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对多元复合函数偏导数求解的研究

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摘　要

本文主要讨论多元复合函数的偏导数问题，主要对中间变量是隐函数的情况进行讨论。文章主线是从隐函数的存在性问题出发，并结合“链式求导法则、矩阵求多元抽象复合函数二阶偏导数”等技巧和方法，获得了求解多元复合函数的一、二阶偏导数的理论性定理。

关键词

多元复合函数，隐函数，链式法则，偏导数

Research on Solving Partial Derivatives of Multivariate Compound Functions

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Abstract

In this article, we mainly discuss the partial derivatives of multivariate compound functions, and mainly discuss the case that the intermediate variables are implicit functions. The main line of this article starts from the existence of implicit function, and combines the techniques and methods of “chain derivative rule, using matrix to find the second partial derivative of multiple abstract composite function”. The theoretical theorems for solving the first and second partial derivatives of multivariate compound functions are obtained.

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Keywords

Multivariate Compound Functions, Implicit Function, The Chain Rule, Partial Derivatives

1. 引言

在现已知的多元函数变量代换的方法与应用中，都是仅对复合函数一阶偏导数的代换方法进行了讨论，然而对高阶可微函数的高阶偏导数(如：二阶偏导数)的代换求解公式极少提及。但当我们处理实际问题时，往往会有很多这方面的理论需求，如：多元复合函数的高精度近似计算、多元函数求极值问题等。另外，在对一般复合函数偏导数的计算过程中，常以链式求导法则为主，但在许多具体的问题，例如：函数 $\xi=\xi(z,r)$，$\eta=\eta(z,r)$，$z=(u,v)$，$r=(u,v)$，z,r 为物理域中的变量，而 ξ,η 为计算域中的变量，利用计算域中的结果来讨论自变量变化问题，如果使用链式法则进行计算，求解过程就会极为复杂。故而，为了解决类似的这样一类问题，我们可从隐函数的存在唯一性定理出发，结合[1] [2]中的方法，可得到本文的结果。

2. 预备知识

为后期得到我们的结论，我们先给出如下的隐函数存在唯一性定理和隐函数组定理。

引理 2.1 [3]若函数 $F(x,y)$ 满足下列条件：

1) F 在以 $P_0(x_0,y_0)$ 为内点的某一区域 $D\subset R^2$ 上连续；

2) $F(x_0,y_0)=0$ (通常称为初始条件)；

3) F 在 D 内存在连续的偏导数 $F_y(x,y)$；

4) $F_y(x_0,y_0)\neq 0$。

则

i) 存在点 p_0 的某领域 $U(P_0)\subset D$，在 $U(P_0)$ 上方程 $F(x,y)=0$ 唯一地决定了一个定义在某区间 $[x_0-\alpha,x_0+\alpha]$ 上的隐函数 $y=f(x)$，使得当 $x\in[x_0-\alpha,x_0+\alpha]$ 时，$(x,f(x))\in U(P_0)$，且 $F(x,f(x))\equiv 0$，$f(x_0)=y_0$。

ii) $y=f(x)$ 在 $[x_0-\alpha,x_0+\alpha]$ 上连续。

引理 2.2 [3]若

1) $F(x,y,u,v)$ 与 $G(x,y,u,v)$ 在以点 $P_0(x_0,y_0,u_0,v_0)$ 为内点的区域 $V\subset R^4$ 上连续；

2) $F(x_0,y_0,u_0,v_0)=0$，$G(x_0,y_0,u_0,v_0)=0$ (初始条件)；

3) 在 V 上 F，G 具有一阶连续偏导数；

4) $J=\dfrac{\partial(F,G)}{\partial(u,v)}$ 在点 P_0 不等于 0。

i) 存在点 P_0 的某一(四维)邻域 $U(P_0)\subset V$，在 $U(P_0)\subset V$ 上，方程组 $\begin{cases}F(x,y,u,v)=0\\G(x,y,u,v)=0\end{cases}$，唯一确定了定义在点 $Q_0(x_0,y_0)$ 的某一(二维空间)邻域 $U(Q_0)$ 上的两个二元隐函数 $u=f(x,y),v=g(x,y)$，使得

$u_0 = f(x_0, y_0), v_0 = g(x_0, y_0)$，且当 $(x,y)\in U(Q_0)$ 时

$$(x,y,f(x,y),g(x,y))\in U(P_0),\quad F(x,y,f(x,y),g(x,y))\equiv 0,\quad G(x,y,f(x,y),g(x,y))\equiv 0;$$

ii) $f(x,y), g(x,y)$ 在 $U(Q_0)$ 上连续；

iii) $f(x,y), g(x,y)$ 在 $U(Q_0)$ 上有一阶连续偏导，且

$$\frac{\partial u}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)},\quad \frac{\partial v}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)},\quad \frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)},\quad \frac{\partial v}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)}$$

为了得到我们的结果，我们先导出如下定理 2.1：

定理 2.1 设函数 $\xi=\xi(z,r), \eta=\eta(z,r)$ 关于变量 z,r 具有连续的一阶偏导数，记

$$J=\begin{pmatrix}\frac{\partial \xi}{\partial z} & \frac{\partial \xi}{\partial r}\\ \frac{\partial \eta}{\partial z} & \frac{\partial \eta}{\partial r}\end{pmatrix} \tag{2.1}$$

若 $|J|\neq 0$，则存在唯一的函数 $z=z(\xi,\eta), r=r(\xi,\eta)$ 满足上述等式并且关于变量 ξ,η 具有连续的一阶偏导数

$$\begin{pmatrix}\frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta}\\ \frac{\partial r}{\partial \xi} & \frac{\partial r}{\partial \eta}\end{pmatrix}=\frac{1}{|J|}\begin{pmatrix}\frac{\partial \eta}{\partial r} & -\frac{\partial \eta}{\partial z}\\ -\frac{\partial \xi}{\partial r} & \frac{\partial \xi}{\partial z}\end{pmatrix} \tag{2.2}$$

其中 $|J|$ 矩阵 J 的行列式。

证明：假设存在两个不同的坐标系，中间变量为 (z,r)，自变量为 (ξ,η)，两个坐标系存在以下的对应关系：

$$\begin{cases}\xi=\xi(z,r)\\ \eta=\eta(z,r)\end{cases} \tag{2.3}$$

进一步，令

$$\begin{cases}F(\xi,\eta,\xi(z,r),\eta(z,r))=\xi-\xi(z,r)=0\\ G(\xi,\eta,\xi(z,r),\eta(z,r))=\eta-\eta(z,r)=0\end{cases} \tag{2.4}$$

则由引理 2.2 可知，(2.4)所对应的 Jacobi 行列式不为 0，即

$$\frac{\partial(F,G)}{\partial(\xi,\eta)}=\begin{vmatrix}F_\xi & F_\eta\\ G_\xi & G_\eta\end{vmatrix}\neq 0$$

亦即

$$\frac{\partial(F,G)}{\partial(z,r)}=\begin{vmatrix}\frac{\partial F}{\partial z} & \frac{\partial F}{\partial r}\\ \frac{\partial G}{\partial z} & \frac{\partial G}{\partial r}\end{vmatrix}=\begin{vmatrix}-\frac{\partial \xi}{\partial z} & \frac{\partial \xi}{\partial r}\\ -\frac{\partial \eta}{\partial z} & \frac{\partial \eta}{\partial r}\end{vmatrix}\neq 0 \tag{2.5}$$

进而对(2.3)式两边分别关于 ξ,η 求偏导，则有

$$\frac{\partial \xi}{\partial z}\cdot\frac{\partial z}{\partial \xi}+\frac{\partial \xi}{\partial r}\cdot\frac{\partial r}{\partial \xi}=1 \tag{2.6}$$

$$\frac{\partial \xi}{\partial z}\cdot\frac{\partial z}{\partial \eta}+\frac{\partial \xi}{\partial r}\cdot\frac{\partial r}{\partial \eta}=0 \tag{2.7}$$

$$\frac{\partial \eta}{\partial z}\cdot\frac{\partial z}{\partial \xi}+\frac{\partial \eta}{\partial r}\cdot\frac{\partial r}{\partial \xi}=0 \tag{2.8}$$

$$\frac{\partial \eta}{\partial z}\cdot\frac{\partial z}{\partial \eta}+\frac{\partial \eta}{\partial r}\cdot\frac{\partial r}{\partial \eta}=1 \tag{2.9}$$

结合(2.6)~(2.9)，易得

$$\begin{pmatrix}\frac{\partial \xi}{\partial z} & \frac{\partial \xi}{\partial r}\\ \frac{\partial \eta}{\partial z} & \frac{\partial \eta}{\partial r}\end{pmatrix}\begin{pmatrix}\frac{\partial z}{\partial \xi}\\ \frac{\partial r}{\partial \xi}\end{pmatrix}=J\begin{pmatrix}\frac{\partial z}{\partial \xi}\\ \frac{\partial r}{\partial \xi}\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix} \tag{2.10}$$

$$\begin{pmatrix}\frac{\partial \xi}{\partial z} & \frac{\partial \xi}{\partial r}\\ \frac{\partial \eta}{\partial z} & \frac{\partial \eta}{\partial r}\end{pmatrix}\begin{pmatrix}\frac{\partial z}{\partial \eta}\\ \frac{\partial r}{\partial \eta}\end{pmatrix}=J\begin{pmatrix}\frac{\partial z}{\partial \eta}\\ \frac{\partial r}{\partial \eta}\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix} \tag{2.11}$$

于是有

$$\begin{pmatrix}\frac{\partial z}{\partial \xi}\\ \frac{\partial r}{\partial \xi}\end{pmatrix}=J^{-1}\begin{pmatrix}1\\0\end{pmatrix}=\frac{1}{|J|}\begin{pmatrix}\frac{\partial \eta}{\partial r}\\ -\frac{\partial \xi}{\partial r}\end{pmatrix} \tag{2.12}$$

$$\begin{pmatrix}\frac{\partial z}{\partial \xi}\\ \frac{\partial r}{\partial \xi}\end{pmatrix}=J^{-1}\begin{pmatrix}1\\0\end{pmatrix}=\frac{1}{|J|}\begin{pmatrix}-\frac{\partial \eta}{\partial z}\\ \frac{\partial \xi}{\partial z}\end{pmatrix} \tag{2.13}$$

由(2.12)和(2.13)可知(2.2)成立，证毕。

3. 多元复合函数的一阶、二阶偏导数

遍及本节，我们有下述假设：

假设 $\omega=f(z,r)$，且中间变量与自变量之间满足如下关系：

$$\begin{cases}\xi=\xi(z,r)\\ \eta=\eta(z,r)\end{cases}$$

进而对上式两边分别关于 ξ,η 求偏导，由定理 2.1，得到：

$$\begin{cases}\frac{\partial z}{\partial \xi}=\frac{1}{|J|}\frac{\partial \eta}{\partial r}\\ \frac{\partial r}{\partial \xi}=-\frac{1}{|J|}\frac{\partial z}{\partial \eta}\\ \frac{\partial \eta}{\partial z}=-\frac{1}{|J|}\frac{\partial r}{\partial \xi}\\ \frac{\partial \eta}{\partial r}=\frac{1}{|J|}\frac{\partial z}{\partial \xi}\end{cases} \tag{3.1}$$

其中

$$J=\begin{pmatrix}\frac{\partial z}{\partial \xi} & \frac{\partial r}{\partial \xi}\\ \frac{\partial z}{\partial \eta} & \frac{\partial r}{\partial \eta}\end{pmatrix},\ |J|=\frac{\partial z}{\partial \xi}\frac{\partial r}{\partial \eta}-\frac{\partial r}{\partial \xi}\frac{\partial z}{\partial \eta} \tag{3.2}$$

接下来，我们首先导出多元复合函数一阶偏导数定理。

定理 3.1 设 $w=f(u,v)$，函数 $x=x(u,v), y=y(u,v)$ 关于变量 u,v 具有连续的一阶偏导数，记：

$$J=\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix} \tag{3.3}$$

若 $|J|\neq 0$，则

$$\begin{pmatrix}\frac{\partial w}{\partial x}\\ \frac{\partial w}{\partial y}\end{pmatrix}^{\mathrm{T}}=\frac{1}{|J|}\begin{pmatrix}\frac{\partial f}{\partial u} & \frac{\partial f}{\partial v}\end{pmatrix}J^{-1} \tag{3.4}$$

证明：依据链式求导法则，有

$$\frac{\partial w}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}=\begin{pmatrix}\frac{\partial f}{\partial u} & \frac{\partial f}{\partial v}\end{pmatrix}\begin{pmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial v}{\partial x}\end{pmatrix} \tag{3.5}$$

$$\frac{\partial w}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}=\begin{pmatrix}\frac{\partial f}{\partial u} & \frac{\partial f}{\partial v}\end{pmatrix}\begin{pmatrix}\frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial y}\end{pmatrix} \tag{3.6}$$

进一步，通过(3.1)式，可以得到

$$\frac{\partial w}{\partial x}=\frac{1}{|J|}\begin{pmatrix}\frac{\partial f}{\partial u} & \frac{\partial f}{\partial v}\end{pmatrix}\begin{pmatrix}\frac{\partial y}{\partial v}\\ -\frac{\partial y}{\partial u}\end{pmatrix} \tag{3.7}$$

$$\frac{\partial w}{\partial y}=\frac{1}{|J|}\begin{pmatrix}\frac{\partial f}{\partial u} & \frac{\partial f}{\partial v}\end{pmatrix}\begin{pmatrix}-\frac{\partial x}{\partial v}\\ \frac{\partial x}{\partial u}\end{pmatrix} \tag{3.8}$$

结合(3.7)~(3.8)，易得(3.4)，证毕。

进一步，我们导出多元复合函数的二阶偏导数定理。

由文献[1]中矩阵求多元抽象复合函数二阶偏导数的方法，可以知道

$$\left(\frac{\partial^2}{\partial z^2}\right)=\begin{pmatrix}\frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z}\end{pmatrix}\begin{bmatrix}\frac{\partial^2}{\partial \xi^2} & \frac{\partial^2}{\partial \xi\partial \eta}\\ \frac{\partial^2}{\partial \eta\partial \xi} & \frac{\partial^2}{\partial \eta^2}\end{bmatrix}\begin{pmatrix}\frac{\partial \xi}{\partial z}\\ \frac{\partial \eta}{\partial z}\end{pmatrix}+\begin{pmatrix}\frac{\partial}{\partial \xi} & \frac{\partial}{\partial \eta}\end{pmatrix}\begin{pmatrix}\frac{\partial^2 \xi}{\partial z^2}\\ \frac{\partial^2 \eta}{\partial z^2}\end{pmatrix} \tag{3.9}$$

$$\begin{pmatrix}\dfrac{\partial^2}{\partial\xi^2}\end{pmatrix}=\begin{pmatrix}\dfrac{\partial z}{\partial\xi} & \dfrac{\partial\gamma}{\partial\xi}\end{pmatrix}\begin{bmatrix}\dfrac{\partial^2}{\partial z^2} & \dfrac{\partial^2}{\partial z\partial\gamma}\\ \dfrac{\partial^2}{\partial z\partial\gamma} & \dfrac{\partial^2}{\partial\gamma^2}\end{bmatrix}\begin{pmatrix}\dfrac{\partial z}{\partial\xi}\\ \dfrac{\partial\gamma}{\partial\xi}\end{pmatrix}+\begin{pmatrix}\dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial\gamma}\end{pmatrix}\begin{pmatrix}\dfrac{\partial^2 z}{\partial\xi^2}\\ \dfrac{\partial^2\gamma}{\partial\xi^2}\end{pmatrix} \tag{3.10}$$

进而我们可以得到如下关于多元复合函数求二阶偏导数的定理。

定理 3.2 设 $\omega=f(z,r),\xi=\xi(z,r),\eta=\eta(z,r)$，且 f,ξ,η 均有连续的二阶偏导数。若记：

$$B=\begin{pmatrix}\left(\dfrac{\partial r}{\partial\eta}\right)^2 & -\dfrac{\partial r}{\partial\xi}\dfrac{\partial r}{\partial\eta} & -\dfrac{\partial r}{\partial\xi}\dfrac{\partial r}{\partial\eta} & \left(\dfrac{\partial r}{\partial\xi}\right)^2\\ -\dfrac{\partial z}{\partial\eta}\dfrac{\partial r}{\partial\eta} & \dfrac{\partial z}{\partial\xi}\dfrac{\partial r}{\partial\eta} & \dfrac{\partial r}{\partial\xi}\dfrac{\partial z}{\partial\eta} & -\dfrac{\partial z}{\partial\xi}\dfrac{\partial r}{\partial\xi}\\ -\dfrac{\partial z}{\partial\eta}\dfrac{\partial r}{\partial\eta} & \dfrac{\partial z}{\partial\eta}\dfrac{\partial r}{\partial\xi} & \dfrac{\partial z}{\partial\xi}\dfrac{\partial r}{\partial\eta} & -\dfrac{\partial z}{\partial\xi}\dfrac{\partial r}{\partial\xi}\\ \left(\dfrac{\partial z}{\partial\eta}\right)^2 & -\dfrac{\partial z}{\partial\xi}\dfrac{\partial z}{\partial\eta} & -\dfrac{\partial z}{\partial\xi}\dfrac{\partial z}{\partial\eta} & \left(\dfrac{\partial z}{\partial\xi}\right)^2\end{pmatrix} \tag{3.11}$$

且满足 $|J|\neq 0$，则

$$\begin{pmatrix}\dfrac{\partial^2\xi}{\partial z^2}\\ \dfrac{\partial^2\xi}{\partial z\partial r}\\ \dfrac{\partial^2\xi}{\partial r\partial z}\\ \dfrac{\partial^2\xi}{\partial r^2}\end{pmatrix}=-\frac{1}{|J|^3}\frac{\partial r}{\partial\eta}B\begin{pmatrix}\dfrac{\partial^2 z}{\partial\xi^2}\\ \dfrac{\partial^2 z}{\partial\xi\partial\eta}\\ \dfrac{\partial^2 z}{\partial\eta\partial\xi}\\ \dfrac{\partial^2 z}{\partial\eta^2}\end{pmatrix}+\frac{1}{|J|^3}\frac{\partial z}{\partial\eta}B\begin{pmatrix}\dfrac{\partial^2 r}{\partial\xi^2}\\ \dfrac{\partial^2 r}{\partial\xi\partial\eta}\\ \dfrac{\partial^2 r}{\partial\eta\partial\xi}\\ \dfrac{\partial^2 r}{\partial\eta^2}\end{pmatrix} \tag{3.12}$$

$$\begin{pmatrix}\dfrac{\partial^2\eta}{\partial z^2}\\ \dfrac{\partial^2\eta}{\partial z\partial r}\\ \dfrac{\partial^2\eta}{\partial r\partial z}\\ \dfrac{\partial^2\eta}{\partial r^2}\end{pmatrix}=\frac{1}{|J|^3}\frac{\partial r}{\partial\xi}B\begin{pmatrix}\dfrac{\partial^2 z}{\partial\xi^2}\\ \dfrac{\partial^2 z}{\partial\xi\partial\eta}\\ \dfrac{\partial^2 z}{\partial\eta\partial\xi}\\ \dfrac{\partial^2 z}{\partial\eta^2}\end{pmatrix}-\frac{1}{|J|^3}\frac{\partial z}{\partial\xi}B\begin{pmatrix}\dfrac{\partial^2 r}{\partial\xi^2}\\ \dfrac{\partial^2 r}{\partial\xi\partial\eta}\\ \dfrac{\partial^2 r}{\partial\eta\partial\xi}\\ \dfrac{\partial^2 r}{\partial\eta^2}\end{pmatrix} \tag{3.13}$$

证明：对(2.6)和(2.7)两边分别关于 ξ,η 求二阶偏导，易得

$$\frac{\partial\xi}{\partial z}\cdot\frac{\partial^2 z}{\partial\xi^2}+\frac{\partial z}{\partial\xi}\left[\frac{\partial^2\xi}{\partial z^2}\cdot\frac{\partial z}{\partial\xi}+\frac{\partial^2\xi}{\partial z\partial r}\cdot\frac{\partial r}{\partial\xi}\right]+\frac{\partial\xi}{\partial r}\cdot\frac{\partial^2 r}{\partial\xi^2}+\frac{\partial r}{\partial\xi}\left[\frac{\partial^2\xi}{\partial r\partial z}\cdot\frac{\partial z}{\partial\xi}+\frac{\partial^2\xi}{\partial r^2}\cdot\frac{\partial r}{\partial\xi}\right]=0$$

$$\frac{\partial\xi}{\partial z}\cdot\frac{\partial^2 z}{\partial\xi\partial\eta}+\frac{\partial z}{\partial\xi}\left[\frac{\partial^2\xi}{\partial z^2}\cdot\frac{\partial z}{\partial\eta}+\frac{\partial^2\xi}{\partial z\partial r}\cdot\frac{\partial r}{\partial\eta}\right]+\frac{\partial\xi}{\partial r}\cdot\frac{\partial^2 r}{\partial\xi\partial\eta}+\frac{\partial r}{\partial\xi}\left[\frac{\partial^2\xi}{\partial r\partial z}\cdot\frac{\partial z}{\partial\eta}+\frac{\partial^2\xi}{\partial r^2}\cdot\frac{\partial r}{\partial\eta}\right]=0$$

$$\frac{\partial \xi}{\partial z}\cdot\frac{\partial^2 z}{\partial \eta \partial \xi}+\frac{\partial z}{\partial \eta}\left[\frac{\partial^2 \xi}{\partial z^2}\cdot\frac{\partial z}{\partial \xi}+\frac{\partial^2 \xi}{\partial z \partial r}\cdot\frac{\partial r}{\partial \xi}\right]+\frac{\partial \xi}{\partial r}\cdot\frac{\partial^2 r}{\partial \xi \partial \eta}+\frac{\partial r}{\partial \eta}\left[\frac{\partial^2 \xi}{\partial r \partial z}\cdot\frac{\partial z}{\partial \xi}+\frac{\partial^2 \xi}{\partial r^2}\cdot\frac{\partial r}{\partial \xi}\right]=0$$

$$\frac{\partial \xi}{\partial z}\cdot\frac{\partial^2 z}{\partial \eta^2}+\frac{\partial z}{\partial \eta}\left[\frac{\partial^2 \xi}{\partial z^2}\cdot\frac{\partial z}{\partial \eta}+\frac{\partial^2 \xi}{\partial z \partial r}\cdot\frac{\partial r}{\partial \eta}\right]+\frac{\partial \xi}{\partial r}\cdot\frac{\partial^2 r}{\partial \eta^2}+\frac{\partial r}{\partial \eta}\left[\frac{\partial^2 \xi}{\partial r \partial z}\cdot\frac{\partial z}{\partial \eta}+\frac{\partial^2 \xi}{\partial r^2}\cdot\frac{\partial r}{\partial \eta}\right]=0$$

联立上面 4 个式子，可得

$$\begin{pmatrix} \left(\frac{\partial z}{\partial \xi}\right)^2 & \frac{\partial z}{\partial \xi}\frac{\partial r}{\partial \xi} & \frac{\partial r}{\partial \xi}\frac{\partial z}{\partial \xi} & \left(\frac{\partial r}{\partial \xi}\right)^2 \\ \frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \xi}\frac{\partial r}{\partial \eta} & \frac{\partial r}{\partial \xi}\frac{\partial z}{\partial \eta} & \frac{\partial r}{\partial \xi}\frac{\partial r}{\partial \eta} \\ \frac{\partial z}{\partial \eta}\frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta}\frac{\partial r}{\partial \xi} & \frac{\partial r}{\partial \eta}\frac{\partial z}{\partial \xi} & \frac{\partial r}{\partial \eta}\frac{\partial r}{\partial \xi} \\ \left(\frac{\partial z}{\partial \eta}\right)^2 & \frac{\partial z}{\partial \eta}\frac{\partial r}{\partial \eta} & \frac{\partial r}{\partial \eta}\frac{\partial z}{\partial \eta} & \left(\frac{\partial r}{\partial \eta}\right)^2 \end{pmatrix}\begin{pmatrix} \frac{\partial^2 \xi}{\partial z^2} \\ \frac{\partial^2 \xi}{\partial z \partial r} \\ \frac{\partial^2 \xi}{\partial r \partial z} \\ \frac{\partial^2 \xi}{\partial r^2} \end{pmatrix} = -\frac{\partial \xi}{\partial z}\cdot\begin{pmatrix} \frac{\partial^2 z}{\partial \xi^2} \\ \frac{\partial^2 z}{\partial \xi \partial \eta} \\ \frac{\partial^2 z}{\partial \eta \partial \xi} \\ \frac{\partial^2 z}{\partial \eta^2} \end{pmatrix} - \frac{\partial \xi}{\partial r}\cdot\begin{pmatrix} \frac{\partial^2 r}{\partial \xi^2} \\ \frac{\partial^2 r}{\partial \xi \partial \eta} \\ \frac{\partial^2 r}{\partial \eta \partial \xi} \\ \frac{\partial^2 r}{\partial \eta^2} \end{pmatrix} \tag{3.14}$$

记

$$A=\begin{pmatrix} \left(\frac{\partial z}{\partial \xi}\right)^2 & \frac{\partial z}{\partial \xi}\frac{\partial r}{\partial \xi} & \frac{\partial r}{\partial \xi}\frac{\partial z}{\partial \xi} & \left(\frac{\partial r}{\partial \xi}\right)^2 \\ \frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \xi}\frac{\partial r}{\partial \eta} & \frac{\partial r}{\partial \xi}\frac{\partial z}{\partial \eta} & \frac{\partial r}{\partial \xi}\frac{\partial r}{\partial \eta} \\ \frac{\partial z}{\partial \eta}\frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta}\frac{\partial r}{\partial \xi} & \frac{\partial r}{\partial \eta}\frac{\partial z}{\partial \xi} & \frac{\partial r}{\partial \eta}\frac{\partial r}{\partial \xi} \\ \left(\frac{\partial z}{\partial \eta}\right)^2 & \frac{\partial z}{\partial \eta}\frac{\partial r}{\partial \eta} & \frac{\partial r}{\partial \eta}\frac{\partial z}{\partial \eta} & \left(\frac{\partial r}{\partial \eta}\right)^2 \end{pmatrix} \tag{3.15}$$

为求 A^{-1}，依据分块矩阵表示方法，A 可写为：

$$\begin{aligned} A&=\begin{pmatrix} \left(\frac{\partial z}{\partial \xi}\right)^2 & \frac{\partial z}{\partial \xi}\frac{\partial r}{\partial \xi} & \frac{\partial r}{\partial \xi}\frac{\partial z}{\partial \xi} & \left(\frac{\partial r}{\partial \xi}\right)^2 \\ \frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \xi}\frac{\partial r}{\partial \eta} & \frac{\partial r}{\partial \xi}\frac{\partial z}{\partial \eta} & \frac{\partial r}{\partial \xi}\frac{\partial r}{\partial \eta} \\ \frac{\partial z}{\partial \eta}\frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta}\frac{\partial r}{\partial \xi} & \frac{\partial r}{\partial \eta}\frac{\partial z}{\partial \xi} & \frac{\partial r}{\partial \eta}\frac{\partial r}{\partial \xi} \\ \left(\frac{\partial z}{\partial \eta}\right)^2 & \frac{\partial z}{\partial \eta}\frac{\partial r}{\partial \eta} & \frac{\partial r}{\partial \eta}\frac{\partial z}{\partial \eta} & \left(\frac{\partial r}{\partial \eta}\right)^2 \end{pmatrix} \\ &=\begin{pmatrix} \frac{\partial z}{\partial \xi}J & \frac{\partial r}{\partial \xi}J \\ \frac{\partial z}{\partial \eta}J & \frac{\partial r}{\partial \eta}J \end{pmatrix}=\begin{pmatrix} \frac{\partial z}{\partial \xi}E_2 & \frac{\partial r}{\partial \xi}E_2 \\ \frac{\partial z}{\partial \eta}E_2 & \frac{\partial r}{\partial \eta}E_2 \end{pmatrix}\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \end{aligned} \tag{3.16}$$

其中 E_2 是 2×2 阶单位矩阵，J 是(3.3)中所定义的 Jacobi 矩阵。

进一步，结合(3.1)式，我们有

$$A^{-1}=\frac{1}{|J|}\begin{pmatrix}J^{-1} & 0\\ 0 & J^{-1}\end{pmatrix}\begin{pmatrix}\frac{\partial r}{\partial \eta}E_2 & -\frac{\partial r}{\partial \xi}E_2\\ -\frac{\partial z}{\partial \eta}E_2 & \frac{\partial z}{\partial \xi}E_2\end{pmatrix},\ J^{-1}=\frac{1}{|J|}\begin{pmatrix}\frac{\partial r}{\partial \eta} & -\frac{\partial r}{\partial \xi}\\ -\frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \xi}\end{pmatrix} \tag{3.17}$$

则可得到

$$A^{-1}=\frac{1}{|J|^2}\begin{pmatrix}\left(\frac{\partial r}{\partial \eta}\right)^2 & -\frac{\partial r}{\partial \xi}\frac{\partial r}{\partial \eta} & -\frac{\partial r}{\partial \xi}\frac{\partial r}{\partial \eta} & \left(\frac{\partial r}{\partial \xi}\right)^2\\ -\frac{\partial z}{\partial \eta}\frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \xi}\frac{\partial r}{\partial \eta} & \frac{\partial r}{\partial \xi}\frac{\partial z}{\partial \eta} & -\frac{\partial z}{\partial \xi}\frac{\partial r}{\partial \xi}\\ -\frac{\partial z}{\partial \eta}\frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta}\frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi}\frac{\partial r}{\partial \eta} & -\frac{\partial z}{\partial \xi}\frac{\partial r}{\partial \xi}\\ \left(\frac{\partial z}{\partial \eta}\right)^2 & -\frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \eta} & -\frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \eta} & \left(\frac{\partial z}{\partial \xi}\right)^2\end{pmatrix} \tag{3.18}$$

结合(3.14)，易得

$$\begin{pmatrix}\frac{\partial^2 \xi}{\partial z^2}\\ \frac{\partial^2 \xi}{\partial z\partial r}\\ \frac{\partial^2 \xi}{\partial r\partial z}\\ \frac{\partial^2 \xi}{\partial r^2}\end{pmatrix}=-\frac{\partial \xi}{\partial z}\cdot A^{-1}\begin{pmatrix}\frac{\partial^2 z}{\partial \xi^2}\\ \frac{\partial^2 z}{\partial \xi\partial \eta}\\ \frac{\partial^2 z}{\partial \eta\partial \xi}\\ \frac{\partial^2 z}{\partial \eta^2}\end{pmatrix}-\frac{\partial \xi}{\partial r}\cdot A^{-1}\begin{pmatrix}\frac{\partial^2 r}{\partial \xi^2}\\ \frac{\partial^2 r}{\partial \xi\partial \eta}\\ \frac{\partial^2 r}{\partial \eta\partial \xi}\\ \frac{\partial^2 r}{\partial \eta^2}\end{pmatrix} \tag{3.19}$$

(3.19)左边可化为

$$\begin{aligned}\begin{pmatrix}\frac{\partial^2 \xi}{\partial z^2}\\ \frac{\partial^2 \xi}{\partial z\partial r}\\ \frac{\partial^2 \xi}{\partial r\partial z}\\ \frac{\partial^2 \xi}{\partial r^2}\end{pmatrix}=&-\frac{1}{|J|^3}\frac{\partial r}{\partial \eta}\begin{pmatrix}\left(\frac{\partial r}{\partial \eta}\right)^2 & -\frac{\partial r}{\partial \xi}\frac{\partial r}{\partial \eta} & -\frac{\partial r}{\partial \xi}\frac{\partial r}{\partial \eta} & \left(\frac{\partial r}{\partial \xi}\right)^2\\ -\frac{\partial z}{\partial \eta}\frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \xi}\frac{\partial r}{\partial \eta} & \frac{\partial r}{\partial \xi}\frac{\partial z}{\partial \eta} & -\frac{\partial z}{\partial \xi}\frac{\partial r}{\partial \xi}\\ -\frac{\partial z}{\partial \eta}\frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta}\frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi}\frac{\partial r}{\partial \eta} & -\frac{\partial z}{\partial \xi}\frac{\partial r}{\partial \xi}\\ \left(\frac{\partial z}{\partial \eta}\right)^2 & -\frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \eta} & -\frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \eta} & \left(\frac{\partial z}{\partial \xi}\right)^2\end{pmatrix}\begin{pmatrix}\frac{\partial^2 z}{\partial \xi^2}\\ \frac{\partial^2 z}{\partial \xi\partial \eta}\\ \frac{\partial^2 z}{\partial \eta\partial \xi}\\ \frac{\partial^2 z}{\partial \eta^2}\end{pmatrix}\\ &+\frac{1}{|J|^3}\frac{\partial z}{\partial \eta}\begin{pmatrix}\left(\frac{\partial r}{\partial \eta}\right)^2 & -\frac{\partial r}{\partial \xi}\frac{\partial r}{\partial \eta} & -\frac{\partial r}{\partial \xi}\frac{\partial r}{\partial \eta} & \left(\frac{\partial r}{\partial \xi}\right)^2\\ -\frac{\partial z}{\partial \eta}\frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \xi}\frac{\partial r}{\partial \eta} & \frac{\partial r}{\partial \xi}\frac{\partial z}{\partial \eta} & -\frac{\partial z}{\partial \xi}\frac{\partial r}{\partial \xi}\\ -\frac{\partial z}{\partial \eta}\frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta}\frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi}\frac{\partial r}{\partial \eta} & -\frac{\partial z}{\partial \xi}\frac{\partial r}{\partial \xi}\\ \left(\frac{\partial z}{\partial \eta}\right)^2 & -\frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \eta} & -\frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \eta} & \left(\frac{\partial z}{\partial \xi}\right)^2\end{pmatrix}\begin{pmatrix}\frac{\partial^2 r}{\partial \xi^2}\\ \frac{\partial^2 r}{\partial \xi\partial \eta}\\ \frac{\partial^2 r}{\partial \eta\partial \xi}\\ \frac{\partial^2 r}{\partial \eta^2}\end{pmatrix}\end{aligned} \tag{3.20}$$

同理，对(2.8)和(2.9)两边分别关于 ξ,η 求偏导，可得

$$\begin{pmatrix} \frac{\partial^2\eta}{\partial z^2} \\ \frac{\partial^2\eta}{\partial z\partial r} \\ \frac{\partial^2\eta}{\partial r\partial z} \\ \frac{\partial^2\eta}{\partial r^2} \end{pmatrix} = \frac{1}{|J|^3}\frac{\partial r}{\partial\xi}\begin{pmatrix} \left(\frac{\partial r}{\partial\eta}\right)^2 & -\frac{\partial r}{\partial\xi}\frac{\partial r}{\partial\eta} & -\frac{\partial r}{\partial\xi}\frac{\partial r}{\partial\eta} & \left(\frac{\partial r}{\partial\xi}\right)^2 \\ -\frac{\partial z}{\partial\eta}\frac{\partial r}{\partial\eta} & \frac{\partial z}{\partial\xi}\frac{\partial r}{\partial\eta} & \frac{\partial r}{\partial\xi}\frac{\partial z}{\partial\eta} & -\frac{\partial z}{\partial\xi}\frac{\partial r}{\partial\xi} \\ -\frac{\partial z}{\partial\eta}\frac{\partial r}{\partial\eta} & \frac{\partial z}{\partial\eta}\frac{\partial r}{\partial\xi} & \frac{\partial z}{\partial\xi}\frac{\partial r}{\partial\eta} & -\frac{\partial z}{\partial\xi}\frac{\partial r}{\partial\xi} \\ \left(\frac{\partial z}{\partial\eta}\right)^2 & -\frac{\partial z}{\partial\xi}\frac{\partial z}{\partial\eta} & -\frac{\partial z}{\partial\xi}\frac{\partial z}{\partial\eta} & \left(\frac{\partial z}{\partial\xi}\right)^2 \end{pmatrix}\begin{pmatrix} \frac{\partial^2 z}{\partial\xi^2} \\ \frac{\partial^2 z}{\partial\xi\partial\eta} \\ \frac{\partial^2 z}{\partial\eta\partial\xi} \\ \frac{\partial^2 z}{\partial\eta^2} \end{pmatrix} - \frac{1}{|J|^3}\frac{\partial z}{\partial\xi}\begin{pmatrix} \left(\frac{\partial r}{\partial\eta}\right)^2 & -\frac{\partial r}{\partial\xi}\frac{\partial r}{\partial\eta} & -\frac{\partial r}{\partial\xi}\frac{\partial r}{\partial\eta} & \left(\frac{\partial r}{\partial\xi}\right)^2 \\ -\frac{\partial z}{\partial\eta}\frac{\partial r}{\partial\eta} & \frac{\partial z}{\partial\xi}\frac{\partial r}{\partial\eta} & \frac{\partial r}{\partial\xi}\frac{\partial z}{\partial\eta} & -\frac{\partial z}{\partial\xi}\frac{\partial r}{\partial\xi} \\ -\frac{\partial z}{\partial\eta}\frac{\partial r}{\partial\eta} & \frac{\partial z}{\partial\eta}\frac{\partial r}{\partial\xi} & \frac{\partial z}{\partial\xi}\frac{\partial r}{\partial\eta} & -\frac{\partial z}{\partial\xi}\frac{\partial r}{\partial\xi} \\ \left(\frac{\partial z}{\partial\eta}\right)^2 & -\frac{\partial z}{\partial\xi}\frac{\partial z}{\partial\eta} & -\frac{\partial z}{\partial\xi}\frac{\partial z}{\partial\eta} & \left(\frac{\partial z}{\partial\xi}\right)^2 \end{pmatrix}\begin{pmatrix} \frac{\partial^2 r}{\partial\xi^2} \\ \frac{\partial^2 r}{\partial\xi\partial\eta} \\ \frac{\partial^2 r}{\partial\eta\partial\xi} \\ \frac{\partial^2 r}{\partial\eta^2} \end{pmatrix} \tag{3.21}$$

证毕。

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