Herz 空间, 并证明了次线性算子的有界性. 2022 年, 史鹏伟和陶双平讨论了参数型 Littlewood-Paley 算子在极大变指标 Herz 空间上的有界性 [11]. 受上述启发, 本文主要讨论了极大变指标 Herz 空间上双线性分数次积分算子的有界性.

定义 1 [12] 设 $p(x) \in \mathcal{P}(\mathbb{R}^n)$, 变指标 Lebesgue 空间 $L^{p(\cdot)}(\mathbb{R}^n)$ 定义为

$$L^{p(\cdot)}(\mathbb{R}^n) := \left\{ f \text{ 是可测函数: 存在常数 } \lambda > 0, \text{ 使得 } \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}x < \infty \right\},$$

其中 Luxemburg-Nakano 范数为

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}x \leqslant 1 \right\}.$$

用 $\mathcal{P}(\mathbb{R}^n)$ 表示 $\mathbb{R}^n$ 上所有满足下列条件的可测函数 $p(x)$ 组成的集合:

$p^- := ess\inf\{p(x): x\in\mathbb{R}^n\}\leqslant 1,\ \ p^+ := ess\sup\{p(x): x\in\mathbb{R}^n\}<\infty.$

定义 2 [10] 设 $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, 若存在 $p(0)\in(1,\infty)$ 及常数 $C_0>0$, 使得

$$|p(x)-p(0)|\leqslant\frac{C_0}{\log|x|},\ |x|\leqslant\frac{1}{2},\tag{1}$$

则称 $p(\cdot)$ 在原点处满足 log-Hölder 连续, 记作 $p(\cdot)\in\mathcal{P}_0^{\log}(\mathbb{R}^n)$; 若存在 $p(\infty)=\lim\limits_{x\to\infty}p(x)>1$ 及常数 $C_\infty>0$, 使得

$$|p(x)-p(\infty)|\leqslant\frac{C_\infty}{\log(e+|x|)},\ x\in\mathbb{R}^n,\tag{2}$$

则称 $p(\cdot)$ 在无穷远处满足 log-Hölder 连续, 记作 $p(\cdot)\in\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$.

定义 3 [10] 设 $\alpha(\cdot)\in L^\infty(\mathbb{R}^n):\mathbb{R}^n\to\mathbb{R}$ 为可测函数, $1<q<\infty,\ p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. 齐次变指标 Herz 空间 $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)$ 定义为

$$\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n):=\Big\{f\in L_{\rm loc}^{p(\cdot)}(\mathbb{R}^n\backslash\{0\}):\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)}<\infty\Big\},$$

其中

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)}=\Big\{\sum_{k=-\infty}^{\infty}\|2^{k\alpha(\cdot)}f\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q\Big\}^{\frac{1}{q}}.$$

这里 $B_k=\{x\in\mathbb{R}^n:|x|\leqslant 2^k\}$, $E_k=B_k\backslash B_{k-1},\ \chi_k=\chi_{E_k},\ k\in\mathbb{Z},\ \chi_{E_k}$ 表示 E_k 的特征函数.

定义 4 [13] 设 $\theta>0,\ \alpha(\cdot)\in L^\infty(\mathbb{R}^n),\ 1\leqslant q<\infty,\ p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. 齐次极大变指标 Herz 空间 $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q),\theta}(\mathbb{R}^n)$ 定义为

$$\dot{K}_{p(\cdot)}^{\alpha(\cdot),q),\theta}(\mathbb{R}^n):=\Big\{f\in L_{\rm loc}^{p(\cdot)}(\mathbb{R}^n\backslash\{0\}):\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q),\theta}(\mathbb{R}^n)}<\infty\Big\},$$

其中

$$\begin{aligned}\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q),\theta}(\mathbb{R}^n)}&=\sup_{\varepsilon>0}\Big(\varepsilon^\theta\sum_{k=\mathbb{Z}}\|2^{k\alpha(\cdot)}f\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q(1+\varepsilon)}\Big)^{\frac{1}{q(1+\varepsilon)}}\\&=\sup_{\varepsilon>0}\varepsilon^{\frac{\theta}{q(1+\varepsilon)}}\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(1+\varepsilon)}(\mathbb{R}^n)}.\end{aligned}$$

我们注意到, 当 $0<q_1\leqslant q_2<\infty$ 时, 则有 $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q_1}(\mathbb{R}^n)\subset\dot{K}_{p(\cdot)}^{\alpha(\cdot),q_2}(\mathbb{R}^n)$ 成立; 当 $\alpha(\cdot)$ 为常数时, 有 $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}(\mathbb{R}^n)\subset\dot{K}_{p(\cdot)}^{\alpha(\cdot),q),\theta}(\mathbb{R}^n)$ 成立, 其中 $q>1,\ \theta>0$.

设 $0<\beta<2n$, 双线性分数次积分算子 BI_β 定义为 [3]

$$BI_\beta(f_1,f_2)(x)=\int_{\mathbb{R}^{2n}}\frac{f_1(y_1)f_2(y_2)}{(|x-y_1|+|x-y_2|)^{2n-\beta}}{\rm d}y_1{\rm d}y_2.$$

本文的主要结果如下:

定理 1 设 $0<\beta<2n,\ 1<p_i^-\leqslant p_i^+<\infty$, 令 $\frac{1}{r_i(x)}=\frac{1}{p_i(x)}-\frac{\beta}{2n},\ \frac{1}{p(x)}=\frac{1}{p_1(x)}+\frac{1}{p_2(x)}-\frac{\beta}{n},$

$p(\cdot),\ p_i(\cdot) \in \mathcal{P}_0^{\log}(\mathbb{R}^n)\bigcap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ 并且满足 $p_i(0) \leqslant p_i(\infty)$. 设 $\theta > 0,\ 1 < q_i < \infty,\ \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, $\alpha(x) = \alpha_1(x) + \alpha_2(x),\ \alpha_i(\cdot) \in L^\infty(\mathbb{R})\bigcap \mathcal{P}_0^{\log}(\mathbb{R}^n)\bigcap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, 若 $\frac{\beta}{2} - \frac{n}{p_i(\infty)} < \alpha_i(0) \leqslant \alpha_i(\infty) < n(1-\frac{1}{p_i(0)})$, 则存在与 f_i 无关的常数 $C>0$, 使得对任意 $f_i \in \dot{K}_{p_i(\cdot)}^{\alpha_i(\cdot),q_i),\theta}(\mathbb{R}^n)$, 其中 $i=1,2$. 有

$$\|BI_\beta(f_1,f_2)\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q),\theta}(\mathbb{R}^n)} \leqslant C\|f_1\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1),\theta}(\mathbb{R}^n)}\|f_2\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2),\theta}(\mathbb{R}^n)}.$$

全文中, C 表示一个不依赖于主要参数的正常数, 但其值在不同地方可能不尽相同.

2. 定理的证明

引理 1 [14] 设 $p(\cdot): \mathbb{R}^n \to [1,\infty)$, 如果 $f \in L^{p(\cdot)}(\mathbb{R}^n),\ g \in L^{p'(\cdot)}(\mathbb{R}^n)$, 那么有

$$\int_{\mathbb{R}^n} |f(x)g(x)|\mathrm{d}x \leqslant r_p\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}$$

其中 $r_p = 1 + \frac{1}{p_-} - \frac{1}{p_+}$.

引理 2 [14] 设 $p(\cdot),\ r_1(\cdot),\ r_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, 满足 $\frac{1}{p(x)} = \frac{1}{r_1(x)} + \frac{1}{r_2(x)}$, 那么当 $f \in L^{r_1(\cdot)}(\mathbb{R}^n)$, $g \in L^{r_2(\cdot)}(\mathbb{R}^n)$ 时, 有 $fg \in L^{p(\cdot)}(\mathbb{R}^n)$, 并且

$$\|fg\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|f\|_{L^{r_1(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{r_2(\cdot)}(\mathbb{R}^n)}.$$

引理 3 [15] 设 $D>1,\ p(\cdot) \in \mathcal{P}_0^{\log}(\mathbb{R}^n)\bigcap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, 存在 $c_0,\ c_\infty \geqslant 1$ 仅依赖于常数 D, 则

$$\frac{1}{c_0}r^{\frac{n}{p(0)}} \leqslant \|\chi_{B(0,D_r)\backslash B(0,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant c_0 r^{\frac{n}{p(0)}},\quad r \in (0,1], \tag{3}$$

$$\frac{1}{c_\infty}r^{\frac{n}{p(\infty)}} \leqslant \|\chi_{B(0,D_r)\backslash B(0,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant c_\infty r^{\frac{n}{p(\infty)}},\quad r \in [1,\infty). \tag{4}$$

引理 4 [16] 设 $1 < p_i^- \leqslant p_i^+ < \infty,\ p_i(\cdot) \in \mathcal{P}_0^{\log}(\mathbb{R}^n)\bigcap \mathcal{P}_\infty^{\log}(\mathbb{R}^n),\ \frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)} - \frac{\beta}{n}$. 则

$$\|BI_\beta(f_1,f_2)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}.$$

定理 1 的证明 设 $f_i \in \dot{K}_{p_i(\cdot)}^{\alpha_i(\cdot),q_i),\theta}(\mathbb{R}^n)$, 对 f_i 进行如下分解

$$f_i(x) = \sum_{l_i=-\infty}^{\infty} f_i(x)\chi_{l_i}(x) = \sum_{l_i=-\infty}^{\infty} f_{l_i}(x),\quad i=1,2,\ l_i \in \mathbb{Z}.$$

则有

$$
\begin{aligned}
&\|BI_\beta(f_1,f_2)\|_{\dot{K}^{\alpha(\cdot),q),\theta}_{p(\cdot)}(\mathbb{R}^n)}\\
&=\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=-\infty}^{\infty}\|2^{k\alpha(\cdot)}BI_\beta(f_1,f_2)\chi_k\|^{q(1+\varepsilon)}_{L^{p(\cdot)}(\mathbb{R}^n)}\right\}^{\frac{1}{q(1+\varepsilon)}}\\
&\leqslant C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=-\infty}^{\infty}2^{k\alpha(\cdot)q(1+\varepsilon)}\left(\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(1+\varepsilon)}\right\}^{\frac{1}{q(1+\varepsilon)}}\\
&=C(V_1+V_2+\cdots+V_9),
\end{aligned}
$$

其中

$$V_1=\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=-\infty}^{\infty}2^{k\alpha(\cdot)q(1+\varepsilon)}\left(\sum_{l_1=-\infty}^{k-2}\sum_{l_2=-\infty}^{k-2}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(1+\varepsilon)}\right\}^{\frac{1}{q(1+\varepsilon)}},$$

$$V_2=\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=-\infty}^{\infty}2^{k\alpha(\cdot)q(1+\varepsilon)}\left(\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k-1}^{k+1}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(1+\varepsilon)}\right\}^{\frac{1}{q(1+\varepsilon)}},$$

$$V_3=\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=-\infty}^{\infty}2^{k\alpha(\cdot)q(1+\varepsilon)}\left(\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k+2}^{\infty}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(1+\varepsilon)}\right\}^{\frac{1}{q(1+\varepsilon)}},$$

$$V_4=\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=-\infty}^{\infty}2^{k\alpha(\cdot)q(1+\varepsilon)}\left(\sum_{l_1=k-1}^{k+1}\sum_{l_2=-\infty}^{k-2}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(1+\varepsilon)}\right\}^{\frac{1}{q(1+\varepsilon)}},$$

$$V_5=\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=-\infty}^{\infty}2^{k\alpha(\cdot)q(1+\varepsilon)}\left(\sum_{l_1=k-1}^{k+1}\sum_{l_2=k-1}^{k+1}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(1+\varepsilon)}\right\}^{\frac{1}{q(1+\varepsilon)}},$$

$$V_6=\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=-\infty}^{\infty}2^{k\alpha(\cdot)q(1+\varepsilon)}\left(\sum_{l_1=k-1}^{k+1}\sum_{l_2=k+2}^{\infty}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(1+\varepsilon)}\right\}^{\frac{1}{q(1+\varepsilon)}},$$

$$V_7=\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=-\infty}^{\infty}2^{k\alpha(\cdot)q(1+\varepsilon)}\left(\sum_{l_1=k+2}^{\infty}\sum_{l_2=-\infty}^{k-2}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(1+\varepsilon)}\right\}^{\frac{1}{q(1+\varepsilon)}},$$

$$V_8=\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=-\infty}^{\infty}2^{k\alpha(\cdot)q(1+\varepsilon)}\left(\sum_{l_1=k+2}^{\infty}\sum_{l_2=k-1}^{k+1}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(1+\varepsilon)}\right\}^{\frac{1}{q(1+\varepsilon)}},$$

$$V_9=\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=-\infty}^{\infty}2^{k\alpha(\cdot)q(1+\varepsilon)}\left(\sum_{l_1=k+2}^{\infty}\sum_{l_2=k+2}^{\infty}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(1+\varepsilon)}\right\}^{\frac{1}{q(1+\varepsilon)}}.$$

利用 f_1 与 f_2 的对称性可知, V_2 的估计类似于 V_4 的估计, V_3 的估计类似 V_7 的估计, V_6 的估计类似于 V_8 的估计, 所以只需估计 V_1, V_2, V_3, V_5, V_6 和 V_9 即可.

首先估计 V_1. 注意到当 $k<0$, $x\in E_k$ 时, 有 $2^{k\alpha(x)}\approx 2^{k\alpha(0)}$; 当 $k>0$, $x\in E_k$ 时, 有 $2^{k\alpha(x)}\approx 2^{k\alpha(\infty)}$. 利用 Minkowski 不等式, 可得

$$
\begin{aligned}
V_1 \leqslant& C\sup_{\varepsilon>0}\left\{\varepsilon^{\theta}\sum_{k=-\infty}^{-1}2^{k\alpha(0)q(1+\varepsilon)}\left(\sum_{l_1=-\infty}^{k-2}\sum_{l_2=-\infty}^{k-2}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(1+\varepsilon)}\right\}^{\frac{1}{q(1+\varepsilon)}}\\
&+C\sup_{\varepsilon>0}\left\{\varepsilon^{\theta}\sum_{k=0}^{\infty}2^{k\alpha(\infty)q(1+\varepsilon)}\left(\sum_{l_1=-\infty}^{k-2}\sum_{l_2=-\infty}^{k-2}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(1+\varepsilon)}\right\}^{\frac{1}{q(1+\varepsilon)}}\\
&:=V_{11}+V_{12}.
\end{aligned}
$$

当 $l_i\leqslant k-2$, $x\in E_k$, $y_i\in E_{l_i}$ 时, 有 $|x-y_i|\geqslant|x|-|y_i|>2^{k-1}-2^{l_i}>2^{k-2}$, 其中 $i=1,2$. 则

$$
\begin{aligned}
|BI_\beta(f_{l_1},f_{l_2})(x)|&\leqslant\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f_{l_1}(y_1)||f_{l_2}(y_2)|}{(|x-y_1|+|x-y_2|)^{2n-\beta}}\mathrm{d}y_1\mathrm{d}y_2\\
&\leqslant C2^{-k(2n-\beta)}\|f_{l_1}\|_{L^1(\mathbb{R}^n)}\|f_{l_2}\|_{L^1(\mathbb{R}^n)}.
\end{aligned}
$$

注意到 $\frac{1}{p(x)}=\frac{1}{r_1(x)}+\frac{1}{r_2(x)}$, $\frac{1}{r_i(x)}=\frac{1}{p_i(x)}-\frac{\beta}{2n}$. 利用 Hölder 不等式及引理 1-3, 可得

$$
\begin{aligned}
&\sum_{l_1=-\infty}^{k-2}\sum_{l_2=-\infty}^{k-2}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C\sum_{l_1=-\infty}^{k-2}\sum_{l_2=-\infty}^{k-2}2^{-k(2n-\beta)}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{l_1}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{l_2}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C\sum_{l_1=-\infty}^{k-2}\sum_{l_2=-\infty}^{k-2}2^{-k(2n-\beta)}2^{\frac{l_1n}{p_1'(0)}}2^{\frac{l_2n}{p_2'(0)}}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{r_1(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{r_2(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C\sum_{l_1=-\infty}^{k-2}\sum_{l_2=-\infty}^{k-2}2^{-k(2n-\beta)}2^{\frac{l_1n}{p_1'(0)}}2^{\frac{l_2n}{p_2'(0)}}2^{\frac{kn}{r_1(0)}}2^{\frac{kn}{r_2(0)}}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C\left(\sum_{l_1=-\infty}^{k-2}2^{(k-l_1)(\frac{n}{p_1(0)}-n)}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)\left(\sum_{l_2=-\infty}^{k-2}2^{(k-l_2)(\frac{n}{p_2(0)}-n)}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right).
\end{aligned}\tag{5}
$$

利用 Hölder 不等式和已知条件 $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$, 得

$$
\begin{aligned}
V_{11}\leqslant& C\sup_{\varepsilon>0}\left\{\varepsilon^{\theta}\sum_{k=-\infty}^{-1}\left(\sum_{l_1=-\infty}^{k-2}2^{k\alpha_1(0)+(k-l_1)(\frac{n}{p_1(0)}-n)}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{q_1(1+\varepsilon)}\right\}^{\frac{1}{q_1(1+\varepsilon)}}\\
&\times\sup_{\varepsilon>0}\left\{\varepsilon^{\theta}\sum_{k=-\infty}^{-1}\left(\sum_{l_2=-\infty}^{k-2}2^{k\alpha_2(0)+(k-l_2)(\frac{n}{p_2(0)}-n)}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_2(1+\varepsilon)}\right\}^{\frac{1}{q_2(1+\varepsilon)}}\\
&:=V_{111}V_{112}.
\end{aligned}
$$

由于 $a_1 := \alpha_1(0) + \frac{n}{p_1(0)} - n < 0$ 且 $2^{-q_1(1+\varepsilon)} < 2^{-q_1}$, 利用 Hölder 不等式和 Fubini 定理, 得

$$
\begin{aligned}
V_{111} \leqslant C \sup_{\varepsilon>0} \Bigg\{ & \varepsilon^{\theta} \sum_{k=-\infty}^{-1} \left(\sum_{l_1=-\infty}^{k-2} 2^{a_1(k-l_1)\frac{q_1(1+\varepsilon)}{2}} 2^{l_1\alpha_1(0)q_1(1+\varepsilon)} \|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1(1+\varepsilon)} \right) \\
& \times \left(\sum_{l_1=-\infty}^{k-2} 2^{a_1(k-l_1)\frac{q_1(1+\varepsilon)'}{2}} \right)^{\frac{q_1(1+\varepsilon)}{(q_1(1+\varepsilon))'}} \Bigg\}^{\frac{1}{q_1(1+\varepsilon)}} \\
& \leqslant C \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{l_1=-\infty}^{-1} 2^{l_1\alpha_1(0)q_1(1+\varepsilon)} \|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1(1+\varepsilon)} \sum_{k=l_1+2}^{-1} 2^{a_1(k-l_1)\frac{q_1(1+\varepsilon)}{2}} \right)^{\frac{1}{q_1(1+\varepsilon)}} \\
& \leqslant C \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{l_1=-\infty}^{-1} 2^{l_1\alpha_1(0)q_1(1+\varepsilon)} \|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1(1+\varepsilon)} \right)^{\frac{1}{q_1(1+\varepsilon)}} \\
& \leqslant C \|f_1\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1),\theta}(\mathbb{R}^n)}.
\end{aligned}
$$

同理可得 $V_{112} \leqslant C\|f_2\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2),\theta}(\mathbb{R}^n)}$, 因此 $V_{11} \leqslant C\|f_1\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot),q_1),\theta}(\mathbb{R}^n)}\|f_2\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot),q_2),\theta}(\mathbb{R}^n)}$.

再考虑 V_{12} 的估计. 利用 Minkowski 不等式, 可得

$$
\begin{aligned}
V_{12} \leqslant & C \sup_{\varepsilon>0} \left\{ \varepsilon^{\theta} \sum_{k=0}^{\infty} 2^{k\alpha(\infty)q(1+\varepsilon)} \left(\sum_{l_1=-\infty}^{-1} \sum_{l_2=-\infty}^{-1} \|BI_{\alpha}(f_{l_1}, f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{q(1+\varepsilon)} \right\}^{\frac{1}{q(1+\varepsilon)}} \\
& + C \sup_{\varepsilon>0} \left\{ \varepsilon^{\theta} \sum_{k=0}^{\infty} 2^{k\alpha(\infty)q(1+\varepsilon)} \left(\sum_{l_1=0}^{k-2} \sum_{l_2=0}^{k-2} \|BI_{\alpha}(f_{l_1}, f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{q(1+\varepsilon)} \right\}^{\frac{1}{q(1+\varepsilon)}} \\
:= & V_{121} + V_{122}.
\end{aligned}
$$

类似于 V_{11} 的估计, 由于 $p_i(0) \leqslant p_i(\infty)$, 用 $\alpha_i(\infty)$ 取代 $\alpha_i(0)$, 便可得到 V_{122} 的估计.

对 V_{121}. 类似于不等式(5), 由 Hölder 不等式, 引理 1-3 和 $p_i(0) \leqslant p_i(\infty)$, 可得

$$
\begin{aligned}
& \sum_{l_1=-\infty}^{-1} \sum_{l_2=-\infty}^{-1} \|BI_{\beta}(f_{l_1}, f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
& \leqslant C \sum_{l_1=-\infty}^{-1} \sum_{l_2=-\infty}^{-1} 2^{-k(2n-\beta)} \|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{l_1}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \|\chi_{l_2}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)} \|\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
& \leqslant C \sum_{l_1=-\infty}^{-1} \sum_{l_2=-\infty}^{-1} 2^{-k(2n-\beta)} 2^{\frac{l_1 n}{p_1'(0)}} 2^{\frac{l_2 n}{p_2'(0)}} 2^{\frac{kn}{r_1(\infty)}} 2^{\frac{kn}{r_2(\infty)}} \|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
& \leqslant C \left(\sum_{l_1=-\infty}^{-1} 2^{(k-l_1)(\frac{n}{p_1(0)}-n)} \|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right) \left(\sum_{l_2=-\infty}^{-1} 2^{(k-l_2)(\frac{n}{p_2(0)}-n)} \|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right). \qquad (6)
\end{aligned}
$$

注意到 $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$. 利用 Hölder 不等式, 得

$$\begin{aligned}V_{121}\leqslant& C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=0}^{\infty}\left(\sum_{l_1=-\infty}^{-1}2^{k\alpha_1(\infty)+(k-l_1)(\frac{n}{p_1(0)}-n)}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{q_1(1+\varepsilon)}\right\}^{\frac{1}{q_1(1+\varepsilon)}}\\&\times\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=0}^{\infty}\left(\sum_{l_2=-\infty}^{-1}2^{k\alpha_2(\infty)+(k-l_2)(\frac{n}{p_2(0)}-n)}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_2(1+\varepsilon)}\right\}^{\frac{1}{q_2(1+\varepsilon)}}\\&:=D_1D_2.\end{aligned}$$

由已知条件 $b_1:=\alpha_1(0)+\frac{n}{p_1(0)}-n<0$. 利用 Hölder 不等式和 Fubini 定理, 可得

$$\begin{aligned}D_1\leqslant& C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=0}^{\infty}\left(\sum_{l_1=-\infty}^{-1}2^{b_1(k-l_1)\frac{q_1(1+\varepsilon)}{2}}2^{l_1\alpha_1(\infty)q_1(1+\varepsilon)}\|f_{l_1}\|^{q_1(1+\varepsilon)}_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)\right.\\&\left.\times\left(\sum_{l_1=-\infty}^{-1}2^{b_1(k-l_1)\frac{q_1(1+\varepsilon)'}{2}}\right)^{\frac{q_1(1+\varepsilon)}{(q_1(1+\varepsilon))'}}\right\}^{\frac{1}{q_1(1+\varepsilon)}}\\&\leqslant C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\left(\sum_{k=-\infty}^{-1}2^{-l_1b_1\frac{q_1(1+\varepsilon)}{2}}2^{l_1\alpha_1(\infty)q_1(1+\varepsilon)}\|f_{l_1}\|^{q_1(1+\varepsilon)}_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)\left(\sum_{k=0}^{\infty}2^{kb_1\frac{q_1(1+\varepsilon)}{2}}\right)\right\}^{\frac{1}{q_1(1+\varepsilon)}}\\&\leqslant C\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{l_1=-\infty}^{-1}2^{l_1\alpha_1(\infty)q_1(1+\varepsilon)}\|f_{l_1}\|^{q_1(1+\varepsilon)}_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q_1(1+\varepsilon)}}\\&\leqslant C\|f_1\|_{\dot{K}^{\alpha_1(\cdot),q_1),\theta}_{p_1(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

类似于 D_1 的估计, 不难得到 D_2 的估计. 结合 V_{11} 和 V_{12} 的估计, 有

$$V_1\leqslant C\|f_1\|_{\dot{K}^{\alpha_1(\cdot),q_1),\theta}_{p_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{\dot{K}^{\alpha_2(\cdot),q_2),\theta}_{p_2(\cdot)}(\mathbb{R}^n)}.$$

其次估计 V_2. 由 Minkowski 不等式, 可得

$$\begin{aligned}V_2\leqslant& C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=-\infty}^{-1}2^{k\alpha(0)q(1+\varepsilon)}\left(\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k-1}^{k+1}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(1+\varepsilon)}\right\}^{\frac{1}{q(1+\varepsilon)}}\\&+C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=0}^{\infty}2^{k\alpha(\infty)q(1+\varepsilon)}\left(\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k-1}^{k+1}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(1+\varepsilon)}\right\}^{\frac{1}{q(1+\varepsilon)}}\\&:=V_{21}+V_{22}.\end{aligned}$$

当 $l_1\leqslant k-2$, $k-1\leqslant l_2\leqslant k+1$, $x\in E_k$, $y_1\in E_{l_1}$ 和 $y_2\in E_{l_2}$ 时, 有

$$|x-y_1|+|x-y_2|\geqslant|x-y_1|\geqslant|x|-|y_1|>2^{k-2},$$

所以

$$|BI_\beta(f_{l_1},f_{l_2})(x)| \leqslant C2^{-k(2n-\beta)}\|f_{l_1}\|_{L^1(\mathbb{R}^n)}\|f_{l_2}\|_{L^1(\mathbb{R}^n)}.$$

注意到 $\frac{1}{p(x)}=\frac{1}{r_1(x)}+\frac{1}{r_2(x)}$, $\frac{1}{r_i(x)}=\frac{1}{p_i(x)}-\frac{\beta}{2n}$. 类似于不等式(5), 利用 Hölder 不等式及引理 1-3, 可得

$$\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k-1}^{k+1}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\left(\sum_{l_1=-\infty}^{-1}2^{(k-l_1)(\frac{n}{p_1(0)}-n)}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right) \times\left(\sum_{l_2=k-1}^{k+1}2^{(k-l_2)(\frac{n}{p_2(0)}-n)}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right).$$

利用 Hölder 不等式和 $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$, 得

$$\begin{aligned}V_{21} &\leqslant C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{l_1=-\infty}^{k-2}2^{k\alpha_1(0)+(k-l_1)(\frac{n}{p_1(0)}-n)}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{q_1(1+\varepsilon)}\right\}^{\frac{1}{q_1(1+\varepsilon)}}\\ &\quad\times\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{l_2=k-1}^{k+1}2^{k\alpha_2(0)+(k-l_2)(\frac{n}{p_2(0)}-n)}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_2(1+\varepsilon)}\right\}^{\frac{1}{q_2(1+\varepsilon)}}\\ &:= V_{211}V_{212}.\end{aligned}$$

由于 $V_{211}=V_{111}$, 只需估计 V_{212}.

$$\begin{aligned}V_{212} &\leqslant C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=-\infty}^{-1}2^{k\alpha_2(0)q_2(1+\varepsilon)}\left(\sum_{l_2=k-1}^{k+1}2^{(k-l_2)(\frac{n}{p_2(0)}-n)}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_2(1+\varepsilon)}\right\}^{\frac{1}{q_2(1+\varepsilon)}}\\ &\leqslant C\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=-\infty}^{-1}2^{k\alpha_2(0)q_2(1+\varepsilon)}\|f_{l_2}\|^{q_2(1+\varepsilon)}_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q_2(1+\varepsilon)}}\\ &\leqslant C\|f_2\|_{\dot{K}^{\alpha_2(\cdot),q_2),\theta}_{p_2(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

对 V_{22} 估计如下. 由 Minkowski 不等式, 可得

$$\begin{aligned}V_{22} &\leqslant C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=0}^{\infty}2^{k\alpha(\infty)q(1+\varepsilon)}\left(\sum_{l_1=-\infty}^{-1}\sum_{l_2=k-1}^{k+1}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(1+\varepsilon)}\right\}^{\frac{1}{q(1+\varepsilon)}}\\ &\quad+C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=0}^{\infty}2^{k\alpha(\infty)q(1+\varepsilon)}\left(\sum_{l_1=0}^{k-2}\sum_{l_2=k-1}^{k+1}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(1+\varepsilon)}\right\}^{\frac{1}{q(1+\varepsilon)}}\\ &:= V_{221}+V_{222}.\end{aligned}$$

类似于不等式(6)的估计, 利用 Hölder 不等式, $p_2(0) \leqslant p_2(\infty)$ 及引理 1-3, 可得

$$\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k-1}^{k+1}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\left(\sum_{l_1=-\infty}^{-1}2^{(k-l_1)(\frac{n}{p_1(0)}-n)}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)$$
$$\times\left(\sum_{l_2=k-1}^{k+1}2^{(k-l_2)(\frac{n}{p_2(\infty)}-n)}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right).$$

注意到 $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$. 利用 Hölder 不等式, 得

$$V_{221} \leqslant C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=0}^{\infty}\left(\sum_{l_1=-\infty}^{-1}2^{k\alpha_1(\infty)+(k-l_1)(\frac{n}{p_1(0)}-n)}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{q_1(1+\varepsilon)}\right\}^{\frac{1}{q_1(1+\varepsilon)}}$$
$$\times\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=0}^{\infty}\left(\sum_{l_2=k-1}^{k+1}2^{k\alpha_2(\infty)+(k-l_2)(\frac{n}{p_2(\infty)}-n)}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_2(1+\varepsilon)}\right\}^{\frac{1}{q_2(1+\varepsilon)}}$$
$$:= E_1E_2.$$

由于 E_1 等价于 D_1, 只需用 $\alpha_1(\infty)$ 代替 $\alpha_1(0)$; 而 E_2 的估计与 V_{212} 的估计相似, 只需用 $\alpha_2(\infty)$ 和 $p_2(\infty)$ 分别代替 $\alpha_2(0)$ 和 $p_2(0)$.

对 V_{222}. 利用 Hölder 不等式, 引理 1-3 和 $p_i(0) \leqslant p_i(\infty)$, 可得

$$\sum_{l_1=0}^{k-2}\sum_{l_2=k-1}^{k+1}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\left(\sum_{l_1=0}^{k-2}2^{(k-l_1)(\frac{n}{p_1(\infty)}-n)}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)$$
$$\times\left(\sum_{l_2=k-1}^{k+1}2^{(k-l_2)(\frac{n}{p_2(\infty)}-n)}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right).$$

由 Hölder 不等式和 $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$, 得

$$V_{222} \leqslant C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=0}^{\infty}\left(\sum_{l_1=0}^{k-2}2^{k\alpha_1(\infty)+(k-l_1)(\frac{n}{p_1(\infty)}-n)}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{q_1(1+\varepsilon)}\right\}^{\frac{1}{q_1(1+\varepsilon)}}$$
$$\times\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=0}^{\infty}\left(\sum_{l_2=k-1}^{k+1}2^{k\alpha_2(\infty)+(k-l_2)(\frac{n}{p_2(\infty)}-n)}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_2(1+\varepsilon)}\right\}^{\frac{1}{q_2(1+\varepsilon)}}$$
$$:= F_1F_2.$$

由于 $F_2=E_2$, 而 F_1 的估计与 V_{212} 的估计相似, 因此 V_{22} 估计完毕. 结合 V_{21} 和 V_{22} 的估计, 得

$$V_2 \leqslant C\|f_1\|_{\dot{K}^{\alpha_1(\cdot),q_1),\theta}_{p_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{\dot{K}^{\alpha_2(\cdot),q_2),\theta}_{p_2(\cdot)}(\mathbb{R}^n)}.$$

下面估计 V_3.

$$V_3 \leqslant C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=-\infty}^{-1}2^{k\alpha(0)q(1+\varepsilon)}\left(\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k+2}^{\infty}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(1+\varepsilon)}\right\}^{\frac{1}{q(1+\varepsilon)}}$$

$$+C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=0}^{\infty}2^{k\alpha(\infty)q(1+\varepsilon)}\left(\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k+2}^{\infty}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(1+\varepsilon)}\right\}^{\frac{1}{q(1+\varepsilon)}}$$

$$:= V_{31}+V_{32}.$$

当 $l_1\leqslant k-2, l_2\geqslant k+2, x\in E_k,\ y_1\in E_{l_1}$ 及 $y_2\in E_{l_2}$ 时, 有

$$|x-y_1|\geqslant|x|-|y_1|>2^{k-2},$$

和

$$|x-y_2|\geqslant|y_2|-|x|>2^{l_2-1}-2^k>2^{l_2-1}-2^{l_2-2}>2^{l_2-2},$$

所以有

$$|BI_\beta(f_{l_1},f_{l_2})(x)|\leqslant C2^{-k(n-\frac{\beta}{2})}\|f_{l_1}\|_{L^1(\mathbb{R}^n)}2^{-l_2(n-\frac{\beta}{2})}\|f_{l_2}\|_{L^1(\mathbb{R}^n)}.$$

注意到 $\frac{1}{p(x)}=\frac{1}{r_1(x)}+\frac{1}{r_2(x)}$, $\frac{1}{r_i(x)}=\frac{1}{p_i(x)}-\frac{\beta}{2n}$ 和 $p_2(0)\leqslant p_2(\infty)$. 由 Hölder 不等式及引理 1-3, 得

$$\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k+2}^{\infty}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

$$\leqslant C\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k+2}^{\infty}2^{-k(n-\frac{\beta}{2})}\|f_{l_1}\|_{L^1(\mathbb{R}^n)}2^{-l_2(n-\frac{\beta}{2})}\|f_{l_2}\|_{L^1(\mathbb{R}^n)}\|\chi_k\|_{L^{r_1(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{r_2(\cdot)}(\mathbb{R}^n)}$$

$$\leqslant C\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k+2}^{\infty}2^{-k(n-\frac{\beta}{2})}2^{\frac{l_1n}{p_1'(0)}}2^{-l_2(n-\frac{\beta}{2})}2^{\frac{l_2n}{p_2'(\infty)}}2^{\frac{kn}{r_1(0)}}2^{\frac{kn}{r_2(0)}}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}$$

$$\leqslant C\left(\sum_{l_1=-\infty}^{k-2}2^{(k-l_1)(\frac{n}{p_1(0)}-n)}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)\left(\sum_{l_2=k+2}^{\infty}2^{(k-l_2)(\frac{n}{p_2(\infty)}-\frac{\beta}{2})}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right). \tag{7}$$

先看 V_{31}. 利用 Hölder 不等式和 $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$, 得

$$V_{31}\leqslant C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{l_1=-\infty}^{k-2}2^{k\alpha_1(0)+(k-l_1)(\frac{n}{p_1(0)}-n)}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{q_1(1+\varepsilon)}\right\}^{\frac{1}{q_1(1+\varepsilon)}}$$

$$\times\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{l_2=k+2}^{\infty}2^{k\alpha_2(0)+(k-l_2)(\frac{n}{p_2(\infty)}-\frac{\beta}{2})}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_2(1+\varepsilon)}\right\}^{\frac{1}{q_2(1+\varepsilon)}}$$

$$:= V_{311}V_{312}.$$

由于 $V_{311}=V_{111}$, 只需估计 V_{312}.

$$
\begin{aligned}
V_{312} &\leqslant C\sup_{\varepsilon>0}\left\{\varepsilon^{\theta}\sum_{k=-\infty}^{-1}\left(\sum_{l_2=k+2}^{-1}2^{(k-l_2)(\alpha_2(0)+\frac{n}{p_2(\infty)}-\frac{\beta}{2})}2^{l_2\alpha_2(0)}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_2(1+\varepsilon)}\right\}^{\frac{1}{q_2(1+\varepsilon)}}\\
&\quad+C\sup_{\varepsilon>0}\left\{\varepsilon^{\theta}\sum_{k=-\infty}^{-1}\left(\sum_{l_2=0}^{\infty}2^{(k-l_2)(\alpha_2(0)+\frac{n}{p_2(\infty)}-\frac{\beta}{2})}2^{l_2\alpha_2(0)}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_2(1+\varepsilon)}\right\}^{\frac{1}{q_2(1+\varepsilon)}}\\
&:=G_1+G_2.
\end{aligned}
$$

由已知条件 $d_2:=\alpha_2(0)+\frac{n}{p_2(\infty)}-\frac{\beta}{2}>0$. 利用 Hölder 不等式和 Fubini 定理, 可得

$$
\begin{aligned}
G_1 &\leqslant C\sup_{\varepsilon>0}\left\{\varepsilon^{\theta}\sum_{k=-\infty}^{-1}\left(\sum_{l_2=k+2}^{-1}2^{l_2\alpha_2(0)q_2(1+\varepsilon)}\|f_{l_2}\|^{q_2(1+\varepsilon)}_{L^{p_2(\cdot)}(\mathbb{R}^n)}2^{\frac{(k-l_2)d_2q_2(1+\varepsilon)}{2}}\right)\right.\\
&\quad\left.\times\left(\sum_{l_2=k+2}^{-1}2^{\frac{(k-l_2)d_2(q_2(1+\varepsilon))'}{2}}\right)^{\frac{q_2(1+\varepsilon)}{(q_2(1+\varepsilon))'}}\right\}^{\frac{1}{q_2(1+\varepsilon)}}\\
&\leqslant C\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{-1}\sum_{l_2=k+2}^{-1}2^{l_2\alpha_2(0)}2^{\frac{(k-l_2)d_2q_2(1+\varepsilon)}{2}}\|f_{l_2}\|^{q_2(1+\varepsilon)}_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q_2(1+\varepsilon)}}\\
&\leqslant C\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{l_2=-\infty}^{-1}2^{l_2\alpha_2(0)q_2(1+\varepsilon)}\|f_{l_2}\|^{q_2(1+\varepsilon)}_{L^{p_2(\cdot)}(\mathbb{R}^n)}\sum_{k=-\infty}^{l_2-2}2^{\frac{(k-l_2)d_2}{2}}\right)^{\frac{1}{q_2(1+\varepsilon)}}\\
&\leqslant C\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{l_2=-\infty}^{-1}2^{l_2\alpha_2(0)q_2(1+\varepsilon)}\|f_{l_2}\|^{q_2(1+\varepsilon)}_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q_2(1+\varepsilon)}}\\
&\leqslant C\|f_2\|_{\dot{K}^{\alpha_2(\cdot),q_2),\theta}_{p_2(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

对于 G_2. 注意到 $d_2>0,\ \alpha_2(0)\leqslant\alpha_2(\infty)$. 由 Hölder 不等式和 Fubini 定理, 可得

$$
\begin{aligned}
G_2 &\leqslant C\sup_{\varepsilon>0}\left\{\varepsilon^{\theta}\sum_{k=-\infty}^{-1}\left(\sum_{l_2=0}^{\infty}2^{l_2\alpha_2(0)q_2(1+\varepsilon)}\|f_{l_2}\|^{q_2(1+\varepsilon)}_{L^{p_2(\cdot)}(\mathbb{R}^n)}2^{\frac{(k-l_2)d_2q_2(1+\varepsilon)}{2}}\right)\right.\\
&\quad\left.\times\left(\sum_{l_2=0}^{\infty}2^{\frac{(k-l_2)d_2(q_2(1+\varepsilon))'}{2}}\right)^{\frac{q_2(1+\varepsilon)}{(q_2(1+\varepsilon))'}}\right\}^{\frac{1}{q_2(1+\varepsilon)}}\\
&\leqslant C\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{-1}\sum_{l_2=0}^{\infty}2^{l_2\alpha_2(0)q_2(1+\varepsilon)}\|f_{l_2}\|^{q_2(1+\varepsilon)}_{L^{p_2(\cdot)}(\mathbb{R}^n)}2^{\frac{(k-l_2)d_2q_2(1+\varepsilon)}{2}}\right)^{\frac{1}{q_2(1+\varepsilon)}}\\
&\leqslant C\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{-1}\sum_{l_2=0}^{\infty}2^{\frac{(k-l_2)d_2q_2(1+\varepsilon)}{2}}\sum_{j=l_2}^{\infty}2^{j\alpha_2(\infty)q_2(1+\varepsilon)}\|f_2\chi_j\|^{q_2(1+\varepsilon)}_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q_2(1+\varepsilon)}}\\
&\leqslant C\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{-1}\sum_{l_2=0}^{\infty}2^{\frac{(k-l_2)d_2q_2(1+\varepsilon)}{2}}\|f_2\|^{q_2(1+\varepsilon)}_{\dot{K}^{\alpha_2(\cdot),q_2),\theta}_{p_2(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q_2(1+\varepsilon)}}\\
&\leqslant C\|f_2\|_{\dot{K}^{\alpha_2(\cdot),q_2),\theta}_{p_2(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

因此得到了 V_{312} 的估计. 结合 V_{311} 和 V_{312} 的估计, 可得

$$V_{31}\leqslant C\|f_1\|_{\dot{K}^{\alpha_1(\cdot),q_1),\theta}_{p_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{\dot{K}^{\alpha_2(\cdot),q_2),\theta}_{p_2(\cdot)}(\mathbb{R}^n)}.$$

再估计 V_{32}.

$$\begin{aligned}V_{32}\leqslant& C\sup_{\varepsilon>0}\left\{\varepsilon^{\theta}\sum_{k=0}^{\infty}2^{k\alpha(\infty)q(1+\varepsilon)}\left(\sum_{l_1=-\infty}^{-1}\sum_{l_2=k+2}^{\infty}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(1+\varepsilon)}\right\}^{\frac{1}{q(1+\varepsilon)}}\\&+C\sup_{\varepsilon>0}\left\{\varepsilon^{\theta}\sum_{k=0}^{\infty}2^{k\alpha(\infty)q(1+\varepsilon)}\left(\sum_{l_1=0}^{k-2}\sum_{l_2=k+2}^{\infty}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(1+\varepsilon)}\right\}^{\frac{1}{q(1+\varepsilon)}}\\&:=V_{321}+V_{322}.\end{aligned}$$

注意到 $\frac{1}{p(x)}=\frac{1}{r_1(x)}+\frac{1}{r_2(x)}$, $\frac{1}{r_i(x)}=\frac{1}{p_i(x)}-\frac{\beta}{2n}$ 和 $p_1(0)\leqslant p_1(\infty)$. 由 Hölder 不等式及引理 1-3, 得

$$\begin{aligned}\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k+2}^{\infty}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\leqslant& C\left(\sum_{l_1=-\infty}^{k-2}2^{(k-l_1)(\frac{n}{p_1(0)}-n)}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)\\&\times\left(\sum_{l_2=k+2}^{\infty}2^{(k-l_2)(\frac{n}{p_2(\infty)}-\frac{\beta}{2})}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right).\end{aligned}$$

由 Hölder 不等式和 $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$, 可得

$$\begin{aligned}V_{321}\leqslant& C\sup_{\varepsilon>0}\left\{\varepsilon^{\theta}\sum_{k=0}^{\infty}\left(\sum_{l_1=-\infty}^{-1}2^{(k-l_1)(\alpha_1(\infty)-\frac{n}{p_1'(0)})}2^{l_1\alpha_1(\infty)}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{q_1(1+\varepsilon)}\right\}^{\frac{1}{q_1(1+\varepsilon)}}\\&\times\sup_{\varepsilon>0}\left\{\varepsilon^{\theta}\sum_{k=0}^{\infty}\left(\sum_{l_2=k+2}^{\infty}2^{(k-l_2)(\alpha_2(\infty)+\frac{n}{p_2(\infty)}-\frac{\beta}{2})}2^{l_2\alpha_2(\infty)}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_2(1+\varepsilon)}\right\}^{\frac{1}{q_2(1+\varepsilon)}}\\&:=H_1H_2.\end{aligned}$$

因为 $H_1=D_1$, 而 H_2 的估计与 V_{312} 的估计相似, 只需用 $\alpha_2(\infty)$ 代替 $\alpha_2(0)$.

对 V_{322}, 注意到 $\frac{1}{p(x)}=\frac{1}{r_1(x)}+\frac{1}{r_2(x)}$, $\frac{1}{r_i(x)}=\frac{1}{p_i(x)}-\frac{\beta}{2n}$. 由 Hölder 不等式及引理 1-3, 得

$$\begin{aligned}\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k+2}^{\infty}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\leqslant& C\left(\sum_{l_1=-\infty}^{k-2}2^{(k-l_1)(\frac{n}{p_1(\infty)}-n)}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)\\&\times\left(\sum_{l_2=k+2}^{\infty}2^{(k-l_2)(\frac{n}{p_2(\infty)}-\frac{\beta}{2})}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right).\end{aligned}$$

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由 Hölder 不等式, $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$, 可得

$$\begin{aligned}V_{322}\leqslant{}& C\sup_{\varepsilon>0}\left\{\varepsilon^{\theta}\sum_{k=0}^{\infty}\left(\sum_{l_1=0}^{k-2}2^{(k-l_1)(\alpha_1(\infty)+\frac{n}{p_1(\infty)}-n)}2^{l_1\alpha_1(\infty)}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{q_1(1+\varepsilon)}\right\}^{\frac{1}{q_1(1+\varepsilon)}}\\&\times\sup_{\varepsilon>0}\left\{\varepsilon^{\theta}\sum_{k=0}^{\infty}\left(\sum_{l_2=k+2}^{\infty}2^{(k-l_2)(\alpha_2(\infty)+\frac{n}{p_2(\infty)}-\frac{\beta}{2})}2^{l_2\alpha_2(\infty)}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_2(1+\varepsilon)}\right\}^{\frac{1}{q_2(1+\varepsilon)}}\\&:=S_1S_2.\end{aligned}$$

因为 $S_2=H_2$, $S_1=F_1$, 结合 V_{31} 和 V_{32} 的估计, 有

$$V_3\leqslant C\|f_1\|_{\dot{K}^{\alpha_1(\cdot),q_1),\theta}_{p_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{\dot{K}^{\alpha_2(\cdot),q_2),\theta}_{p_2(\cdot)}(\mathbb{R}^n)}.$$

对于 V_5. 通过 Minkowski 不等式, 可得

$$\begin{aligned}V_5\leqslant{}& C\sup_{\varepsilon>0}\left\{\varepsilon^{\theta}\sum_{k=-\infty}^{-1}2^{k\alpha(0)q(1+\varepsilon)}\left(\sum_{l_1=k-1}^{k+1}\sum_{l_2=k-1}^{k+1}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(1+\varepsilon)}\right\}^{\frac{1}{q(1+\varepsilon)}}\\&+C\sup_{\varepsilon>0}\left\{\varepsilon^{\theta}\sum_{k=0}^{\infty}2^{k\alpha(\infty)q(1+\varepsilon)}\left(\sum_{l_1=k+1}^{k-1}\sum_{l_2=k-1}^{k+1}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(1+\varepsilon)}\right\}^{\frac{1}{q(1+\varepsilon)}}\\&:=V_{51}+V_{52}.\end{aligned}$$

注意到 $\frac{1}{p(x)}=\frac{1}{r_1(x)}+\frac{1}{r_2(x)}$, $\frac{1}{r_i(x)}=\frac{1}{p_i(x)}-\frac{\beta}{2n}$. 利用引理 3-4, 得

$$\sum_{l_1=k-1}^{k+1}\sum_{l_2=k-1}^{k+1}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\leqslant C\left(\sum_{l_1=k-1}^{k+1}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)\left(\sum_{l_2=k-1}^{k+1}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right).$$

注意到 $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$. 利用 Hölder 不等式, 得

$$\begin{aligned}\mathrm{V}_{51}\leqslant{}& C\sup_{\varepsilon>0}\left\{\varepsilon^{\theta}\sum_{k=-\infty}^{-1}\left(\sum_{l_1=k-1}^{k+1}2^{k\alpha_1(0)}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{q_1(1+\varepsilon)}\right\}^{\frac{1}{q_1(1+\varepsilon)}}\\&\times\sup_{\varepsilon>0}\left\{\varepsilon^{\theta}\sum_{k=-\infty}^{-1}\left(\sum_{l_1=k-1}^{k+1}2^{k\alpha_2(0)}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_2(1+\varepsilon)}\right\}^{\frac{1}{q_2(1+\varepsilon)}}\\&\leqslant C\|f_1\|_{\dot{K}^{\alpha_1(\cdot),q_1),\theta}_{p_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{\dot{K}^{\alpha_2(\cdot),q_2),\theta}_{p_2(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

类似于 V_{51} 的估计, 只需将 $\alpha_i(0)$ 用 $\alpha_i(\infty)$ 代替, 我们便可得到 V_{52} 的估计, 所以

$$V_5\leqslant C\|f_1\|_{\dot{K}^{\alpha_1(\cdot),q_1),\theta}_{p_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{\dot{K}^{\alpha_2(\cdot),q_2),\theta}_{p_2(\cdot)}(\mathbb{R}^n)}.$$

再来看 V_6. 通过 Minkowski 不等式, 可得

$$V_6 \leqslant C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=-\infty}^{-1}2^{k\alpha(0)q(1+\varepsilon)}\left(\sum_{l_1=k-1}^{k+1}\sum_{l_2=k+2}^{\infty}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(1+\varepsilon)}\right\}^{\frac{1}{q(1+\varepsilon)}}$$

$$+C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=0}^{\infty}2^{k\alpha(\infty)q(1+\varepsilon)}\left(\sum_{l_1=k+1}^{k-1}\sum_{l_2=k+2}^{\infty}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{q(1+\varepsilon)}\right\}^{\frac{1}{q(1+\varepsilon)}}$$

$$:= V_{61}+V_{62}.$$

当 $k-1\leqslant l_1\leqslant k+1$, $l_2\geqslant k+2$, $x\in E_k$, $y_1\in E_{l_1}$ 及 $y_2\in E_{l_2}$ 时, 有 $|x-y_2|\geqslant|y_2|-|x|>2^{l_2-2}$. 则

$$|BI_\beta(f_{l_1},f_{l_2})(x)|\leqslant C2^{-k(n-\frac{\beta}{2})}\|f_{l_1}\|_{L^1(\mathbb{R}^n)}2^{-l_2(n-\frac{\beta}{2})}\|f_{l_2}\|_{L^1(\mathbb{R}^n)}.$$

注意到 $\frac{1}{p(x)}=\frac{1}{r_1(x)}+\frac{1}{r_2(x)}$, $\frac{1}{r_i(x)}=\frac{1}{p_i(x)}-\frac{\beta}{2n}$. 由 Hölder 不等式, 引理 1-3 及 $p_2(0)\leqslant p_2(\infty)$, 得

$$\sum_{l_1=k-1}^{k+1}\sum_{l_2=k+2}^{\infty}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\leqslant C\left(\sum_{l_1=k-1}^{k+1}2^{(k-l_1)(\frac{n}{p_1(0)}-n)}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)$$
$$\times\left(\sum_{l_2=k+2}^{\infty}2^{(k-l_2)(\frac{n}{p_2(\infty)}-\frac{\beta}{2})}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right).$$

通过 Hölder 不等式, $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$, 可得

$$V_{61}\leqslant C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{l_1=k-1}^{k+1}2^{k\alpha_1(0)+(k-l_1)(\frac{n}{p_1(0)}-n)}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{q_1(1+\varepsilon)}\right\}^{\frac{1}{q_1(1+\varepsilon)}}$$
$$\times\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{l_2=k+2}^{\infty}2^{(k-l_2)(\alpha_2(0)+\frac{n}{p_2(\infty)}-\frac{\beta}{2})}2^{l_2\alpha_2(0)}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_2(1+\varepsilon)}\right\}^{\frac{1}{q_2(1+\varepsilon)}}$$
$$:= V_{611}V_{612}.$$

注意到 V_{611} 的估计与 V_{212} 的估计相似, V_{612} 的估计与 V_{312} 的估计相似, 因此我们便完成了 V_{61} 的估计.

对 V_{62}. 同理用 $\alpha_i(\infty)$ 代替 $\alpha_i(0)$, 用 $p_1(\infty)$ 代替 $p_1(0)$, 可得

$$V_{62}\leqslant C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=0}^{\infty}\left(\sum_{l_2=k-1}^{k+1}2^{k\alpha_1(\infty)+(k-l_1)(\frac{n}{p_1(\infty)}-n)}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{q_1(1+\varepsilon)}\right\}^{\frac{1}{q_1(1+\varepsilon)}}$$
$$\times\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=0}^{\infty}\left(\sum_{l_2=k+2}^{\infty}2^{(k-l_2)(\alpha_2(\infty)+\frac{n}{p_2(\infty)}-\frac{\beta}{2})}2^{l_2\alpha_2(\infty)}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_2(1+\varepsilon)}\right\}^{\frac{1}{q_2(1+\varepsilon)}}$$
$$:= V_{621}V_{622}.$$

因为 V_{621} 的估计与 F_1 的估计相似, V_{622} 的估计与 S_2 的估计相似, 所以

$$V_6 \leqslant C\|f_1\|_{\dot{K}^{\alpha_1(\cdot),q_1),\theta}_{p_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{\dot{K}^{\alpha_2(\cdot),q_2),\theta}_{p_2(\cdot)}(\mathbb{R}^n)}.$$

最后估计 V_9. 通过 Minkowski 不等式, 可得

$$\begin{aligned}
V_9 \leqslant & C\sup_{\varepsilon>0}\left\{\varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q(1+\varepsilon)}\left(\sum_{l_1=k+2}^{\infty}\sum_{l_2=k+2}^{\infty}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|^{q(1+\varepsilon)}_{L^{p(\cdot)}(\mathbb{R}^n)}\right)\right\}^{\frac{1}{q(1+\varepsilon)}} \\
& + C\sup_{\varepsilon>0}\left\{\varepsilon^\theta \sum_{k=0}^{\infty} 2^{k\alpha(\infty)q(1+\varepsilon)}\left(\sum_{l_1=k+2}^{\infty}\sum_{l_2=k+2}^{\infty}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|^{q(1+\varepsilon)}_{L^{p(\cdot)}(\mathbb{R}^n)}\right)\right\}^{\frac{1}{q(1+\varepsilon)}} \\
& := V_{91}+V_{92}.
\end{aligned}$$

当 $l_i \geqslant k+2$, $x\in E_k$, $y_i \in E_{l_i}$ 时, 有 $|x-y_i| \geqslant |y_i|-|x| \geqslant C2^{l_i}$, 其中 $i=1,2$. 则

$$|BI_\beta(f_1,f_2)(x)| \leqslant C2^{-l_1(n-\frac{\beta}{2})}\|f_{l_1}\|_{L^1(\mathbb{R}^n)}2^{-l_2(n-\frac{\beta}{2})}\|f_{l_2}\|_{L^1(\mathbb{R}^n)}.$$

注意到 $\frac{1}{p(x)}=\frac{1}{r_1(x)}+\frac{1}{r_2(x)}$, $\frac{1}{r_i(x)}=\frac{1}{p_i(x)}-\frac{\beta}{2n}$ 和 $p_i(0)\leqslant p_i(\infty)$. 由 Hölder 不等式和引理 1-3, 得

$$\begin{aligned}
\sum_{l_1=k+2}^{\infty}\sum_{l_2=k+2}^{\infty}\|BI_\beta(f_{l_1},f_{l_2})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant & C\left(\sum_{l_1=k+2}^{\infty}2^{(k-l_1)(\frac{n}{p_1(\infty)}-\frac{\beta}{2})}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right) \\
& \times\left(\sum_{l_2=k+2}^{\infty}2^{(k-l_2)(\frac{n}{p_2(\infty)}-\frac{\beta}{2})}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right).
\end{aligned}$$

利用 Hölder 不等式和 $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$, 可得

$$\begin{aligned}
V_{91} \leqslant & C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{l_1=k+2}^{\infty}2^{(k-l_1)(\alpha_1(0)+\frac{n}{p_1(\infty)}-\frac{\beta}{2})}2^{l_1\alpha_1(0)}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{q_1(1+\varepsilon)}\right\}^{\frac{1}{q_1(1+\varepsilon)}} \\
& \times\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{l_2=k+2}^{\infty}2^{(k-l_2)(\alpha_2(0)+\frac{n}{p_2(\infty)}-\frac{\beta}{2})}2^{l_2\alpha_2(0)}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_2(1+\varepsilon)}\right\}^{\frac{1}{q_2(1+\varepsilon)}} \\
& := V_{911}V_{912}.
\end{aligned}$$

类似于 V_{312} 的估计, 不难得到 V_{911} 和 V_{912} 的估计.

下面估计 V_{92}.

$$\begin{aligned}
V_{92} \leqslant & C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=0}^{\infty}\left(\sum_{l_1=k+2}^{\infty}2^{(k-l_1)(\alpha_1(\infty)+\frac{n}{p_1(\infty)}-\frac{\beta}{2})}2^{l_1\alpha_1(\infty)}\|f_{l_1}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{q_1(1+\varepsilon)}\right\}^{\frac{1}{q_1(1+\varepsilon)}} \\
& \times\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=0}^{\infty}\left(\sum_{l_2=k+2}^{\infty}2^{(k-l_2)(\alpha_2(\infty)+\frac{n}{p_2(\infty)}-\frac{\beta}{2})}2^{l_2\alpha_2(\infty)}\|f_{l_2}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_2(1+\varepsilon)}\right\}^{\frac{1}{q_2(1+\varepsilon)}} \\
& := V_{921}V_{922}.
\end{aligned}$$

类似于 V_{322} 的估计, 便可得到 V_{921} 和 V_{922} 的估计, 所以

$$V_9 \leqslant C\|f_1\|_{\dot{K}^{\alpha_1(\cdot),q_1),\theta}_{p_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{\dot{K}^{\alpha_2(\cdot),q_2),\theta}_{p_2(\cdot)}(\mathbb{R}^n)}.$$

结合上述对 $V_i(i=1,2,\cdots,9)$ 的估计, 有

$$\|BI_\beta(f_1,f_2)\|_{\dot{K}^{\alpha(\cdot),q),\theta}_{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|f_1\|_{\dot{K}^{\alpha_1(\cdot),q_1),\theta}_{p_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{\dot{K}^{\alpha_2(\cdot),q_2),\theta}_{p_2(\cdot)}(\mathbb{R}^n)}.$$

定理 1 证毕.

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