Pure Mathematics 理论数学, 2013, 3, 31-40
http://dx.doi.org/10.12677/pm.2013.31006 Published Online January 2013 (http://www.hanspub.org/journal/pm.html)

Pseudo Almost Periodic Solution of Bidirectional Cohen-Grossberg Neural Networks with Distributed Delays*

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Received: Oct. 14th, 2012; revised: Oct. 29th, 2012; accepted: Nov. 7th, 2012

Abstract: In this paper, by using fixed point theorem and inequality techniques, a new sufficient condition is obtained to ensure the existence and unique for the pseudo almost periodic solution of a class bidirectional Cohen-Grossberg neural networks with distributed delays. An example is given to illustrate the correctness of our discussions.

Keywords: Bidirectional Cohen-Grossberg Neural Networks; Pseudo Almost Periodic Solution; Distributed Delays

具分布时滞双向 Cohen-Grossberg 模型的伪概周期解*

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收稿日期：2012 年 10 月 14 日；修回日期：2012 年 10 月 29 日；录用日期：2012 年 11 月 7 日

摘　要：利用不动点定理和不等式分析技巧，研究了一类具分布时滞的双向 Cohen-Grossberg 神经网络模型，得到了一个新的充分条件，确保该模型伪概周期解存在性与唯一性，最后用一个实例说明所得结论的正确性。

关键词：双向 C-G 神经网络；伪概周期解；分布时滞

1. 引言

Cohen-Grossberg[1] (C-G)在 1983 年，提出了以下神经网络模型：

$$x_i'(t)=-a_i\left(x_i(t)\right)\left[b_i\left(x_i(t)\right)-\sum_{j=1}^{n}c_{ij}g_j\left(x_j(t)\right)-I_i\right]. \tag{1.1}$$

由于它在模式识别、自动化、图像与信息处理等领域的潜在应用，近几十年，吸引了大量的学者与专家关注，参见文献[2-6]及所引文献。在文献[7]中，作者考虑了双向 C-G 神经网络，通过构造 Lyapunov 泛函，讨论了如下时滞模型：

$$\begin{cases} x_i'(t)=-a_i\left(x_i(t)\right)\left[b_i\left(x_i(t)\right)-\sum\limits_{j=1}^{m}p_{ij}g_j\left(\mu_j y_j\left(t-\tau_j\right)\right)\right], i=1,2,\cdots,n \\ y_j'(t)=-c_j\left(y_j(t)\right)\left[d_j\left(y_j(t)\right)-\sum\limits_{i=1}^{n}q_{ji}f_i\left(\lambda_i x_i\left(t-\sigma_i\right)\right)\right], j=1,2,\cdots,m. \end{cases} \tag{1.2}$$

*基金项目：湖南省自科基金项目(11JJ3005)；湖南省教育厅质量工程项目(2011)。

得到了其概周期解的渐进稳定性。曹进德[8]等人讨论了时变时滞的双向 C-G 模型：

$$\begin{cases} x_i'(t) = -a_i\left(x_i(t)\right)\left[b_i\left(x_i(t)\right) - \sum_{j=1}^{m} p_{ij}(t)\, g_j\left(\mu_j y_j\left(t-\tau_{ij}(t)\right)\right) - I_i\right], i = 1,2,\cdots,n \\ y_j'(t) = -c_j\left(y_j(t)\right)\left[d_j\left(y_j(t)\right) - \sum_{i=1}^{n} q_{ji}(t)\, f_i\left(\lambda_i x_i\left(t-\sigma_{ji}(t)\right)\right) - J_j\right], j = 1,2,\cdots,m. \end{cases} \tag{1.3}$$

得到了其不动点的存在性、唯一性与全局指数稳定性的几个充分条件。向红军[9]等人讨论了具分布时滞的 C-G 神经网络模型的伪概周期解。文献[10]的作者研究了一类具分布时滞的双向 C-G 神经网络：

$$\begin{cases} x_i'(t) = -a_i\left(x_i(t)\right)\left[b_i\left(x_i(t)\right) - \sum_{j=1}^{m} p_{ij}(t)\int_{-\infty}^{t} K_{ij}(t-s)\, g_j\left(y_j(s)\right)\mathrm{d}s - I_i(t)\right] \\ y_j'(t) = -c_j\left(y_j(t)\right)\left[d_j\left(y_j(t)\right) - \sum_{i=1}^{n} q_{ji}(t)\int_{-\infty}^{t} L_{ji}(t-s)\, f_i\left(x_i(s)\right)\mathrm{d}s - J_j(t)\right] \end{cases} \tag{1.4}$$

初始条件：

$$x_i(s) = \phi_i(s), y_j(s) = \psi_j(s), s \in (-\infty, 0], i = 1,2,\cdots,n, j = 1,2,\cdots,m. \tag{1.5}$$

的概周期解。而据我们所掌握的知识，具分布时滞双向 C-G 神经网络模型的伪概周期解研究很少。基于上述讨论，本文主要研究具分布时滞的双向 C-G 神经网络模型(1.4)的伪概周期解。这里 $i = 1,2,\cdots,n$； $j = 1,2,\cdots,m$； $n, m \ge 2$ 分别表示第 F_X 层与第 F_Y 层中神经元的个数； $x_i(t)$ 与 $y_j(t)$ 分别表示第 i 个神经元与第 j 个神经元的状态变量； $a_i(\cdot)$、 $c_j(\cdot)$ 表示强化函数； $b_i(\cdot)$、 $d_j(\cdot)$ 表示行为函数； $p_{ij}(t)$ 表示 t 时刻第 i 个神经元对第 j 个神经元的连接权值， $q_{ji}(t)$ 表示第 j 个神经元对第 i 个神经元的连接权值； f_i, g_j 分别表示第 F_X 层第 i 个神经元与第 F_Y 层第 j 个神经元的激励函数；核 $K_{ij}(\cdot)$ 与 $L_{ji}(\cdot)$ 都是 $[0,+\infty) \to [0,+\infty)$ 上的连续函数； $I_i(t)$ 与 $J_j(t)$ 分别表示 t 时刻第 F_X 层第 i 个神经元与第 F_Y 层第 j 个神经元的外部输入。

2. 定义与假设

设 $BC(R, R^n)$ 表示从 R 到 R^n 的有界连续函数集。$\left(BC(R,R^n), \|\cdot\|_\infty\right)$ 是一个 Banach 空间，这里 $\|\cdot\|_\infty$ 表示 $\|f\|_\infty$ 的上确界范数，即 $\|f\|_\infty = \sup_{t\in R}\|f(t)\|$。

定义 1[11] 令 $s \in N$ 且 $f \in C(R, R^s)$，若对 $\forall \varepsilon > 0$，$\exists l_\varepsilon > 0$，$\forall \alpha \in R$，$\exists \delta \in [\alpha, \alpha + l_\varepsilon]$，$\|f(\cdot+\delta) - f(\cdot)\|_\infty \le \varepsilon$，则称 f 是一个概周期解。

$AP(R, R^s)$ 表示从 R 到 R^s 的概周期函数集，显然 $AP(R, R^s)$ 是具上确界范数的 Banach 空间。$PAP_0(R, R^s)$ 表示具有以下性质的函数集：

$$PAP_0(R, X) = \left\{ f \in BC(R, R^s) \,\middle|\, \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T}\|f(t)\|\mathrm{d}t = 0 \right\}。$$

定义 2[12] 设函数 $f \in BC(R, R^s)$，且 f 可以表示为 $f = h + \varphi$，这里 $h \in AP(R, R^s), \varphi \in PAP_0(R, R^s)$，则称 f 为伪概周期函数。这类函数所组成的集合记为 $PAP(R, R^s)$。

注 1. 定义 2 的伪概周期 f 中的 h 为概周期函数， φ 表示遍历扰动。伪概周期函数 f 的分解唯一。显然 $\left(PAP(R,R^s), \|\cdot\|_\infty\right)$ 是一个 Banach 空间，且有 $AP(R, R^s) \subsetneq PAP(R, R^s) \subset BC(R, R^s)$。例如函数 $\varphi(t) = \sin^2 t + \sin^2\sqrt{3}t + \exp\left(-t^2\cos^2 t\right)$ 是伪概周期函数而不是概周期函数。

为方便起见，全文做如下假设：

$(H1)$：强化函数 $a_i(\cdot)$ 与 $c_j(\cdot)$ 连续有界，且存在正常数 a_i^-、 a_i^+ 和 c_j^-、 c_j^+ 满足：

$$0 < a_i^- \le a_i(x) \le a_i^+,\ \forall x \in R, i = 1,2,\cdots,n,\ 0 < c_j^- \le c_j(y) \le c_j^+,\ \forall y \in R, j = 1,2,\cdots,m.$$

$(H2)$：行为函数 $b_i\left(x_i(t)\right), d_j\left(y_j(t)\right)$ 连续且满足 $b_i(0)=0, d_j(0)=0$，存在常数 τ_i, σ_j，使得：

$$\frac{b_i(u)-b_i(\tilde{u})}{u-\tilde{u}} \geq \tau_i > 0, \forall u, \tilde{u} \in R, u \neq \tilde{u}, \frac{d_j(v)-d_j(\tilde{v})}{v-\tilde{v}} \geq \sigma_j > 0, \forall v, \tilde{v} \in R, v \neq \tilde{v}\text{。}$$

这里 $i=1,2,\cdots,n; j=1,2,\cdots,m$ 。

$(H3)$：激励函数 $f_i(x)$，$g_j(x)$ 满足 Lipschitzti 条件，即存在 $\lambda_i > 0, \delta_j > 0$，使：

$$\lambda_i = \sup_{u \neq \tilde{u}}\left|\frac{f_i(u)-f_i(\tilde{u})}{u-\tilde{u}}\right|, \forall u, \tilde{u} \in R, u \neq \tilde{u}, \delta_j = \sup_{v \neq \tilde{v}}\left|\frac{g_j(v)-g_j(\tilde{v})}{v-\tilde{v}}\right|, \forall v, \tilde{v} \in R, v \neq \tilde{v}\text{。}$$

这里 $i=1,2,\cdots,n; j=1,2,\cdots,m$ 。

$(H4)$：对一切 $0 \leq i \leq n, 0 \leq j \leq m,$ 核 $K_{ij}(\cdot)$ 与 $L_{ji}(\cdot)$ 都是 $[0,+\infty) \to [0,+\infty)$ 的分段连续可积函数，满足：

$$\int_0^{+\infty} K_{ij}(s)\mathrm{d}s = 1, \int_0^{+\infty} \exp(\beta s) K_{ij}(s)\mathrm{d}s < \infty, \int_0^{+\infty} L_{ji}(s)\mathrm{d}s = 1, \int_0^{+\infty} \exp(\beta s) L_{ji}(s)\mathrm{d}s < \infty,$$

这里 β 是正实数，$i=1,2,\cdots,n; j=1,2,\cdots,m$ 。

$(H5)$：$p_{ij}(t), q_{ji}(t), I_i(t), J_j(t)$ 是 R 上的连续伪概周期函数。为了方便，引入以下符号：

$$p_{ij} = \sup_{t \in R}\left\{\left|p_{ij}(t)\right|\right\}, q_{ji} = \sup_{t \in R}\left\{\left|q_{ji}(t)\right|\right\}, I_i = \sup_{t \in R}\left\{\left|I_i(t)\right|\right\}, J_j = \sup_{t \in R}\left\{\left|J_j(t)\right|\right\},\ I = \max\left\{\max_i\left\{\frac{I_i}{\tau_i a_i^-}\right\}, \max_j\left\{\frac{J_j}{\sigma_j c_j^-}\right\}\right\}.$$

这里 $i=1,2,\cdots,n+m$ 。

3. 引理

由 $(H1)$，$\dfrac{1}{a_i(x_i)}$ 与 $\dfrac{1}{c_j(y_j)}$ 的原函数存在，我们分别选择 $\dfrac{1}{a_i(x_i)}$ 与 $\dfrac{1}{c_j(y_j)}$ 的一个满足 $F_i(0)=0$，$G_j(0)=0$ 的原函数为 $F_i(x_i)$ 与 $G_j(y_j)$，即：$F_i'(x_i)=\dfrac{1}{a_i(x_i)}$，$G_j'(y_j)=\dfrac{1}{c_j(y_j)}$ 。又因为 $a_i(x_i)>0, c_j(y_j)>0$，所以 $F_i(x_i)$ 与 $G_j(y_j)$ 是分别关于 x_i 与 y_j 的增函数，故 $F_i(x_i)$，$G_j(y_j)$ 存在连续可导的反函数。这样：

$$\left(F_i^{-1}\right)'(x_i) = a_i(x_i), \left(G_j^{-1}\right)'(y_j) = c_j(y_j) \tag{3.1}$$

这里 $\left(F_i^{-1}\right)'(x_i), \left(G_j^{-1}\right)'(y_j)$ 分别表示 $F_i^{-1}(x_i)$ 与 $G_j^{-1}(y_j)$ 关于 x_i 与 y_j 的导数。而复合函数 $b_i\left(F_i^{-1}(x_i)\right)$ 与 $d_j\left(G_j^{-1}(y_j)\right)$ 都是可微的。为了简便，令 $u_i(t)=F_i\left(x_i(t)\right)$，$v_j(t)=G_j\left(y_j(t)\right)$ 易得：

$$u_i'(t) = F_i'(x_i)x_i'(t) = \frac{x_i'(t)}{a_i\left(x_i(t)\right)}, v_j'(t) = G_j'(y_j)y_j'(t) = \frac{y_j'(t)}{c_j\left(y_j(t)\right)}$$

而 $x_i(t)=F_i^{-1}\left(u_i(t)\right), y_j(t)=G_j^{-1}\left(v_j(t)\right)$ 。将这些等式代入(1.4)中得：

$$\begin{cases} u_i'(t) = -b_i\left(F_i^{-1}\left(u_i(t)\right)\right) + \sum_{j=1}^{m} p_{ij}(t)\int_{-\infty}^{t} K_{ij}(t-s) g_j\left(G_j^{-1}\left(v_j(s)\right)\right)\mathrm{d}s + I_i(t), t>0, \\ v_j'(t) = -d_j\left(G_j^{-1}\left(v_j(t)\right)\right) + \sum_{i=1}^{n} q_{ji}(t)\int_{-\infty}^{t} L_{ji}(t-s) f_i\left(F_i^{-1}\left(u_i(s)\right)\right)\mathrm{d}s + J_j(t), t>0, \\ u_i(t) = F_i\left(\phi_i(t)\right) = \phi_i(t), t \leq 0, \\ v_j(t) = G_j\left(\psi_j(t)\right) = \psi_j(t), t \leq 0. \end{cases} \tag{3.2}$$

由 $(H2)$ 得：

$$\begin{cases} b_i\left(F_i^{-1}\left(u_i(t)\right)\right)=\left[b_i\left(F_i^{-1}\left(\theta_i u_i(t)\right)\right)\right]' u_i(t)=\tilde{b}_i\left(u_i(t)\right)u_i(t), \\ d_j\left(G_j^{-1}\left(v_j(t)\right)\right)=\left[d_j\left(G_j^{-1}\left(\vartheta_j v_j(t)\right)\right)\right]' v_j(t)=\tilde{d}_j\left(v_j(t)\right)v_j(t). \end{cases} \tag{3.3}$$

这里 $0\le\theta_i\le 1,\ 0\le\vartheta_j\le 1$。

这样(3.2)式可以改成：

$$\begin{cases} u_i'(t)=-\tilde{b}_i\left(u_i(t)\right)u_i(t)+\sum_{j=1}^{m}p_{ij}(t)\int_{-\infty}^{t}K_{ij}(t-s)\,g_j\left(G_j^{-1}\left(v_j(s)\right)\right)\mathrm{d}s+I_i(t),t>0, \\ v_j'(t)=-\tilde{d}_j\left(v_j(t)\right)v_j(t)+\sum_{i=1}^{n}q_{ji}(t)\int_{-\infty}^{t}L_{ji}(t-s)\,f_i\left(F_i^{-1}\left(u_i(s)\right)\right)\mathrm{d}s+J_j(t),t>0, \\ u_i(t)=F_i\left(\phi_i(t)\right)=\phi_i(t),t\le 0, \\ v_j(t)=G_j\left(\psi_j(t)\right)=\psi_j(t),t\le 0. \end{cases} \tag{3.4}$$

故(1.4)存在唯一的伪概周解当且仅当(3.2)或(3.4)存在唯一的伪概周期解，这里我们仅考虑(3.4)，由拉格朗日中值公式和(3.1)易得：

$$\begin{cases} \left|F_i^{-1}(u)-F_i^{-1}(v)\right|=\left|\left(F_i^{-1}\right)'\left(v+\theta(u-v)\right)(u-v)\right|=\left|a_i\left(v+\theta(u-v)\right)\right|\left|u-v\right|, \\ \left|G_j^{-1}(u)-G_j^{-1}(v)\right|=\left|\left(G_j^{-1}\right)'\left(v+\vartheta(u-v)\right)(u-v)\right|=\left|c_j\left(v+\vartheta(u-v)\right)\right|\left|u-v\right|; \end{cases}$$

这里 $0\le\theta\le 1,\ 0\le\vartheta\le 1$。由 $(H1)$ 得

$$\begin{cases} a_i^-\left|u-v\right|\le\left|F_i^{-1}(u)-F_i^{-1}(v)\right|\le a_i^+\left|u-v\right|, \\ c_j^-\left|u-v\right|\le\left|G_j^{-1}(u)-G_j^{-1}(v)\right|\le c_j^+\left|u-v\right|. \end{cases} \tag{3.5}$$

联立 $(H2)$ 与(3.5)式，易得：

$(H6)$： $b_i'\left(F_i^{-1}(\cdot)\right)\ge\tau_i a_i^-, d_j'\left(G_j^{-1}(\cdot)\right)\ge\sigma_j c_j^-$， 这里 $i=1,2,\cdots,n;\ j=1,2,\cdots,m$ 。

引理 1 如果 $\varphi\in PAP\left(R,R^s\right)$，则对 $\forall h\in R$，都有 $\varphi(\cdot-h)\in PAP\left(R,R^s\right)$。

证明：由前面的定义可知 $\varphi=\varphi_1+\varphi_2$，其中 $\varphi_1\in AP\left(R,R^s\right)$， $\varphi_2\in PAP_0\left(R,R^s\right)$。显然：

$$\varphi(\cdot-h)=\varphi_1(\cdot-h)+\varphi_2(\cdot-h)$$

易知 $\varphi_1(\cdot-h)\in AP\left(R,R^s\right)$，且

$$0\le\frac{1}{2T}\int_{-T}^{T}\left\|\varphi_2(t-h)\right\|\mathrm{d}t=\frac{1}{2T}\int_{-T-h}^{T-h}\left\|\varphi_2(t)\right\|\mathrm{d}t\le\frac{T+h}{2T(T+h)}\int_{-(T+h)}^{T+h}\left\|\varphi_2(t)\right\|\mathrm{d}t$$

这表明 $\varphi_2(\cdot-h)\in PAP_0\left(R,R^s\right)$，所以 $\varphi(\cdot-h)\in PAP\left(R,R^s\right)$。

引理 2 若假设(H3)~(H5)成立，当 $\varphi,\psi\in PAP\left(R,R^s\right)$，则对一切 i,j 有

$$P_{ij}:t\to\int_{-\infty}^{t}K_{ij}(t-s)\varphi_j(s)\mathrm{d}s\in PAP(R,R),Q_{ji}:t\to\int_{-\infty}^{t}L_{ji}(t-s)\psi_i(s)\mathrm{d}s\in PAP(R,R)\text{。}$$

证明：由定义得： $\varphi_j=\varphi_j^1+\varphi_j^2,\psi_i=\psi_i^1+\psi_i^2$

$$P_{ij}(t)=\int_{-\infty}^{t}K_{ij}(t-s)\left(\varphi_j^1(s)+\varphi_j^2(s)\right)\mathrm{d}s=\int_{-\infty}^{t}K_{ij}(t-s)\varphi_j^1(s)\mathrm{d}s+\int_{-\infty}^{t}K_{ij}(t-s)\varphi_j^2(s)\mathrm{d}s=P_{ij}^1+P_{ij}^2,$$

$$Q_{ji}(t)=\int_{-\infty}^{t}L_{ji}(t-s)\left(\psi_i^1(s)-\psi_i^2\right)\mathrm{d}s=\int_{-\infty}^{t}L_{ji}(t-s)\psi_i^1(s)\mathrm{d}s+\int_{-\infty}^{t}L_{ji}(t-s)\psi_i^2(s)\mathrm{d}s=Q_{ji}^1+Q_{ji}^2.$$

先证明 $P_{ij}^1,Q_{ji}^1\in AP(R,R)$。对 $\forall\varepsilon>0,\ \exists l_\varepsilon>0,\ \forall\alpha\in R,\ \exists\delta\in\left[\alpha,\alpha+l_\varepsilon\right]$，

$$\left|\varphi_j^1(\cdot+\delta)-\varphi_j^1(\cdot)\right|_\infty \le \varepsilon,\ \left|\psi_i^1(\cdot+\delta)-\psi_i^1(\cdot)\right|_\infty \le \varepsilon,$$

故

$$\begin{aligned}\left|P_{ij}^1(t+\delta)-P_{ij}^1(t)\right| &= \left|\int_{-\infty}^{t+\delta}K_{ij}(t+\delta-s)\varphi_j^1(s)\mathrm{d}s-\int_{-\infty}^{t}K_{ij}(t-s)\varphi_j^1(s)\mathrm{d}s\right| \\ &\le \int_{-\infty}^{t}\left|K_{ij}(t-s)\right|\left|\varphi_j^1(s+\delta)-\varphi_j^1(s)\right|\mathrm{d}s \le \varepsilon\int_{-\infty}^{t}\left|K_{ij}(t-s)\right|\mathrm{d}s \le \varepsilon \\ \left|Q_{ji}^1(t+\delta)-Q_{ji}^1(t)\right| &= \left|\int_{-\infty}^{t+\delta}L_{ji}(t+\delta-s)\psi_i^1(s)\mathrm{d}s-\int_{-\infty}^{t}L_{ji}(t-s)\psi_i^1(s)\mathrm{d}s\right| \\ &\le \int_{-\infty}^{t}\left|L_{ji}(t-s)\right|\left|\psi_i^1(s+\delta)-\psi_i^1(s)\right|\mathrm{d}s \le \varepsilon\int_{-\infty}^{t}\left|L_{ji}(t-s)\right|\mathrm{d}s \le \varepsilon\end{aligned}$$

于是证得 $P_{ij}^1, Q_{ji}^1 \in AP(R,R)$。再来证明 $P_{ij}^2, Q_{ji}^2 \in PAP_0(R,R)$，

$$\begin{aligned}\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left\|P_{ij}^2\right\|\mathrm{d}t &= \sup_{t\in R}\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left|\int_{-\infty}^{t}K_{ij}(t-s)\varphi_j^2(s)\mathrm{d}s\right|\mathrm{d}t \\ &\le \sup_{t\in R}\lim_{T\to+\infty}\frac{1}{2T}\int_0^{+\infty}\left|K_{ij}(u)\right|\left(\int_{-(T+u)}^{T+u}\left|\varphi_j^2(v)\right|\mathrm{d}v\right)\mathrm{d}u = 0\end{aligned}$$

$$\begin{aligned}\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left\|Q_{ji}^2\right\|\mathrm{d}t &= \sup_{t\in R}\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left|\int_{-\infty}^{t}L_{ji}(t-s)\psi_i^2(s)\mathrm{d}s\right|\mathrm{d}t \\ &\le \sup_{t\in R}\lim_{T\to+\infty}\frac{1}{2T}\int_0^{+\infty}\left|L_{ji}(u)\right|\left(\int_{-(T+u)}^{T+u}\left|\psi_i^2(v)\right|\mathrm{d}v\right)\mathrm{d}u = 0\end{aligned}$$

于是证得 $P_{ij}^2, Q_{ji}^2 \in PAP_0(R,R)$，故 $P_{ij}(t), Q_{ji}(t) \in PAP(R,R)$。

引理 3 若假设(H2)-(H6)成立，定义非线性映射 Γ：对任意

$$\varphi = (\varphi_1,\cdots,\varphi_{n+m})^{\mathrm{T}} = \left(\phi(t)^{\mathrm{T}},\psi(t)^{\mathrm{T}}\right)^{\mathrm{T}} = (\phi_1,\cdots,\phi_n,\psi_1,\cdots,\psi_m)^{\mathrm{T}} \in PAP\left(R,R^{n+m}\right)$$

$$\begin{cases}u_\phi(t) = \int_{-\infty}^{t}\exp\left(-\int_s^t\tilde{b}_i(u_i(w))\mathrm{d}w\right)\left[\sum_{j=1}^{m}p_{ij}(s)\int_{-\infty}^{t}K_{ij}(s-w)g_j\left(G_j^{-1}(\psi_j(s))\right)\mathrm{d}w+I_i(s)\right]\mathrm{d}s,\\ v_\psi(t) = \int_{-\infty}^{t}\exp\left(-\int_s^t\tilde{d}_j(v_j(w))\mathrm{d}w\right)\left[\sum_{i=1}^{n}q_{ji}(s)\int_{-\infty}^{t}L_{ji}(s-w)f_i\left(F_i^{-1}(\phi_i(w))\right)\mathrm{d}w+J_j(s)\right]\mathrm{d}s.\end{cases} \tag{3.6}$$

令：

$$D_{ij} = \sum_{j=1}^{m}p_{ij}(s)\int_{-\infty}^{t}K_{ij}(s-w)g_j\left(G_j^{-1}(\psi_j(s))\right)\mathrm{d}w+I_i(s), E_{ji} = \sum_{i=1}^{n}q_{ji}(s)\int_{-\infty}^{t}L_{ji}(s-w)f_i\left(F_i^{-1}(\phi_i(w))\right)\mathrm{d}w+J_j(s),$$

Γ 是 $PAP\left(R,R^{n+m}\right)$ 上的自映射。

证明：令 $\varphi,\psi \in PAP\left(R,R^{n+m}\right)$，由引理 2 知函数 $P_{ij}: t\to\int_{-\infty}^{t}K_{ij}(t-s)\varphi_j(s)\mathrm{d}s \in PAP\left(R,R^{n+m}\right)$, $Q_{ji}: t\to\int_{-\infty}^{t}L_{ji}(t-s)\psi_j(s)\mathrm{d}s \in PAP\left(R,R^{n+m}\right)$，由(H2)-(H5)和[13]中的组合定理知 $\Phi_{ij}: s\to D_{ij}, \Phi_{ij}\in PAP(R,R), \Psi_{ji}: s\to E_{ji}, \Psi_{ji}\in PAP(R,R)$；由前面的定义 $\Phi_{ij} = \Phi_{ij}^1+\Phi_{ij}^2, \Psi_{ji} = \Psi_{ji}^1+\Psi_{ji}^2$，其中 $\Phi_{ij}^1 \in AP(R,R),\ \Psi_{ji}^1 \in AP(R,R)$ 和 $\Phi_{ij}^2 \in PAP_0(R,R),\ \Psi_{ji}^2 \in PAP_0(R,R)$，对一切 $i=1,2,\cdots,n,\ j=1,2,\cdots,m$。

$$\begin{cases}\left(\Gamma_j\psi_j\right) = \int_{-\infty}^{t}\exp\left(-\int_s^t\tilde{b}_i(u_i(w))\mathrm{d}w\right)\Phi_{ij}^1(s)\mathrm{d}s+\int_{-\infty}^{t}\exp\left(-\int_s^t\tilde{b}_i(u_i(w))\mathrm{d}w\right)\Phi_{ij}^2(s)\mathrm{d}s = \Theta_1+\Theta_2,\\ \left(\Gamma_i\phi_i\right) = \int_{-\infty}^{t}\exp\left(-\int_s^t\tilde{d}_j(v_j(w))\mathrm{d}w\right)\Psi_{ji}^1(s)\mathrm{d}s+\int_{-\infty}^{t}\exp\left(-\int_s^t\tilde{d}_j(v_j(w))\mathrm{d}w\right)\Psi_{ji}^2(s)\mathrm{d}s = \Lambda_1+\Lambda_2.\end{cases}$$

其中 $\Theta_1: t\to\int_{-\infty}^{t}\exp\left(-\int_s^t\tilde{b}_i(u_i(w))\mathrm{d}w\right)\Phi_{ij}^1(s)\mathrm{d}s; \Lambda_1: t\to\int_{-\infty}^{t}\exp\left(-\int_s^t\tilde{d}_j(v_j(w))\mathrm{d}w\right)\Psi_{ji}^1(s)\mathrm{d}s$。

首先证明：$\Theta_1,\Lambda_1\in AP(R,R)$。由于 Φ_{ij}^1，Ψ_{ji}^1 是概周期函数，所以对 $\forall\varepsilon>0$，存在常数 l_i,l_j，使得在任意区间 $[\alpha,\alpha+l_i]$，$[\alpha,\alpha+l_j]$ 上有常数 δ_i^1，δ_j^2 满足：$\sup\limits_{t\in R}\left|\Phi_{ij}^1(t+\delta_i^1)-\Phi_{ij}^1(t)\right|<\varepsilon,\sup\limits_{t\in R}\left|\Psi_{ji}^1(t+\delta_j^2)-\Psi_{ji}^1(t)\right|<\varepsilon$。对一切 $0\le i\le n,0\le j\le m$。有：

$$\begin{aligned}\left|\Theta_1(t+\delta_i^1)-\Theta_1(t)\right|&=\left|\int_{-\infty}^{t+\delta_i^1}\exp\left(-\int_s^t\tilde{b}_i(u_i(w))\mathrm{d}w\right)\Phi_{ij}^1(s)\mathrm{d}s-\int_{-\infty}^{t}\exp\left(-\int_s^t\tilde{b}_i(u_i(w))\mathrm{d}w\right)\Phi_{ij}^1(s)\mathrm{d}s\right|\\&\le\left|\int_{-\infty}^{t+\delta_i^1}\exp\left(-\tau_ia_i^-(t+\delta_i^1-s)\right)\Phi_{ij}^1(s)\mathrm{d}s-\int_{-\infty}^{t}\exp\left(-\tau_ia_i^-(t-s)\right)\Phi_{ij}^1(s)\mathrm{d}s\right|\\&=\left|\int_{-\infty}^{t}\exp\left(-\tau_ia_i^-(t-s)\right)\Phi_{ij}^1(s+\delta_i^1)\mathrm{d}s-\int_{-\infty}^{t}\exp\left(-\tau_ia_i^-(t-s)\right)\Phi_{ij}^1(s)\mathrm{d}s\right|\\&\le\int_{-\infty}^{t}\exp\left(-\tau_ia_i^-(t-s)\right)\left|\Phi_{ij}^1(s+\delta_i^1)-\Phi_{ij}^1(s)\right|\mathrm{d}s\le\frac{\varepsilon}{\tau_ia_i^-}\end{aligned}$$

$$\begin{aligned}\left|\Lambda_1(t+\delta_j^2)-\Lambda_1(t)\right|&=\left|\int_{-\infty}^{t+\delta_j^2}\exp\left(-\int_s^t\tilde{d}_j(v_j(w))\mathrm{d}w\right)\Psi_{ji}^1(s)\mathrm{d}s-\int_{-\infty}^{t}\exp\left(-\int_s^t\tilde{d}_j(v_j(w))\mathrm{d}w\right)\Psi_{ji}^1(s)\mathrm{d}s\right|\\&\le\left|\int_{-\infty}^{t+\delta_j^2}\exp\left(-\sigma_jc_j^-(t+\delta_j^2-s)\right)\Psi_{ji}^1(s)\mathrm{d}s-\int_{-\infty}^{t}\exp\left(-\sigma_jc_j^-(t-s)\right)\Psi_{ji}^1(s)\mathrm{d}s\right|\\&=\left|\int_{-\infty}^{t}\exp\left(-\sigma_jc_j^-(t-s)\right)\Psi_{ji}^1(s+\delta_j^2)\mathrm{d}s-\int_{-\infty}^{t}\exp\left(-\sigma_jc_j^-(t-s)\right)\Psi_{ji}^1(s)\mathrm{d}s\right|\\&\le\int_{-\infty}^{t}\exp\left(\sigma_jc_j^-(t-s)\right)\left|\Psi_{ji}^1(s+\delta_j^2)-\Psi_{ji}^1(s)\right|\mathrm{d}s\le\frac{\varepsilon}{\sigma_jc_j^-}\end{aligned}$$

所以 $\Theta,\Lambda_1\in AP(R,R)$。接下来证明 $\Theta_2,\Lambda_2\in PAP_0(R,R)$，对一切 $1\le i\le n,\ 1\le j\le m$。即要证：

$$\begin{cases}\lim\limits_{T\to+\infty}\dfrac{1}{2T}\displaystyle\int_{-T}^{T}\left|\int_{-\infty}^{t}\exp\left(-\int_s^t\tilde{b}_i(u_i(w))\mathrm{d}w\right)\Phi_{ij}^2(s)\mathrm{d}s\right|\mathrm{d}t=0\\\lim\limits_{T\to+\infty}\dfrac{1}{2T}\displaystyle\int_{-T}^{T}\left|\int_{-\infty}^{t}\exp\left(-\int_s^t\tilde{d}_j(v_j(w))\mathrm{d}w\right)\Psi_{ji}^2(s)\mathrm{d}s\right|\mathrm{d}t=0\end{cases}$$

显然

$$\begin{cases}\lim\limits_{T\to+\infty}\dfrac{1}{2T}\displaystyle\int_{-T}^{T}\left|\int_{-\infty}^{t}\exp\left(-\int_s^t\tilde{b}_i(u_i(w))\mathrm{d}w\right)\Phi_{ij}^2(s)\mathrm{d}s\right|\mathrm{d}t\le I_1+I_2\\\lim\limits_{T\to+\infty}\dfrac{1}{2T}\displaystyle\int_{-T}^{T}\left|\int_{-\infty}^{t}\exp\left(-\int_s^t\tilde{d}_j(v_j(w))\mathrm{d}w\right)\Psi_{ji}^2(s)\mathrm{d}s\right|\mathrm{d}t\le \mathrm{M}_1+\mathrm{M}_2\end{cases}$$

这里

$$I_1=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left|\int_{-T}^{t}\exp\left(-\int_s^t\tilde{b}_i(u_i(w))\mathrm{d}w\right)\Phi_{ij}^2(s)\mathrm{d}s\right|\mathrm{d}t,\quad I_2=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{-T}\left|\int_{-\infty}^{-T}\exp\left(-\int_s^t\tilde{b}_i(u_i(w))\mathrm{d}w\right)\Phi_{ij}^2(s)\mathrm{d}s\right|\mathrm{d}t,$$

$$\mathrm{M}_1=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left|\int_{-T}^{t}\exp\left(-\int_s^t\tilde{d}_j(v_j(w))\mathrm{d}w\right)\Psi_{ji}^2(s)\mathrm{d}s\right|\mathrm{d}t,\quad \mathrm{M}_2=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left|\int_{-\infty}^{-T}\exp\left(-\int_s^t\tilde{d}_j(v_j(w))\mathrm{d}w\right)\Psi_{ji}^2(s)\mathrm{d}s\right|\mathrm{d}t.$$

显然

$$\begin{aligned}I_1&=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left|\int_{-T}^{t}\exp\left(-\int_s^t\tilde{b}_i(u_i(w))\mathrm{d}w\right)\Phi_{ij}^2(s)\mathrm{d}s\right|\mathrm{d}t\le\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\mathrm{d}t\left(\int_{-T}^{t}\left|\exp\left(-\tau_{ii}a_i^-(t-s)\right)\right|\left\|\Phi_{ij}^2(s)\right\|\mathrm{d}s\right)\\&\le\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left\|\Phi_{ij}^2(s)\right\|\mathrm{d}t\left(\int_0^{t+T}\exp\left(-\tau_ia_i^-\upsilon\right)\mathrm{d}\upsilon\right)\le\lim_{T\to+\infty}\frac{1}{2T\tau_ia_i^-}\int_{-T}^{T}\left\|\Phi_{ij}^2(s)\right\|\mathrm{d}t=0\end{aligned}$$

$$\begin{aligned}\mathrm{M}_1&=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left|\int_{-T}^{t}\exp\left(-\int_s^t\tilde{d}_j(v_j(w))\mathrm{d}w\right)\Psi_{ji}^2(s)\mathrm{d}s\right|\mathrm{d}t\le\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\mathrm{d}t\left(\int_{-T}^{t}\left|\exp\left(-\sigma_jc_j^-(t-s)\right)\right|\left\|\Psi_{ji}^2(s)\right\|\mathrm{d}s\right)\\&\le\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left\|\Psi_{ji}^2(s)\right\|\mathrm{d}t\left(\int_0^{t+T}\exp\left(-\sigma_jc_j^-\zeta\right)\mathrm{d}\zeta\right)\le\lim_{T\to+\infty}\frac{1}{2T\sigma_jc_j^-}\int_{-T}^{T}\left\|\Psi_{ji}^2(s)\right\|\mathrm{d}t=0\end{aligned}$$

同样

$$I_2=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{-T}\left|\int_{-\infty}^{-T}\exp\left(-\int_s^t\tilde{b}_i\left(u_i\left(w\right)\right)\mathrm{d}w\right)\Phi_{ij}^2\left(s\right)\mathrm{d}s\right|\mathrm{d}t\le\lim_{T\to+\infty}\frac{\sup\limits_{t\to R}\left|\Phi_{ij}^2\left(s\right)\right|}{2T}\int_{-T}^{T}\mathrm{d}t\left(\int_{-\infty}^{-T}\left|\exp\left(-\tau_i a_i^-\left(t-s\right)\right)\right|\mathrm{d}s\right)$$

$$=\lim_{T\to+\infty}\frac{\sup\limits_{t\to R}\left|\Phi_{ij}^2\left(s\right)\right|}{2T}\int_{-T}^{T}\mathrm{d}t\left(\int_{\infty}^{t+T}\left|\exp\left(-\tau_i a_i^- u\right)\right|d\left(-u\right)\right)=\lim_{T\to+\infty}\frac{\sup\limits_{t\to R}\left|\Phi_{ij}^2\left(s\right)\right|}{2T}\int_{-T}^{T}\mathrm{d}t\left(\int_{t+T}^{\infty}\left|\exp\left(-\tau_i a_i^- s\right)\right|\mathrm{d}\left(s\right)\right)$$

$$=\lim_{T\to+\infty}\frac{\sup\limits_{t\to R}\left|\Phi_{ij}^2\left(s\right)\right|}{2T}\int_{-T}^{T}\frac{\exp\left(\tau_i a_i^-\left(t+T\right)\right)}{\tau_i a_i^-}\mathrm{d}t=\lim_{T\to+\infty}\frac{\sup\limits_{t\to R}\left|\Phi_{ij}^2\left(s\right)\right|}{2T}\left(1-\exp\left(-\tau_i a_i^-\left(2T\right)\right)\right)=0$$

$$\mathrm{M}_2=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left|\int_{-\infty}^{-T}\exp\left(-\int_s^t\tilde{d}_j\left(v_j\left(w\right)\right)\mathrm{d}w\right)\Psi_{ji}^2\left(s\right)\mathrm{d}s\right|\mathrm{d}t\le\lim_{T\to+\infty}\frac{\sup\limits_{t\to R}\left|\Psi_{ji}^2\left(s\right)\right|}{2T}\int_{-T}^{T}\mathrm{d}t\left(\int_{-\infty}^{-T}\left|\exp\left(-\sigma_j c_j^-\left(t-s\right)\right)\right|\mathrm{d}s\right)$$

$$=\lim_{T\to+\infty}\frac{\sup\limits_{t\to R}\left|\Psi_{ji}^2\left(s\right)\right|}{2T}\int_{-T}^{T}\mathrm{d}t\left(\int_{\infty}^{t+T}\left|\exp\left(-\sigma_j c_j^- v\right)\right|\mathrm{d}\left(-v\right)\right)=\lim_{T\to+\infty}\frac{\sup\limits_{t\to R}\left|\Psi_{ji}^2\left(s\right)\right|}{2T}\int_{-T}^{T}\mathrm{d}t\left(\int_{t+T}^{\infty}\left|\exp\left(-\sigma_j c_j^- s\right)\right|\mathrm{d}s\right)$$

$$=\lim_{T\to+\infty}\frac{\sup\limits_{t\to R}\left|\Psi_{ji}^2\left(s\right)\right|}{2T}\int_{-T}^{T}\frac{\exp\left(-\sigma_j c_j^-\left(t+T\right)\right)}{\sigma_j c_j^-}\mathrm{d}t$$

$$=\lim_{T\to+\infty}\frac{\sup\limits_{t\to R}\left|\Psi_{ji}^2\left(s\right)\right|}{2T}\left(1-\exp\left(-\sigma_j c_j^-\left(2T\right)\right)\right)=0$$

所以有 $\Theta_2,\Lambda_2\in PAP_0\left(R,R\right)$。从而对一切的 $i=1,2,\cdots,n,\ j=1,2,\cdots,m$。$\Gamma_j\psi_j,\Gamma_i\phi_i\in PAP\left(R,R\right)$ 故 $\Gamma\varphi\in PAP\left(R,R^{n+m}\right)$。

4. 主要结果

定理 1. 若假设(H1)~(H6)成立，且 $\eta=\max\left\{\max\limits_i\left\{\frac{1}{\tau_i a_i^-}\sum_{j=1}^m p_{ij}c_j^+\delta_j\right\},\max\limits_j\left\{\frac{1}{\sigma_j c_j^-}\sum_{i=1}^n q_{ji}a_i^+\lambda_i\right\}\right\}<1$，则等式(3.4)有唯一的伪概周期解存在凸集 B^* 中：$B^*=\left\{\varphi\middle|\varphi\in PAP\left(R,R^{n+m}\right),\left\|\varphi-\varphi_0\right\|\le\frac{\eta I}{1-\eta}\right\}$。这里 $\left\|\varphi\right\|=\max\limits_i\left\{\left|\varphi_i\right|\right\}$，$\left|\varphi_i\right|=\sup\limits_{t\in R}\left\{\left|\varphi_i\left(t\right)\right|\right\}$

$$\varphi_0\left(t\right)=\left(\int_{-\infty}^t I_1\left(s\right)\exp\left(-\int_s^t\tilde{b}_1\left(u_1\left(w\right)\right)\mathrm{d}w\right)\mathrm{d}s,\cdots,\int_{-\infty}^t I_n\left(s\right)\exp\left(-\int_s^t\tilde{b}_n\left(u_n\left(w\right)\right)\mathrm{d}w\right)\mathrm{d}s,\right.$$
$$\left.\int_{-\infty}^t J_1\left(s\right)\exp\left(-\int_s^t\tilde{d}_1\left(v_1\left(w\right)\right)\mathrm{d}w\right)\mathrm{d}s,\cdots,\int_{-\infty}^t J_m\left(s\right)\exp\left(-\int_s^t\tilde{d}_m\left(v_m\left(w\right)\right)\mathrm{d}w\right)\mathrm{d}s\right).$$

$$\left\|\varphi_0\right\|=\max\left\{\sup_{t\in R}\max_i\left\{\left|\int_{-\infty}^t I_i\left(s\right)\exp\left(-\int_s^t\tilde{b}_i\left(u_i\left(w\right)\right)\mathrm{d}w\right)\mathrm{d}s\right|\right\},\sup_{t\in R}\max_j\left\{\left|\int_{-\infty}^t J_j\left(s\right)\exp\left(-\int_s^t\tilde{d}_j\left(v_j\left(w\right)\right)\mathrm{d}w\right)\mathrm{d}s\right|\right\}\right\}$$
$$\le\max\left\{\sup_{t\in R}\max_i\left\{\left|\int_{-\infty}^t I_i\left(s\right)\exp\left(\tau_i a_i^-\left(s-t\right)\right)\mathrm{d}s\right|\right\},\sup_{t\in R}\max_j\left\{\left|\int_{-\infty}^t J_j\left(s\right)\exp\left(\sigma_j c_j^-\left(s-t\right)\right)\mathrm{d}s\right|\right\}\right\}$$
$$=\max\left\{\max_i\left\{\frac{I_i}{\tau_{ii}a_i^-}\right\},\max_j\left\{\frac{J_j}{\sigma_j c_j^-}\right\}\right\}=I.$$

对 $\forall\varphi\in B^*$ 有 $\left\|\varphi\right\|\le\left\|\varphi-\varphi_0\right\|+\left\|\varphi_0\right\|\le\frac{\eta I}{1-\eta}+I=\frac{I}{1-\eta}$。

现在我们证明 Γ 是一个从 $B^*\to B^*$ 的恒等映射，对 $\forall\varphi\in B^*$，有

$$\begin{aligned}\left\|\Gamma\varphi-\varphi_0\right\| &= \max\left\{\sup_{t\in R}\max_i\left\{\left|\int_{-\infty}^t \exp\left(-\int_s^t \tilde{b}_i\left(u_i(w)\right)\mathrm{d}w\right)\sum_{j=1}^m p_{ij}(s)\int_{-\infty}^s K_{ij}(s-w)\,g_j\left(G_j^{-1}\left(\varphi_{n+j}(w)\right)\right)\mathrm{d}w\mathrm{d}s\right|\right\},\right.\\
&\qquad \left.\sup_{t\in R}\max_j\left\{\left|\int_{-\infty}^t \exp\left(-\int_s^t \tilde{d}_j\left(v_j(w)\right)\mathrm{d}w\right)\sum_{i=1}^n q_{ji}(s)\int_{-\infty}^s L_{ji}(s-w)\,f_i\left(F_i^{-1}\left(\varphi_i(w)\right)\right)\mathrm{d}w\mathrm{d}s\right|\right\}\right\}\\
&\le \max\left\{\sup_{t\in R}\max_i\left\{\int_{-\infty}^t \exp\left(-\int_s^t \tilde{b}_i\left(u_i(w)\right)\mathrm{d}w\right)\sum_{j=1}^m\left|p_{ij}(s)\right|\left|\int_{-\infty}^s K_{ij}(s-w)\right|\left|g_j\left(G_j^{-1}\left(\varphi_{n+j}(w)\right)\right)\right|\mathrm{d}w\mathrm{d}s\right\},\right.\\
&\qquad \left.\sup_{t\in R}\max_j\left\{\int_{-\infty}^t \exp\left(-\int_s^t \tilde{d}_j\left(v_j(w)\right)\mathrm{d}w\right)\sum_{i=1}^n\left|q_{ji}(s)\right|\left|\int_{-\infty}^s L_{ji}(s-w)\right|\left|f_i\left(F_i^{-1}\left(\varphi_i(w)\right)\right)\right|\mathrm{d}w\mathrm{d}s\right\}\right\}\\
&\le \max\left\{\sup_{t\in R}\max_i\left\{\int_{-\infty}^t \exp\left(-\tau_i a_i^-(t-s)\right)\mathrm{d}s\sum_{j=1}^m p_{ij}c_j^+\delta_j\left|\varphi_{n+j}\right|\right\},\right.\\
&\qquad \left.\sup_{t\in R}\max_j\left\{\int_{-\infty}^t \exp\left(-\sigma_j c_j^-(t-s)\right)\mathrm{d}s\sum_{i=1}^n q_{ji}a_i^+\lambda_i\left|\varphi_i\right|\right\}\right\}\\
&\le \max\left\{\sup_{t\in R}\max_i\left\{\int_{-\infty}^t \exp\left(-\tau_i a_i^-(t-s)\right)\mathrm{d}s\sum_{j=1}^m p_{ij}c_j^+\delta_j\right\}\|\varphi\|,\right.\\
&\qquad \left.\sup_{t\in R}\max_j\left\{\int_{-\infty}^t \exp\left(-\sigma_j c_j^-(t-s)\right)\mathrm{d}s\sum_{i=1}^n q_{ji}a_i^+\lambda_i\right\}\|\varphi\|\right\}\\
&\le \max\left\{\max_i\left\{\frac{1}{\tau_i a_i^-}\sum_{j=1}^m p_{ij}c_j^+\delta_j\right\},\max_j\left\{\frac{1}{\sigma_j c_j^-}\sum_{i=1}^n q_{ji}a_i^+\lambda_i\right\}\right\}\|\varphi\| = \eta\|\varphi\| \le \frac{\eta I}{1-\eta}\end{aligned}$$

这表明$\Gamma(\varphi)(t)\in B^*$，故$\Gamma$是$B^*\to B^*$的恒等映射。下证$\Gamma$是$B^*$上的一个压缩映射，对$\forall\varphi,\psi\in B^*$，有

$$\begin{aligned}\left\|\Gamma(\varphi(t))-\Gamma(\psi(t))\right\| &= \max\left\{\sup_{t\in R}\max_i\left\{\left|\int_{-\infty}^t \exp\left(-\int_s^t \tilde{b}_i\left(u_i(w)\right)\mathrm{d}w\right)\sum_{j=1}^m p_{ij}(s)\right.\right.\right.\\
&\qquad \left.\left.\times\int_{-\infty}^s K_{ij}(s-w)\left[g_j\left(G_j^{-1}\left(\varphi_{n+j}(w)\right)\right)-g_j\left(G_j^{-1}\left(\psi_{n+j}(w)\right)\right)\right]\mathrm{d}w\mathrm{d}s\right|\right\},\\
&\quad \sup_{t\in R}\max_j\left\{\left|\int_{-\infty}^t \exp\left(-\int_s^t \tilde{d}_j\left(v_j(w)\right)\mathrm{d}w\right)\sum_{i=1}^n q_{ji}(s)\right.\right.\\
&\qquad \left.\left.\left.\times\int_{-\infty}^s L_{ji}(s-w)\left[f_i\left(F_i^{-1}\left(\varphi_i(w)\right)\right)-f_i\left(F_i^{-1}\left(\psi_i(w)\right)\right)\right]\mathrm{d}w\mathrm{d}s\right|\right\}\right\}\\
&\le \max\left\{\sup_{t\in R}\max_i\left\{\int_{-\infty}^t \exp\left(-\int_s^t \tilde{b}_i\left(u_i(w)\right)\mathrm{d}w\right)\sum_{j=1}^m\left|p_{ij}(s)\right|\right.\right.\\
&\qquad \left.\times\int_{-\infty}^s K_{ij}(s-w)\left|g_j\left(G_j^{-1}\left(\varphi_{n+j}(w)\right)\right)-g_j\left(G_j^{-1}\left(\psi_{n+j}(w)\right)\right)\right|\mathrm{d}w\mathrm{d}s\right\},\\
&\quad \sup_{t\in R}\max_j\left\{\int_{-\infty}^t \exp\left(-\int_s^t \tilde{d}_j\left(v_j(w)\right)\mathrm{d}w\right)\sum_{i=1}^n\left|q_{ji}(s)\right|\right.\\
&\qquad \left.\left.\times\int_{-\infty}^s L_{ji}(s-w)\left|f_i\left(F_i^{-1}\left(\varphi_i(w)\right)\right)-f_i\left(F_i^{-1}\left(\psi_i(w)\right)\right)\right|\mathrm{d}w\mathrm{d}s\right\}\right\}\\
&\le \left\{\sup_{t\in R}\max_i\left\{\int_{-\infty}^t \exp\left(-\int_s^t \tilde{b}_i\left(u_i(w)\right)\mathrm{d}w\right)\sum_{j=1}^m p_{ij}c_j^+\delta_j\left|\varphi_{n+j}(s)-\psi_{n+j}(s)\right|\mathrm{d}s\right\},\right.\\
&\quad \left.\sup_{t\in R}\max_j\left\{\int_{-\infty}^t \exp\left(-\int_s^t \tilde{d}_j\left(v_j(w)\right)\mathrm{d}w\right)\sum_{j=1}^m q_{ji}a_i^+\lambda_i\left|\varphi_i(s)-\psi_i(s)\right|\mathrm{d}s\right\}\right\}\\
&\le \max\left\{\max_i\left\{\frac{1}{\tau_i a_i^-}\sum_{j=1}^m p_{ij}c_j^+\delta_j\right\},\max_j\left\{\frac{1}{\sigma_{jj}c_j^-}\sum_{i=1}^n q_{ji}a_i^+\lambda_i\right\}\right\}\|\varphi-\psi\| = \eta\|\varphi-\psi\|\end{aligned}$$

由(5.1)知，Γ 是一个压缩映射，这样 Γ 存在一个稳定点 $\varphi^* \in B^*$，$\Gamma\varphi^* = \varphi^*$，$\eta < 1$，$\varphi^*$ 满足(3.6)式。所以 φ^* 是(3.4)式唯一的伪概周期解，从而定理 1 得到证明。

5. 举例

考虑以下模型的伪概周期解：

$$\begin{cases} x_i'(t) = -a_i\left(x_i(t)\right)\left[b_i\left(x_i(t)\right) - \sum_{j=1}^{2} p_{ij}(t)\int_{-\infty}^{t} K_{ij}(t-s)\, g_j\left(y_j(s)\right)\mathrm{d}s - I_i(t)\right], \\ y_j'(t) = -c_j\left(y_j(t)\right)\left[d_j\left(y_j(t)\right) - \sum_{i=1}^{2} q_{ji}(t)\int_{-\infty}^{t} L_{ji}(t-s)\, f_i\left(x_i(s)\right)\mathrm{d}s - J_j(t)\right]. \end{cases}$$

对 $t > 0$，这里

$$\begin{pmatrix} I_1(t) \\ I_2(t) \end{pmatrix} = \begin{pmatrix} \dfrac{\cos t + \exp\left(-t^2\cos^2 t\right)}{20} \\ \dfrac{\sin t + \exp\left(-t^2\cos^2 t\right)}{20} \end{pmatrix}, \quad \begin{pmatrix} J_1(t) \\ J_2(t) \end{pmatrix} = \begin{pmatrix} \dfrac{\sin\sqrt{2}t + \exp\left(-t^2\cos^2 t\right)}{20} \\ \dfrac{\sin t + \sin\sqrt{2}t + \exp\left(-t^2\cos^2 t\right)}{20} \end{pmatrix},$$

$$g_j(x) = f_i(x) = \frac{1}{2}\left(|x+1| - |x-1|\right),\ K_{ij}(s) = \exp(-s),\ L_{ji}(s) = \exp(-2s),$$

$$a_i\left(x_i(t)\right) = \begin{pmatrix} 6 - \cos\left(2x_1(t)\right) & 0 \\ 0 & 4 + \sin\left(x_2(t)\right) \end{pmatrix},$$

$$c_j\left(y_j(t)\right) = \begin{pmatrix} 4 - \sin\left(4y_1(t)\right) & 0 \\ 0 & 6 + \cos\left(5y_2(t)\right) \end{pmatrix}, \quad b_i\left(x_i(t)\right) = \begin{pmatrix} 5x_1(t) & 0 \\ 0 & 6x_2(t) \end{pmatrix},$$

$$d_j\left(y_j(t)\right) = \begin{pmatrix} 5y_1(t) & 0 \\ 0 & 3y_2(t) \end{pmatrix}, \quad p_{ij}(t) = \begin{pmatrix} \cos t & -\sin t \\ -\sin t & \cos t \end{pmatrix}, q_{ji}(t) = \begin{pmatrix} \sin t & -\cos t \\ -\cos t & \sin t \end{pmatrix}.$$

通过计算得：$a_1^- = 5, a_1^+ = 7, a_2^- = 3, a_2^+ = 5, c_1^- = 3, c_1^+ = 5, c_2^- = 5, c_2^+ = 7,$

$$\tau_1 = 5, \tau_2 = 6, \sigma_1 = 5, \sigma_2 = 3, I_1 = \frac{1}{10}, I_2 = \frac{1}{10}, J_1 = \frac{1}{10}, J_2 = \frac{3}{20}.$$

$$p_{ij} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, q_{ji} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \lambda_1 = \lambda_2 = 1, \delta_1 = \delta_2 = 1,\ \int_0^{+\infty} K_{ij}(s)\mathrm{d}s = 1, \int_0^{+\infty} L_{ji}(s)\mathrm{d}s = 1.$$

$$I = \max\left\{\max_i\left\{\frac{I_i}{\tau_i a_i^-}\right\}, \max_j\left\{\frac{J_j}{\sigma_j c_j^-}\right\}\right\} = \frac{1}{100},$$

$$\eta = \max\left\{\max_i\left\{\frac{1}{\tau_i a_i^-}\sum_{j=1}^{m} p_{ij} c_j^+ \delta_j\right\}, \max_j\left\{\frac{1}{\sigma_j c_j^-}\sum_{i=1}^{n} q_{ji} a_i^+ \lambda_i\right\}\right\} = 0.87 < 1.$$

所以由定理 1，(3.6)有唯一的伪概周期解。

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