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具有浓度迁移率和对数势能的修正Cahn-Hillard方程的有限元算法

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摘 要

为了研究具有浓度迁移率和对数势能的修正Cahn-Hilliard方程，在空间上采用混合有限元方法进行离散，时间上采用Crank-Nicolson格式进行离散，对于非线性项采用了凸分裂的方法。证明了数值方法的稳定性，并且给出了误差估计。最后，通过数值算例对理论分析进行了验证。结果表明，理论分析与数值实验相一致。

关键词

修正Cahn-Hilliard方程，浓度迁移率，对数势能

Finite Element Method for the Modified Cahn-Hilliard Equation with the Concentration Mobility and the Logarithmic Potential

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Abstract

In order to study the modified Cahn-Hilliard equation with concentration mobility and logarithmic

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potential energy, the mixed finite element method was used in space, and the Crank-Nicolson scheme was used in time. The convex splitting method is used for nonlinear terms. Furthermore, the stability of the numerical method is proved and the error estimate is given. Finally, a numerical example is given to verify the theoretical analysis. The results show that the theoretical analysis is consistent with the numerical experiment.

Keywords

The Modified Cahn-Hilliard Equation, The Concentration Mobility, The Logarithmic Potential

1. 引言

Cahn-Hilliard 方程最早是由 Cahn 和 Hilliard 在 20 世纪 50 年代提出的[1]，常用来描述二元合金在某种不稳定状态时相的分离和粗化现象[2] [3] [4]。为了抑制粗化现象，Aristotelous 提出修正 Cahn-Hilliard 方程[5]。本文研究的具有浓度迁移率和对数势能的修正 Cahn-Hilliard 方程具有如下形式：

$$\begin{cases} u_t = \nabla \cdot \left(m(u) \nabla w \right), & (x,t) \in \Omega \times (0,T), \\ w = -\varepsilon^2 \Delta u + \phi(u) + \varphi, & (x,t) \in \Omega \times (0,T), \\ -\Delta \varphi = \beta \left(u - \overline{u}_0 \right), & (x,t) \in \Omega \times (0,T), \\ \partial_n u = \partial_n w = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1}$$

其中：$\Omega \in R^d, d = 2$；ε 是给定的参数；$u_t = \dfrac{\partial u}{\partial t}$；$\beta > 0$；$n$ 是单位外法向量。u 是混合物中某种物质的浓度，w 是化学势能，φ 是辅助变量，$\overline{u}_0 = \dfrac{1}{|\Omega|}\int_\Omega u_0(x)\mathrm{d}x$。$m(u)$ 是浓度迁移率，定义如下[6]：对于 $0 < \sigma \le 1$，

$$m(u) = \begin{cases} \dfrac{1}{4}\left[1 - (1-\sigma)(2u-1)^2\right], & 0 \le u \le 1, \\ \dfrac{1}{8}\sigma\left[1 + \mathrm{e}^{\frac{-2(1-\sigma)\left(4u^2 - 4u\right)}{\sigma}}\right], & \text{其他}. \end{cases}$$

容易得到，对于 $\forall u \in R$，存在 $m_1 > m_2 > 0$，使得 $m_1 \ge m(u) \ge m_2$ 成立；对于 $\forall u \in R$，存在 $M > 0$，使得 $|m'(u)| \le M$ 成立。$\Phi(u)$ 是对数势能，定义如下[7]：对于 $\forall \theta \in (0,1)$，

$$\Phi(u) = \frac{\theta}{2}\left((1+u)\ln(1+u) + (1-u)\ln(1-u)\right) + \frac{1}{2}\left(1-u^2\right), \quad u \in (-1,1).$$

$$\phi(u) = \Phi'(u) = \frac{\theta}{2}\left(\ln(1+u) - \ln(1-u)\right) - u.$$

根据正则化思想，考虑下面的对数势能函数[8]：对于 $\forall k \in (0,1)$，

$$\breve{\Phi}(u):=\begin{cases}\frac{\theta}{2}(1+u)\ln(1+u)+\frac{\theta}{4k}(1-u)^2-\frac{\theta k}{4}+\frac{\theta}{2}(1-u)\ln k+\frac{1}{2}(1-u^2), & u\ge 1-k,\\ \frac{\theta}{2}\big((1+u)\ln(1+u)-(1-u)\ln(1-u)\big)+\frac{1}{2}(1-u^2), & |u|<1-k,\\ \frac{\theta}{2}(1-u)\ln(1-u)+\frac{\theta}{4k}(1+u)^2-\frac{\theta k}{4}+\frac{\theta}{2}(1+u)\ln k+\frac{1}{2}(1-u^2), & u\le -1+k.\end{cases}$$

$$\breve{\phi}:=\breve{\Phi}(u)=\begin{cases}\frac{\theta}{2}\ln(1+u)+\frac{\theta}{2}-\frac{\theta}{2k}(1-u)-\frac{\theta}{2}\ln k-u, & u\ge 1-k,\\ \frac{\theta}{2}\big(\ln(1+u)-\ln(1-u)\big)-u, & |u|<1-k,\\ -\frac{\theta}{2}\ln(1-u)-\frac{\theta}{2}+\frac{\theta}{2k}(1+u)+\frac{\theta}{2}\ln k-u, & u\le -1+k.\end{cases}$$

在接下来的研究中，将用正则化的对数势能 $\breve{\Phi}$ 及其导数 $\breve{\phi}$ 来代替 Φ 和 ϕ，为了简单，仍记为 Φ 和 ϕ。自由能函数定义为：

$$E=\int_\Omega\left(\frac{\varepsilon^2}{2}|\nabla u|^2+\Phi(u)\right)\mathrm{d}x+\frac{\beta}{2}\|u-\bar{u}_0\|_{H^{-1}}^2 .$$

2. 数值格式

$L^2(\Omega)$ 是平方可积函数空间，内积为 $(u,v)=\int_\Omega u(x)v(x)\mathrm{d}x$，相应范数为 $\|u\|=\|u\|_{L^2(\Omega)}=\sqrt{(u,u)}$。$H^1(\Omega)$ 是通常的 Sobolev 空间，相应的半范和范数分别为

$$|u|_{H^1}=\left(\int_\Omega|Du|^2\,\mathrm{d}x\right)^{\frac{1}{2}},\ \|u\|_{H^1}=\left(\int_\Omega|u|^2\,\mathrm{d}x+\int_\Omega|Du|^2\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

具有浓度迁移率和对数势能的修正 Cahn-Hilliard 方程的弱解形式为：

$$\begin{cases}(u_t,q)+\big(\nabla\cdot(m(u)\nabla w),\nabla q\big)=0, & \forall q\in H^1(\Omega),\\ (w,v)-\varepsilon^2(\nabla u,\nabla v)-\big(\phi_1(u)-\phi_2(u),v\big)-(\varphi,v)=0, & \forall v\in H^1(\Omega),\\ (\nabla\varphi,\nabla\psi)-\beta(u-\bar{u}_0,\psi)=0, & \forall\psi\in H^1(\Omega).\end{cases}\tag{2}$$

其中

$$\phi_1(u)=\begin{cases}\frac{\theta}{2}\ln(1+u)+\frac{\theta}{2}-\frac{\theta}{2k}(1-u)-\frac{\theta}{2}\ln k, & u\ge 1-k,\\ \frac{\theta}{2}\big(\ln(1+u)-\ln(1-u)\big), & |u|<1-k,\\ -\frac{\theta}{2}\ln(1-u)-\frac{\theta}{2}+\frac{\theta}{2k}(1+u)+\frac{\theta}{2}\ln k, & u\le -1+k.\end{cases}$$

$\phi_2(u)=u$。容易得到 $\forall u\in(-\infty,\infty)$，$0<\phi_1'\le L:=\frac{\theta}{(2-k)k}$，$\phi_2'=1$。

2.1. 半离散格式

把时间区间 $[0,T]$ 进行剖分，$0=t_0<t_1<\cdots<t_N=T$，N 是一个正整数，时间节点满足 $t_i=i\tau$，$i=0,1,\cdots,N$，$\tau=\frac{T}{N}$ 是时间步长。接下来构造具有浓度迁移率和对数势能的修正 Cahn-Hilliard 方程的半

离散格式：给定u^n，求u^{n+1}，w^{n+1}，φ^{n+1}，当$n\ge 1$时，满足

$$\begin{cases}\left(\delta_\tau u^{n+1},q\right)+\left(\nabla\cdot\left(m\left(u^n\right)\nabla w^{n+1}\right),\nabla q\right)=0, & \forall q\in H^1(\Omega),\\ \left(w^{n+1},v\right)-\varepsilon^2\left(\nabla u^{n+1},\nabla v\right)-\left(\varphi^{n+1},v\right)-\left(\phi_1\left(u^{n+1}\right)-\phi_2\left(u^n\right),v\right)=0, & \forall v\in H^1(\Omega),\\ \left(\nabla\varphi^{n+1},\nabla\psi\right)-\beta\left(u^{n+1}-\bar{u}_0,\psi\right)=0, & \forall\psi\in H^1(\Omega).\end{cases} \tag{3}$$

其中$\delta_\tau u^{n+1}=\dfrac{u^{n+1}-u^n}{\tau}$。

2.2. 全离散格式

设$T_h=K$是区域Ω上拟一致剖分，h_i表示网格大小，$h=\max\limits_{0\le i\le N}h_i$，$S_h$是分片连续的有限元空间，定义$S_h=\left\{v_h\in C(\Omega)\middle|v_h|_K\in P_k(x,y),K\in T_h\right\}$，$S_h\subset H^1(\Omega)$。这里$P_k(x,y)$是$x,y$的次数不超过$k\left(\in Z^+\right)$的线性多项式的集合。定义$L_0^2:=\left\{u\in L^2(\Omega)\middle|(u,1)=0\right\}$，$\dot{S}_h:=S_h\cap L_0^2(\Omega)$。接下来构造具有浓度迁移率和对数势能的修正Cahn-Hilliard方程的全离散格式：给定u_h^n，求u_h^{n+1}，w_h^{n+1}，φ_h^{n+1}，当$n\ge 1$时，满足

$$\begin{cases}\left(\delta_\tau u_h^{n+1},q_h\right)+\left(\nabla\cdot\left(m\left(u_h^n\right)\nabla w_h^{n+1}\right),\nabla q_h\right)=0, & \forall q_h\in S_h,\\ \left(w_h^{n+1},v_h\right)-\varepsilon^2\left(\nabla u_h^{n+1},\nabla v_h\right)-\left(\varphi_h^{n+1},v_h\right)-\left(\phi_1\left(u_h^{n+1}\right)-\phi_2\left(u_h^n\right),v_h\right)=0, & \forall v_h\in S_h,\\ \left(\nabla\varphi_h^{n+1},\nabla\psi_h\right)-\beta\left(u_h^{n+1}-\bar{u}_0,\psi_h\right)=0, & \forall\psi_h\in S_h.\end{cases} \tag{4}$$

Ritz 投影算子$R_h:H^1(\Omega)\to S_h$满足$\left(\nabla\left(u-R_hu\right),\nabla v\right)=0$，$\left(R_hu-u,1\right)=0$，其中$u_h^0=R_hu_0$。

3. 稳定性分析

定义 3.1：H^{-1}范数定义如下：

$$\left\|v_h\right\|_{-1,h}:=\sqrt{\left(v_h,\left(-\Delta_h\right)^{-1}v_h\right)}=\sup_{0\ne\chi\in\dot{S}_h}\frac{(v,\chi)}{\|\nabla\chi\|},\quad\forall v_h\in\dot{S}_h.$$

引理 3.1 [9]：设$\zeta,\chi\in\dot{S}_h(\Omega)$，有

$$(\zeta,\chi)_{-1,h}:=\left(\nabla T_h(\zeta),\nabla T_h(\chi)\right)=\left(\zeta,T_h(\chi)\right)=\left(T_h(\zeta),\chi\right).$$

其中$T_h:\dot{S}_h(\Omega)\to\dot{S}_h(\Omega)$是可逆线性算子，$\left(\nabla T_h(\chi),\nabla v\right)=(\chi,v)$，且满足

$$|(\chi,g)|\le\|\chi\|_{-1,h}\|\nabla g\|,$$

其中$g\in S_h(\Omega)$。进一步，下面估计式成立

$$\|\zeta\|_{-1,h}\le C\|\zeta\|,\forall\zeta\in\dot{S}_h.$$

定理 3.1：令$\left(u_h^{n+1},w_h^{n+1}\right)$是(4)的解，对任意的$\tau,h,\varepsilon>0$，下面的不等式成立

$$E\left(u_h^{n+1}\right)+\tau\left\|\sqrt{m\left(u_h^n\right)}\nabla w_h^{n+1}\right\|^2+\frac{\varepsilon^2}{2}\left\|\nabla u_h^{n+1}-\nabla u_h^n\right\|^2+\frac{\beta}{2}\left\|u_h^{n+1}-u_h^n\right\|_{-1,h}^2\le E\left(u_h^n\right). \tag{5}$$

证明：在(4)的第 1 式中，令$q_h=\tau w_h^{n+1}$，得

$$\left(u_h^{n+1}-u_h^n,w_h^{n+1}\right)+\tau\left\|\sqrt{m\left(u_h^n\right)}\nabla w_h^{n+1}\right\|^2=0. \tag{6}$$

在(4)的第 2 式中，令 $v_h=-\left(u_h^{n+1}-u_h^n\right)$，并运用 $2(a-b,a)=a^2-b^2+(a-b)^2$ 得

$$
\begin{aligned}
&-\left(w_h^{n+1},u_h^{n+1}-u_h^n\right)+\frac{\varepsilon^2}{2}\left(\left\|\nabla u_h^{n+1}\right\|^2-\left\|\nabla u_h^n\right\|^2+\left\|\nabla u_h^{n+1}-\nabla u_h^n\right\|^2\right)\\
&+\left(\phi_1\left(u_h^{n+1}\right)-\phi_2\left(u_h^n\right),u_h^{n+1}-u_h^n\right)+\left(\varphi_h^{n+1},u_h^{n+1}-u_h^n\right)=0.
\end{aligned}
\tag{7}
$$

在(4)的第(3)式中，令 $\psi_h=-T_h\left(u_h^{n+1}-u_h^n\right)$，得

$$
-\left(\nabla\varphi_h^{n+1},\nabla T_h\left(u_h^{n+1}-u_h^n\right)\right)+\beta\left(u_h^{n+1}-\bar{u}_0,T_h\left(u_h^{n+1}-u_h^n\right)\right)=0. \tag{8}
$$

将(6)，(7)，(8)相加并利用引理 3.1 得

$$
\begin{aligned}
&\tau\left\|\sqrt{m\left(u_h^n\right)}\nabla w_h^{n+1}\right\|^2+\frac{\varepsilon^2}{2}\left(\left\|\nabla u_h^{n+1}\right\|^2-\left\|\nabla u_h^n\right\|^2+\left\|\nabla u_h^{n+1}-\nabla u_h^n\right\|^2\right)\\
&+\left(\phi_1\left(u_h^{n+1}\right)-\phi_2\left(u_h^n\right),u_h^{n+1}-u_h^n\right)+\beta\left(u_h^{n+1}-\bar{u}_0,u_h^{n+1}-u_h^n\right)_{-1,h}=0.
\end{aligned}
\tag{9}
$$

在 $\Phi(u)$ 的定义中，用 $\Phi_1(u)$ 和 $\Phi_2(u)$ 来表示 $\phi_1(u)$ 和 $\phi_2(u)$ 的原函数所对应的部分，即 $\Phi(u)=\Phi_1(u)-\Phi_2(u)$，$\Phi_1'(u)=\phi_1(u)$，$\Phi_2'(u)=\phi_2(u)$。

然后利用泰勒展开公式，得到：

$$
\Phi_1\left(u_h^n\right)-\Phi_1\left(u_h^{n+1}\right)=\phi_1\left(u_h^{n+1}\right)\left(u_h^n-u_h^{n+1}\right)+\frac{1}{2}\phi_1'(\xi_1)\left(u_h^n-u_h^{n+1}\right)^2\geq\phi_1\left(u_h^{n+1}\right)\left(u_h^n-u_h^{n+1}\right). \tag{10}
$$

$$
\Phi_2\left(u_h^{n+1}\right)-\Phi_2\left(u_h^n\right)=\phi_2\left(u_h^n\right)\left(u_h^{n+1}-u_h^n\right)+\frac{1}{2}\phi_2'(\xi_2)\left(u_h^{n+1}-u_h^n\right)^2\geq\phi_2\left(u_h^n\right)\left(u_h^{n+1}-u_h^n\right). \tag{11}
$$

结合(10)，(11) 得到：

$$
\left(\phi_1\left(u_h^{n+1}\right)-\phi_2\left(u_h^n\right)\right)\left(u_h^{n+1}-u_h^n\right)\geq\Phi\left(u_h^{n+1}\right)-\Phi\left(u_h^n\right). \tag{12}
$$

将(12)代入(9)式，并对(9)式最后一项运用 $2(a-b,a)=a^2-b^2+(a-b)^2$，得

$$
\begin{aligned}
&\tau\left\|\sqrt{m\left(u_h^n\right)}\nabla w_h^{n+1}\right\|^2+\frac{\varepsilon^2}{2}\left(\left\|\nabla u_h^{n+1}\right\|^2-\left\|\nabla u_h^n\right\|^2+\left\|\nabla u_h^{n+1}-\nabla u_h^n\right\|^2\right)\\
&+\left(\Phi\left(u_h^{n+1}\right)-\Phi\left(u_h^n\right),1\right)+\frac{\beta}{2}\left(\left\|u_h^{n+1}-\bar{u}_0\right\|_{-1,h}^2-\left\|u_h^n-\bar{u}_0\right\|_{-1,h}^2+\left\|u_h^{n+1}-u_h^n\right\|_{-1,h}^2\right)\leq 0.
\end{aligned}
\tag{13}
$$

根据自由能函数的定义，则上式变为

$$
E\left(u_h^{n+1}\right)+\tau\left\|\sqrt{m\left(u_h^n\right)}\nabla w_h^{n+1}\right\|^2+\frac{\varepsilon^2}{2}\left\|\nabla u_h^{n+1}-\nabla u_h^n\right\|^2+\frac{\beta}{2}\left\|u_h^{n+1}-u_h^n\right\|_{-1,h}^2\leq E\left(u_h^n\right).
$$

则稳定性得证。

4. 误差估计

为了之后证明的简单，介绍下面一些符号：

$$
\hat{e}_u^{n+1}:=R_hu^{n+1}-u_h^{n+1},\quad \tilde{e}_u^{n+1}:=u^{n+1}-R_hu^{n+1},
$$

$$
\hat{e}_w^{n+1}:=R_hw^{n+1}-w_h^{n+1},\quad \tilde{e}_w^{n+1}:=w^{n+1}-R_hw^{n+1},
$$

$$
\hat{e}_\varphi^{n+1}:=R_h\varphi^{n+1}-\varphi_h^{n+1},\quad \tilde{e}_\varphi^{n+1}:=\varphi^{n+1}-R_h\varphi^{n+1},
$$

$$
\sigma\left(u^{n+1}\right)=\delta_\tau R_hu^{n+1}-u_t^{n+1}.
$$

对于 (u,w,φ)，做如下正则性假设：

$$u\in L^{\infty}\left(0,T;W^{1,5}(\Omega)\right)\cap L^{\infty}\left(0,T;H^{r+1}(\Omega)\right),$$
$$w\in L^{\infty}\left(0,T;H^{r+1}(\Omega)\right),$$
$$\varphi\in L^{\infty}\left(0,T;H^{r+1}(\Omega)\right).$$

引理 4.2 [10]：Ritz 投影算子 R_h 满足下面的估计：

$$\|R_hu\|_1\le C\|u\|_1,\quad \forall u\in H^1(\Omega),$$
$$\|u-R_hu\|+h\|u-R_hu\|_1\le Ch^{r+1}\|u\|_{r+1},\forall u\in H^{r+1}(\Omega).$$

引理 4.3 [11]：假设 (u,w) 是(2)的解，则有下面估计：

$$\left\|\sigma\left(u^{n+1}\right)\right\|^2\le Ch^{2r+2}+C\tau^2.$$

定理 4.2：设初始问题(2)，全离散格式(4)的解分别是 (u,w) 和 $\left(u_h^{n+1},w_h^{n+1}\right)$，存在常数 C,τ,h，则有估计式

$$\sum_{i=1}^{n}C\tau\left\|\nabla\hat{e}_w^{i+1}\right\|^2+\varepsilon^2\left\|\nabla\hat{e}_u^{n+1}\right\|^2+\beta\left\|\nabla\hat{e}_u^{n+1}\right\|_{-1,h}^2\le C\tau^2+Ch^{2r}.\tag{14}$$

证明：对于(3)式，令 $q=q_h$，$v=v_h$，$\psi=\psi_h$，利用 R_h 的性质，在 $t=t_{n+1}$，有

$$\begin{cases}\left(\delta_\tau R_hu^{n+1},q_h\right)+\left(m\left(u^{n+1}\right)\nabla R_hw^{n+1},\nabla q_h\right)=\left(\sigma\left(u^{n+1}\right),q_h\right),\\ \varepsilon^2\left(\nabla R_hu^{n+1},\nabla v_h\right)-\left(R_hw^{n+1},v_h\right)+\left(R_h\varphi^{n+1},v_h\right)\\ =\left(\tilde{e}_w^{n+1},v_h\right)-\left(\tilde{e}_\varphi^{n+1},v_h\right)-\left(\phi_1\left(u^{n+1}\right)-\phi_2\left(u^{n+1}\right),v_h\right),\\ \left(\nabla R_h\varphi^{n+1},\nabla\psi_h\right)-\beta\left(R_hu^{n+1}-\bar{u}_0,\psi_h\right)=\beta\left(\tilde{e}_u^{n+1},\psi_h\right).\end{cases}\tag{15}$$

(15)式减去(4)式得

$$\begin{cases}\left(\delta_\tau\hat{e}_u^{n+1},q_h\right)+\left(m\left(u^{n+1}\right)\nabla R_hw^{n+1},\nabla q_h\right)-\left(m\left(u_h^{n}\right)\nabla w_h^{n+1},\nabla q_h\right)=\left(\sigma\left(u^{n+1}\right),q_h\right),\\ \varepsilon^2\left(\nabla\hat{e}_u^{n+1},\nabla v_h\right)-\left(\hat{e}_w^{n+1},v_h\right)+\left(\hat{e}_\varphi^{n+1},v_h\right)\\ =\left(\tilde{e}_w^{n+1},v_h\right)-\left(\tilde{e}_\varphi^{n+1},v_h\right)-\left(\phi_1\left(u^{n+1}\right)-\phi_2\left(u^{n+1}\right),v_h\right)+\left(\phi_1\left(u_h^{n+1}\right)-\phi_2\left(u_h^{n}\right),v_h\right),\\ \left(\nabla\hat{e}_\varphi^{n+1},\nabla\psi_h\right)-\beta\left(\hat{e}_u^{n+1},\psi_h\right)=\beta\left(\tilde{e}_u^{n+1},\psi_h\right).\end{cases}\tag{16}$$

在(16)式中，令 $q_h=\hat{e}_w^{n+1}$，$v_h=\delta_\tau\hat{e}_u^{n+1}$，$\psi_h=-T_h\left(\delta_\tau\hat{e}_u^{n+1}\right)$，然后三式相加得

$$\begin{aligned}&\left(m\left(u^{n+1}\right)\nabla R_hw^{n+1},\nabla\hat{e}_w^{n+1}\right)-\left(m\left(u_h^{n}\right)\nabla w_h^{n+1},\nabla\hat{e}_w^{n+1}\right)\\ &+\frac{\varepsilon^2}{2\tau}\left(\left\|\nabla\hat{e}_u^{n+1}\right\|^2-\left\|\nabla\hat{e}_u^{n}\right\|^2+\left\|\nabla\hat{e}_u^{n+1}-\nabla\hat{e}_u^{n}\right\|^2\right)+\frac{\beta}{2\tau}\left(\left\|\hat{e}_u^{n+1}\right\|_{-1,h}^2-\left\|\hat{e}_u^{n}\right\|_{-1,h}^2+\left\|\hat{e}_u^{n+1}-\hat{e}_u^{n}\right\|_{-1,h}^2\right)\\ &=\left(\sigma\left(u^{n+1}\right),\hat{e}_w^{n+1}\right)+\left(\tilde{e}_w^{n+1},\delta_\tau\hat{e}_u^{n+1}\right)-\left(\tilde{e}_\varphi^{n+1},\delta_\tau\hat{e}_u^{n+1}\right)-\beta\left(\tilde{e}_u^{n+1},T_h\left(\delta_\tau\hat{e}_u^{n+1}\right)\right)\\ &\quad-\left(\phi_1\left(u^{n+1}\right)-\phi_2\left(u^{n+1}\right),\delta_\tau\hat{e}_u^{n+1}\right)+\left(\phi_1\left(u_h^{n+1}\right)-\phi_2\left(u_h^{n}\right),\delta_\tau\hat{e}_u^{n+1}\right).\end{aligned}\tag{17}$$

其中

$$\begin{aligned}&\left(m\left(u^{n+1}\right)\nabla R_h w^{n+1},\nabla\hat{e}_w^{n+1}\right)-\left(m\left(u_h^n\right)\nabla w_h^{n+1},\nabla\hat{e}_w^{n+1}\right)\\&=\left(\left(m\left(u_h^n\right)+m'(\xi_3)\left(u^{n+1}-u_h^n\right)\right)\nabla R_h w^{n+1},\nabla\hat{e}_w^{n+1}\right)-\left(m\left(u_h^n\right)\nabla w_h^{n+1},\nabla\hat{e}_w^{n+1}\right)\\&=\left\|\sqrt{m\left(u_h^n\right)}\nabla\hat{e}_w^{n+1}\right\|^2+\left(m'(\xi_3)\left(u^{n+1}-u_h^n\right)\nabla R_h w^{n+1},\nabla\hat{e}_w^{n+1}\right).\end{aligned}\tag{18}$$

则(17)式变为

$$\begin{aligned}&\left\|\sqrt{m\left(u_h^n\right)}\nabla\hat{e}_w^{n+1}\right\|^2+\frac{\varepsilon^2}{2\tau}\left(\left\|\nabla\hat{e}_u^{n+1}\right\|^2-\left\|\nabla\hat{e}_u^n\right\|^2+\left\|\nabla\hat{e}_u^{n+1}-\nabla\hat{e}_u^n\right\|^2\right)\\&+\frac{\beta}{2\tau}\left(\left\|\hat{e}_u^{n+1}\right\|_{-1,h}^2-\left\|\hat{e}_u^n\right\|_{-1,h}^2+\left\|\hat{e}_u^{n+1}-\hat{e}_u^n\right\|_{-1,h}^2\right)\\&=\left(\sigma\left(u^{n+1}\right),\hat{e}_w^{n+1}\right)+\left(\tilde{e}_w^{n+1},\delta_\tau\hat{e}_u^{n+1}\right)-\left(\tilde{e}_\varphi^{n+1},\delta_\tau\hat{e}_u^{n+1}\right)-\beta\left(\tilde{e}_u^{n+1},T_h\left(\delta_\tau\hat{e}_u^{n+1}\right)\right)\\&\quad-\left(\phi_1\left(u^{n+1}\right)-\phi_2\left(u^{n+1}\right),\delta_\tau\hat{e}_u^{n+1}\right)+\left(\phi_1\left(u_h^{n+1}\right)-\phi_2\left(u_h^n\right),\delta_\tau\hat{e}_u^{n+1}\right)\\&\quad-\left(m'(\xi_3)\left(u^{n+1}-u_h^n\right)\nabla R_h w^{n+1},\nabla\hat{e}_w^{n+1}\right)\\&=\sum_{i=1}^{6}G_i.\end{aligned}\tag{19}$$

接下来，根据 Cauchy-Schwarz 不等式，庞加莱不等式和 Young 不等式估计 G_i：

$$\begin{aligned}G_1&=\left(\sigma\left(u^{n+1}\right),\hat{e}_w^{n+1}\right)\\&\le\left|\left(\sigma\left(u^{n+1}\right),\hat{e}_w^{n+1}\right)\right|\\&\le C\left\|\sigma\left(u^{n+1}\right)\right\|\left\|\nabla\hat{e}_w^{n+1}\right\|\\&\le\frac{C^2}{m_1}\left\|\sigma\left(u^{n+1}\right)\right\|^2+\frac{m_1}{4}\left\|\nabla\hat{e}_w^{n+1}\right\|^2\\&\le C\tau^2+Ch^{2r+2}+\frac{m_1}{4}\left\|\nabla\hat{e}_w^{n+1}\right\|^2.\end{aligned}\tag{20}$$

$$\begin{aligned}G_2&=\left(\tilde{e}_w^{n+1},\delta_\tau\hat{e}_u^{n+1}\right)\\&\le\left|\left(\tilde{e}_w^{n+1},\delta_\tau\hat{e}_u^{n+1}\right)\right|\\&\le C\left\|\nabla\tilde{e}_w^{n+1}\right\|\left\|\delta_\tau\hat{e}_u^{n+1}\right\|_{-1,h}\\&\le\frac{4C^2}{\alpha}\left\|\nabla\tilde{e}_w^{n+1}\right\|^2+\frac{\alpha}{16}\left\|\delta_\tau\hat{e}_u^{n+1}\right\|_{-1,h}^2\\&\le Ch^{2r}+\frac{\alpha}{16}\left\|\delta_\tau\hat{e}_u^{n+1}\right\|_{-1,h}^2.\end{aligned}\tag{21}$$

$$\begin{aligned}G_3&=-\left(\tilde{e}_\varphi^{n+1},\delta_\tau\hat{e}_u^{n+1}\right)\\&\le\left|\left(\tilde{e}_\varphi^{n+1},\delta_\tau\hat{e}_u^{n+1}\right)\right|\\&\le C\left\|\nabla\tilde{e}_\varphi^{n+1}\right\|\left\|\delta_\tau\hat{e}_u^{n+1}\right\|_{-1,h}\\&\le\frac{4C^2}{\alpha}\left\|\nabla\tilde{e}_\varphi^{n+1}\right\|^2+\frac{\alpha}{16}\left\|\delta_\tau\hat{e}_u^{n+1}\right\|_{-1,h}^2\\&\le Ch^{2r}+\frac{\alpha}{16}\left\|\delta_\tau\hat{e}_u^{n+1}\right\|_{-1,h}^2.\end{aligned}\tag{22}$$

$$\begin{aligned} G_4 &= -\beta\left(\tilde{e}_u^{n+1}, T_h\left(\delta_\tau \hat{e}_u^{n+1}\right)\right) \\ &\le \beta\left|\left(\tilde{e}_u^{n+1}, T_h\left(\delta_\tau \hat{e}_u^{n+1}\right)\right)\right| \\ &\le C\beta\left\|\tilde{e}_u^{n+1}\right\|\left\|T_h\left(\delta_\tau \hat{e}_u^{n+1}\right)\right\| \\ &\le C\beta\left\|\nabla\tilde{e}_u^{n+1}\right\|\left\|\delta_\tau \hat{e}_u^{n+1}\right\|_{-1,h} \\ &\le \frac{4C^2\beta^2}{\alpha}\left\|\nabla\tilde{e}_u^{n+1}\right\|^2 + \frac{\alpha}{16}\left\|\delta_\tau \hat{e}_u^{n+1}\right\|_{-1,h}^2 \\ &\le Ch^{2r} + \frac{\alpha}{16}\left\|\delta_\tau \hat{e}_u^{n+1}\right\|_{-1,h}^2. \end{aligned} \tag{23}$$

运用泰勒展开，有

$$\begin{aligned} \left\|\nabla\tau\delta_\tau u(t)\right\| &= \left\|\int_{t-\tau}^{t}\nabla u_s(s)\,\mathrm{d}s\right\|^2 \\ &\le \left(\int_{t-\tau}^{t}\left\|\nabla u_s(s)\right\|\mathrm{d}s\right)^2 \\ &\le \left(\int_{t-\tau}^{t}\left\|\nabla u_s(s)\right\|^3\mathrm{d}s\right)^{\frac{2}{3}}\left(\int_{t-\tau}^{t}1^{\frac{3}{2}}\,\mathrm{d}s\right)^{\frac{4}{3}} \\ &\le \tau^{\frac{4}{3}}\left(\int_{t-\tau}^{t}\left\|\nabla u_s(s)\right\|^3\mathrm{d}s\right)^{\frac{2}{3}} \\ &\le C\tau^2. \end{aligned} \tag{24}$$

因此

$$\begin{aligned} G_5 &= -\left(\phi_1\left(u^{n+1}\right)-\phi_2\left(u^{n+1}\right),\delta_\tau\hat{e}_u^{n+1}\right)+\left(\phi_1\left(u_h^{n+1}\right)-\phi_2\left(u_h^{n}\right),\delta_\tau\hat{e}_u^{n+1}\right) \\ &= \left(\phi_1\left(u_h^{n+1}\right)-\phi_1\left(u^{n+1}\right),\delta_\tau\hat{e}_u^{n+1}\right)+\left(\phi_2\left(u^{n+1}\right)-\phi_2\left(u_h^{n}\right),\delta_\tau\hat{e}_u^{n+1}\right) \\ &= -\left(\phi_1'(\xi_3)\left(u^{n+1}-u_h^{n+1}\right),\delta_\tau\hat{e}_u^{n+1}\right)+\left(u^{n+1}-u_h^{n},\delta_\tau\hat{e}_u^{n+1}\right) \\ &\le L\left|\left(u^{n+1}-u_h^{n+1},\delta_\tau\hat{e}_u^{n+1}\right)\right|+\tau\left(\delta_\tau u^{n+1},\delta_\tau\hat{e}_u^{n+1}\right)+\left(u^{n}-u_h^{n},\delta_\tau\hat{e}_u^{n+1}\right) \\ &\le \frac{4L^2}{\alpha}\left\|\nabla\left(u^{n+1}-u_h^{n+1}\right)\right\|^2+\frac{\alpha}{16}\left\|\delta_\tau\hat{e}_u^{n+1}\right\|_{-1,h}^2 \\ &\quad +\left\|\nabla\tau\delta_\tau u^{n+1}\right\|\left\|\delta_\tau\hat{e}_u^{n+1}\right\|_{-1,h}+\left\|\nabla\left(u^{n}-u_h^{n}\right)\right\|\left\|\delta_\tau\hat{e}_u^{n+1}\right\|_{-1,h} \\ &\le Ch^{2r}+C\left\|\nabla\hat{e}_u^{n+1}\right\|^2+\frac{\alpha}{16}\left\|\delta_\tau\hat{e}_u^{n+1}\right\|_{-1,h}^2+\frac{4}{\alpha}\left\|\nabla\tau\delta_\tau u^{n+1}\right\|^2+\frac{\alpha}{16}\left\|\delta_\tau\hat{e}_u^{n+1}\right\|_{-1,h}^2 \\ &\quad +Ch^{2r}+C\left\|\nabla\hat{e}_u^{n}\right\|^2+\frac{\alpha}{16}\left\|\delta_\tau\hat{e}_u^{n+1}\right\|_{-1,h}^2 \\ &\le C\tau^2+Ch^{2r}+C\left\|\nabla\hat{e}_u^{n+1}\right\|^2+C\left\|\nabla\hat{e}_u^{n}\right\|^2+\frac{3\alpha}{16}\left\|\delta_\tau\hat{e}_u^{n+1}\right\|_{-1,h}^2. \end{aligned} \tag{25}$$

$$\begin{aligned} G_6 &= -\left(m'(\xi_3)\left(u^{n+1}-u_h^{n}\right)\nabla R_h w^{n+1},\nabla\hat{e}_w^{n+1}\right) \\ &\le \left\|m'(\xi_3)\left(u^{n+1}-u_h^{n}\right)\nabla R_h w^{n+1}\right\|\left\|\nabla\hat{e}_w^{n+1}\right\| \\ &\le \frac{1}{m_1}\left\|m'(\xi_3)\left(u^{n+1}-u_h^{n}\right)\nabla R_h w^{n+1}\right\|^2+\frac{m_1}{4}\left\|\nabla\hat{e}_w^{n+1}\right\|^2 \\ &\le \frac{CM^2}{m_1}\left\|\nabla\left(u^{n+1}-u_h^{n}\right)\right\|^2\left\|\nabla w^{n+1}\right\|_{-1,h}^2+\frac{m_1}{4}\left\|\nabla\hat{e}_w^{n+1}\right\|^2 \\ &\le C\tau^2+Ch^{2r}+C\left\|\nabla\hat{e}_u^{n+1}\right\|^2+\frac{m_1}{4}\left\|\nabla\hat{e}_w^{n+1}\right\|^2. \end{aligned} \tag{26}$$

则(19)式变为

$$\left\|\sqrt{m\left(u_h^n\right)}\nabla\hat{e}_w^{n+1}\right\|^2+\frac{\varepsilon^2}{2\tau}\left(\left\|\nabla\hat{e}_u^{n+1}\right\|^2-\left\|\nabla\hat{e}_u^n\right\|^2+\left\|\nabla\hat{e}_u^{n+1}-\nabla\hat{e}_u^n\right\|^2\right)$$
$$+\frac{\beta}{2\tau}\left(\left\|\hat{e}_u^{n+1}\right\|_{-1,h}^2-\left\|\hat{e}_u^n\right\|_{-1,h}^2+\left\|\hat{e}_u^{n+1}-\hat{e}_u^n\right\|_{-1,h}^2\right) \tag{27}$$
$$\le C\tau^2+Ch^{2r}+C\left\|\nabla\hat{e}_u^{n+1}\right\|^2+C\left\|\nabla\hat{e}_u^n\right\|^2+\frac{m_1}{2}\left\|\nabla\hat{e}_w^{n+1}\right\|^2+\frac{3\alpha}{8}\left\|\delta_\tau\hat{e}_u^{n+1}\right\|_{-1,h}^2.$$

在(16)式的第一式中令 $q_h=T_h\left(\delta_\tau\hat{e}_u^{n+1}\right)$，有

$$\begin{aligned}\left\|\delta_\tau\hat{e}_u^{n+1}\right\|_{-1,h}^2&=\left(\sigma\left(u^{n+1}\right),T_h\left(\delta_\tau\hat{e}_u^{n+1}\right)\right)-\left(m\left(u^{n+1}\right)\nabla R_hw^{n+1},T_h\left(\delta_\tau\hat{e}_u^{n+1}\right)\right)\\&\quad+\left(m\left(u_h^n\right)\nabla w_h^{n+1},T_h\left(\delta_\tau\hat{e}_u^{n+1}\right)\right)\\&=\left(\sigma\left(u^{n+1}\right),T_h\left(\delta_\tau\hat{e}_u^{n+1}\right)\right)-\left(\left(m\left(u_h^n\right)+m'\left(\xi_3\right)\left(u^{n+1}-u_h^n\right)\right)\nabla R_hw^{n+1},T_h\left(\delta_\tau\hat{e}_u^{n+1}\right)\right)\\&\quad+\left(m\left(u_h^n\right)\nabla w_h^{n+1},T_h\left(\delta_\tau\hat{e}_u^{n+1}\right)\right)\\&=\left(\sigma\left(u^{n+1}\right),T_h\left(\delta_\tau\hat{e}_u^{n+1}\right)\right)-\left(m\left(u_h^n\right)\nabla\hat{e}_w^{n+1},T_h\left(\delta_\tau\hat{e}_u^{n+1}\right)\right)\\&\quad-\left(m'\left(\xi_3\right)\left(u^{n+1}-u_h^n\right)\nabla R_hw^{n+1},T_h\left(\delta_\tau\hat{e}_u^{n+1}\right)\right)\\&\le\left\|\sigma\left(u^{n+1}\right)\right\|\left\|T_h\left(\delta_\tau\hat{e}_u^{n+1}\right)\right\|+\left\|m\left(u_h^n\right)\right\|_{L^\infty}\left\|\nabla\hat{e}_w^{n+1}\right\|\left\|\nabla T_h\left(\delta_\tau\hat{e}_u^{n+1}\right)\right\|\\&\quad+\left\|m'\left(\xi_3\right)\left(u^{n+1}-u_h^n\right)\nabla R_hw^{n+1}\right\|\left\|\nabla T_h\left(\delta_\tau\hat{e}_u^{n+1}\right)\right\|\\&\le\frac{3}{2}\left\|\sigma\left(u^{n+1}\right)\right\|^2+\frac{1}{2}\left\|\delta_\tau\hat{e}_u^{n+1}\right\|_{-1,h}^2+\frac{3m_1^2}{2}\left\|\nabla\hat{e}_w^{n+1}\right\|^2+\frac{3}{2}\left\|m'\left(\xi_3\right)\left(u^{n+1}-u_h^n\right)\nabla R_hw^{n+1}\right\|^2\\&\le C\tau^2+Ch^{2r}+Ch^{2r+2}+\frac{3m_1^2}{2}\left\|\nabla\hat{e}_w^{n+1}\right\|^2+C\left\|\nabla\hat{e}_u^{n+1}\right\|^2+\frac{1}{2}\left\|\delta_\tau\hat{e}_u^{n+1}\right\|_{-1,h}^2.\end{aligned} \tag{28}$$

则

$$\left\|\delta_\tau\hat{e}_u^{n+1}\right\|_{-1,h}^2\le C\tau^2+Ch^{2r}+3m_1^2\left\|\nabla\hat{e}_w^{n+1}\right\|^2+C\left\|\nabla\hat{e}_u^{n+1}\right\|^2. \tag{29}$$

将(29)式代入(27)式得

$$\begin{aligned}&\left\|\sqrt{m\left(u_h^n\right)}\nabla\hat{e}_w^{n+1}\right\|^2+\frac{\varepsilon^2}{2\tau}\left(\left\|\nabla\hat{e}_u^{n+1}\right\|^2-\left\|\nabla\hat{e}_u^n\right\|^2\right)+\frac{\beta}{2\tau}\left(\left\|\hat{e}_u^{n+1}\right\|_{-1,h}^2-\left\|\hat{e}_u^n\right\|_{-1,h}^2\right)\\&\le C\tau^2+Ch^{2r}+C\left\|\nabla\hat{e}_u^{n+1}\right\|^2+C\left\|\nabla\hat{e}_u^n\right\|^2+\left(\frac{m_1}{2}+\frac{9m_1^2\alpha}{8}\right)\left\|\nabla\hat{e}_w^{n+1}\right\|^2.\end{aligned} \tag{30}$$

在(30)式中取适当的 $\alpha\left(0<\alpha<\frac{4\left(2m_2-m_1\right)}{9m_1^2}\right)$，并乘以 2τ 得：

$$\begin{aligned}&C\tau\left\|\nabla\hat{e}_w^{n+1}\right\|^2+\varepsilon^2\left(\left\|\nabla\hat{e}_u^{n+1}\right\|^2-\left\|\nabla\hat{e}_u^n\right\|^2\right)+\beta\left(\left\|\hat{e}_u^{n+1}\right\|_{-1,h}^2-\left\|\hat{e}_u^n\right\|_{-1,h}^2\right)\\&\le C\tau\tau^2+C\tau h^{2r}+C\left\|\nabla\hat{e}_u^{n+1}\right\|^2+C\left\|\nabla\hat{e}_u^n\right\|^2.\end{aligned} \tag{31}$$

上式从 1 到 n 求和，根据离散的 Gronwall 不等式，有：

$$\sum_{i=1}^n C\tau\left\|\nabla\hat{e}_w^{i+1}\right\|^2+\varepsilon^2\left\|\nabla\hat{e}_u^{n+1}\right\|^2+\beta\left\|\nabla\hat{e}_u^{n+1}\right\|_{-1,h}^2\le C\tau^2+Ch^{2r}. \tag{32}$$

5. 数值实验

在数值实验部分，采用一些数值算例验证理论分析的正确性和有效性。选择初始条件为

$$u_0 = 0.5 + 0.17\cos(\pi x)\cos(2\pi y) + 0.2\cos(3\pi x)\cos(\pi y)$$

计算区域为 $[-1,1]\times[-1,1]$。在表 1~4 中，选择固定的参数 $\tau = \frac{1}{1000}$，变化的网格步长为 $h = \frac{1}{16}$，$h = \frac{1}{32}$，$h = \frac{1}{64}$，$h = \frac{1}{128}$，$h = \frac{1}{256}$，$k = 0.01$。相对误差 $\|\hat{e}_u\|$ 和 $\|\hat{e}_u\|_{H^1}$ 的空间收敛阶接近于 2 和 1，与理论部分得到的收敛阶保持一致，且不同的 θ 和 β 对收敛阶影响不大。

Table 1. $\varepsilon = 0.09, \theta = 0.1, \beta = 0$
表1. $\varepsilon = 0.09, \theta = 0.1, \beta = 0$

h	$\|\hat{e}_u\|$	rate	$\|\hat{e}_u\|_{H^1}$	rate
1/16	0.0167411		0.428408	
1/32	0.00436806	1.93833	0.214218	0.999908
1/64	0.00111088	1.97529	0.105747	1.01846
1/128	0.00027910	1.99283	0.051690	1.03266
1/256	6.902e-005	2.01582	0.024097	1.10104

Table 2. $\varepsilon = 0.09, \theta = 0.1, \beta = 1$
表2. $\varepsilon = 0.09, \theta = 0.1, \beta = 1$

h	$\|\hat{e}_u\|$	rate	$\|\hat{e}_u\|_{H^1}$	rate
1/16	0.0159265		0.282201	
1/32	0.00442041	1.84918	0.141806	0.992805
1/64	0.0011382	1.95743	0.0700273	1.01793
1/128	0.00028670	1.98912	0.0343081	1.02937
1/256	7.148e-005	2.00387	0.0161321	1.08861

Table 3. $\varepsilon = 0.09, \theta = 0.7, \beta = 0$
表3. $\varepsilon = 0.09, \theta = 0.7, \beta = 0$

h	$\|\hat{e}_u\|$	rate	$\|\hat{e}_u\|_{H^1}$	rate
1/16	0.0163399		0.359055	
1/32	0.00441221	1.88882	0.179362	1.00133
1/64	0.00113252	1.96197	0.0883805	1.02108
1/128	0.00028524	1.98926	0.043175	1.03353
1/256	7.088e-005	2.0088	0.0201539	1.09914

Table 4. $\varepsilon = 0.09, \theta = 0.7, \beta = 1$
表4. $\varepsilon = 0.09, \theta = 0.7, \beta = 1$

h	$\|\hat{e}_u\|$	rate	$\|\hat{e}_u\|_{H^1}$	rate
1/16	0.0141677		0.229437	
1/32	0.00399717	1.82555	0.115215	0.993773
1/64	0.00103366	1.95121	0.0568472	1.01916
1/128	0.00026067	1.98747	0.0278674	1.02851
1/256	6.509e-005	2.00181	0.0131527	1.08322

6. 结论

本文研究了具有浓度迁移率和对数势能的修正 Cahn-Hilliard 方程，在理论分析中证明了它的稳定性和误差估计，并给出了数值算例验证了它的结论。

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