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基于两个独立的指数随机变量之和的相依风险模型的破产问题研究

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摘 要

在本文中我们考虑了经典复合泊松模型的一个推广，该模型的累计索赔额过程的增量是独立的。其中索赔间隔时间的分布是两个独立的指数随机变量之和，在本文中我们考虑了索赔时间与后续索赔规模之间的一种特殊的依赖结构，导出了Gerber-Shiu惩罚函数的积分微分方程，利用Rouché定理研究了林德伯格等式的根，导出了惩罚函数的拉普拉斯变换及其满足的瑕疵更新方程。最后在索赔时间间隔服从两个独立的指数随机变量之和分布时给出破产概率的解析解，对理论结果做数值分析，对不同相依参数下的破产概率进行对比。

关键词

两个独立的指数随机变量之和分布，积分微分方程，林德伯格等式，Gerber-Shiu期望贴现惩罚函数，拉普拉斯变换，瑕疵更新方程

Research on the Ruin Problem of Dependent Risk Model Based on the Sum of Two Independent Exponential Random Variables

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Abstract

In this paper, we consider an extension of the classical compound Poisson model where the in-

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cremental cumulative claims process is independent. The distribution of claim interval follows the sum of two independent random variables. In this paper, we consider a special dependence structure between claim time and subsequent claim scale, derive the Gerber-Shiu penalty function integral differential equation, and use Rouché's theorem to study the roots of Lundberg equation. The Laplace transform of penalty function and its defect renewal equation are derived. Finally, the analytic solution of ruin probability is given when the claim interval follows the sum distribution of two independent exponential random variables. The theoretical results are analyzed numerically, the ruin probabilities under different dependent parameters are compared.

Keywords

Sum Distribution of Two Independent Exponential Random Variables, Integral Differential Equation, Lundberg Equation, Gerber-Shiu Expected Discounted Penalty Function, The Laplace Transform, Defective Renewal Equation

1. 引言

在精算文献中有两个著名的风险模型。即经典的复合泊松风险模型和基于连续时间更新过程的风险模型(后者也称为 Sparre Andersen [1]风险模型)。已被广泛地分析了破产概率和许多破产相关的量，如破产时间的边际分布和联合缺陷分布。在这两种风险模型中，破产时的赤字，破产前的盈余和导致破产的索赔规模已经得到了广泛的研究(Dickson (1992, 1993) [2] [3]，Gerber and Shiu (1997) [4]，Dickson and Hipp (1998) [5]，Rolski *et al*. (1999) [6]和其中的参考文献)，Gerber 和 Shiu (1998) [7]在复合 Poisson 风险模型的框架内提出了一个统一的折现罚函数方法，并对 Sparre Andersen 风险模型的一些类进行了研究(见 Dickson and Hipp (2001 年) [8]，Gerber 和 Shiu (2005 年) [9]和 Li and Garrido (2004 年，2005 年) [10])。

请注意，对于这两个风险模型，明确假设两个连续索赔之间的到达间限时间和索赔金额是独立的。然而，有许多现实世界的情况下，这样的假设是不恰当的。例如，对于一个承保地震造成的损失的业务，索赔之间的时间越长，损失越大，Albrecher 和 Boxma (2004) [11]对经典复合泊松风险模型提出了一个扩展，其中两次索赔之间的时间分布取决于前一次索赔的大小。在本文中，我们考虑对经典复合泊松风险模型的一个扩展，它考虑反向依赖结构，意味着下一次索赔的分布大小取决于上次索赔后的时间。我们将看到，后者的风险模型为破产理论分析提供了一个很好的框架，因为盈余过程增量之间的独立性假设得到了保留，我们提到 Albrecher 和 Teuoels (2006) [12]考虑了区间时间内的任意相关结构和通过 Copula 表示的后续索赔额，他们得到了有限时间和无限时间破产机率的渐近结果。Zhang 和 Liu [13]在索赔相依情况下讨论了分红策略问题。

本文在构造了基于两个独立的指数随机变量之和分布的相依索赔破产概率模型。第一节介绍了风险模型，然后提出了本文研究的是相依索赔模型，间隔时间服从两个独立的指数随机变量之和分布，相依结构由两个指数分布函数复合而成。第二节建立了期望惩罚函数的微分积分方程。第三节讨论了林德伯格方程的根。第四、五节给出了惩罚函数的拉普拉斯变换及其显示表达以及瑕疵更新方程第六节基于索赔相依结构的联合密度函数和间隔时间服从两个独立的指数随机变量之和分布等假设下，得到了破产概率的显式表达式，并与索赔相依指数模型进行了对比。

2. 模型构建

在时刻 t 的盈余被定义为 $U(t)=u+ct-\sum_{k=1}^{N(t)}X_k, t\geq 0$。其中 $U(0)=u$ 表示初始盈余，c 是保费率即单位时间收取的保险费金额；$\{N(t),t\geq 0\}$ 是一个更新过程为到时刻 t 为止发生的索赔次数。

在时刻 t 的盈余被定义为 $U(t)=u+ct-\sum_{k=1}^{N(t)}X_k, t\geq 0$。其中 $U(0)=u$ 表示初始盈余，c 是保费率即单位时间收取的保险费金额；$\{N(t),t\geq 0\}$ 是一个更新过程为到时刻 t 为止发生的索赔次数，$\{X_i, i=1,2,\cdots\}$ 表示第 i 次的索赔额，第 $i-1$ 到第 i 次的索赔间隔时间用 V_i 表示，V_1 表示第一次索赔的时间。索赔额 $\{X_i, i=1,2,\cdots\}$ 是恒正的独立同分布的随机变量序列且与 $\{N(t),t\geq 0\}$ 独立，记 $\{X_i, i=1,2,\cdots\}$ 的密度函数为 f_X，分布函数为 F_X。索赔间隔时间 $\{W_i, i\in\mathbb{N}^+\}$ 形成了独立同分布的随机变量序列，$W=W_1+W_2$，W_j 是两个独立的参数为 $\lambda_j, j=1,2$ 的指数分布，W 的密度函数和分布函数分别为 f_W, F_W 当 $\lambda_1\neq\lambda_2$ 时

$$f_W(t)=\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left(\mathrm{e}^{-\lambda_1 t}-\mathrm{e}^{-\lambda_2 t}\right), F_W(t)=1-\frac{1}{\lambda_2-\lambda_1}\left(\lambda_2\mathrm{e}^{-\lambda_1 t}-\lambda_1\mathrm{e}^{-\lambda_2 t}\right), \tag{1}$$

令 $\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}=\lambda$，

$$f_W(t)=\lambda\left(\mathrm{e}^{-\lambda_1 t}-\mathrm{e}^{-\lambda_2 t}\right) \tag{2}$$

我们定义 $X_k|W_k$ 的密度函数是由两个边际密度函数 f_1 和 f_2 (期望分别为 η_1 和 η_2)复合而成的，其表达式为：

$$f_{X_k|W_k}(x)=\mathrm{e}^{-\beta W_k}f_1(x)+\left(1-\mathrm{e}^{-\beta W_k}\right)f_2(x), x\geq 0, k=1,2,\cdots,$$

其中，β 是正的常数。X_k 的边际分布函数为

$$f_{X_K}(x)=M_W(-\beta)f_1(x)+\left[1-M_W(-\beta)\right]f_2(x)$$

计算得 $M_W(-\beta)=\int_0^{+\infty}\mathrm{e}^{-\beta t}\lambda\left(\mathrm{e}^{-\lambda_1 t}-\mathrm{e}^{-\lambda_2 t}\right)\mathrm{d}t=\lambda\left(\frac{1}{\beta+\lambda_1}-\frac{1}{\beta+\lambda_2}\right)=\frac{\lambda_1\lambda_2}{(\beta+\lambda_1)(\beta+\lambda_2)}$，即

$$f_{X_K}(x)=\frac{\lambda_1\lambda_2}{(\beta+\lambda_1)(\beta+\lambda_2)}f_1(x)+\frac{\beta(\beta+\lambda_1+\lambda_2)}{(\beta+\lambda_1)(\beta+\lambda_2)}f_2(x) \tag{3}$$

为了确保破产不发生，保费率 c 应满足安全负载条件：

$$E\left[cW_j-X_j\right]>0, j=1,2,\cdots,$$

$$E\left[cW_j-X_j\right]=\frac{c(\lambda_1+\lambda_2)}{\lambda_1\lambda_2}-\frac{\lambda_1\lambda_2\eta_1+\beta(\beta+\lambda_1+\lambda_2)\eta_2}{(\beta+\lambda_1)(\beta+\lambda_2)}>0$$

在经典风险理论中，风险模型的盈余为负时的破产，是指理论上的破产。而在实际情况中，保险公司会采取一些补救的措施来缓解财务的紧张。但它仍可以作为预测保险公司财务状况的重要指标之一，破产概率代表着破产的可能性。将 T 定义为破产时间

$$T=\inf\nolimits_{t\geq 0}\{t, U(t)<0\},$$

为保险公司的破产时间，即盈余小于 0 的时刻。当 $U_t\geq 0$ 时 $\tau=\infty$，即理论上的永不破产。破产赤字定义为 $|U_\tau|$，破产前盈余为 $U_{\tau-}$。令

$$\psi(u,t)=P\{\tau<t\,|\,U(0)=u\}$$

为保险公司有限时间破产概率。令

$$\psi(u,t)=P\{\tau<\infty\,|\,U(0)=u\}$$

为无限时间破产概率，又称为最终破产概率。

3. 微分－积分方程

在这一节中我们推导了惩罚函数 $m_\delta(u)$ 的积分微分方程，通过第一次索赔的时间和金额条件，我们有

$$\begin{aligned}m_\delta(u)&=\int_0^\infty\lambda\left(\mathrm{e}^{-\lambda_1 t}-\mathrm{e}^{-\lambda_2 t}\right)\mathrm{e}^{-\delta t}\int_0^{u+ct}m_\delta(u+ct-y)\left(\mathrm{e}^{-\beta t}f_1(x)+\left(1-\mathrm{e}^{-\beta t}\right)f_2(x)\right)\mathrm{d}y\mathrm{d}t\\&+\int_0^\infty\lambda\left(\mathrm{e}^{-\lambda_1 t}-\mathrm{e}^{-\lambda_2 t}\right)\mathrm{e}^{-\delta t}\int_{u+ct}^\infty w(u+ct,y-(u+ct))\left(\mathrm{e}^{-\beta t}f_1(x)+\left(1-\mathrm{e}^{-\beta t}\right)f_2(x)\right)\mathrm{d}y\mathrm{d}t\end{aligned}\tag{4}$$

定理 1 在第二节介绍的风险模型中惩罚函数 $m_\delta(u)$ 满足下列方程

$$\alpha(D)\beta(D)m_\delta(u)=\alpha(D)\left(\sigma_1(u)-\sigma_2(u)\right)+\beta(D)\sigma_2(u)\tag{5}$$

其中

$$\alpha(D)=\left(\frac{\lambda_1+\delta}{\lambda_1}I-\frac{c}{\lambda_1}D\right)\left(\frac{\lambda_2+\delta}{\lambda_2}I-\frac{c}{\lambda_2}D\right)$$
$$\beta(D)=\left(\frac{\lambda_1+\delta+\beta}{\lambda_1}I-\frac{c}{\lambda_1}D\right)\left(\frac{\lambda_2+\delta+\beta}{\lambda_2}I-\frac{c}{\lambda_2}D\right)$$

其中 I 为恒等因子，D 为微分因子

证明：令 $u+ct=t'$

$$\begin{aligned}m_\delta(u)&=\int_u^\infty\frac{\lambda}{c}\left(\mathrm{e}^{-\lambda_1\frac{t-u}{c}}-\mathrm{e}^{-\lambda_2\frac{t-u}{c}}\right)\mathrm{e}^{-\delta\frac{t-u}{c}}\int_0^t m_\delta(t-y)\left(\mathrm{e}^{-\beta\frac{t-u}{c}}f_1(x)+\left(1-\mathrm{e}^{-\beta\frac{t-u}{c}}\right)f_2(x)\right)\mathrm{d}y\mathrm{d}t\\&+\int_u^\infty\frac{\lambda}{c}\left(\mathrm{e}^{-\lambda_1\frac{t-u}{c}}-\mathrm{e}^{-\lambda_2\frac{t-u}{c}}\right)\mathrm{e}^{-\delta\frac{t-u}{c}}\int_t^\infty w(t,y-t)\left(\mathrm{e}^{-\beta\frac{t-u}{c}}f_1(x)+\left(1-\mathrm{e}^{-\beta\frac{t-u}{c}}\right)f_2(x)\right)\mathrm{d}y\mathrm{d}t\\&=\frac{\lambda}{c}\left[\int_u^\infty\left(\mathrm{e}^{-\frac{\lambda_1+\delta+\beta}{c}(t-u)}-\mathrm{e}^{-\frac{\lambda_2+\delta+\beta}{c}(t-u)}\right)\left(\sigma_{1,\delta}(t)-\sigma_{2,\delta}(t)\right)\mathrm{d}t+\int_u^\infty\left(\mathrm{e}^{-\frac{\lambda_1+\delta}{c}(t-u)}-\mathrm{e}^{-\frac{\lambda_2+\delta}{c}(t-u)}\right)\sigma_{2,\delta}(t)\mathrm{d}t\right]\\&=\frac{\lambda}{c}\xi(u)+\frac{\lambda}{c}\eta(u)\end{aligned}$$

其中

$$\gamma_i(t)=\int_t^\infty w(t,y-t)f_i(y)\mathrm{d}y,\sigma_{i,\delta}(t)=\int_0^t m_\delta(t-y)f_i(y)\mathrm{d}y+\gamma_i(t)\tag{6}$$

$$\xi(u)=\int_u^\infty\left(\mathrm{e}^{-\frac{\lambda_1+\delta+\beta}{c}(t-u)}-\mathrm{e}^{-\frac{\lambda_2+\delta+\beta}{c}(t-u)}\right)\left(\sigma_{1,\delta}(t)-\sigma_{2,\delta}(t)\right)\mathrm{d}t\tag{7}$$

$$\eta(u)=\int_u^\infty\left(\mathrm{e}^{-\frac{\lambda_1+\delta}{c}(t-u)}-\mathrm{e}^{-\frac{\lambda_2+\delta}{c}(t-u)}\right)\sigma_{2,\delta}(t)\mathrm{d}t\tag{8}$$

$$D\xi(u)=\int_u^\infty\left(\frac{\lambda_1+\delta+\beta}{c}\mathrm{e}^{-\frac{\lambda_1+\delta+\beta}{c}(t-u)}-\frac{\lambda_2+\delta+\beta}{c}\mathrm{e}^{-\frac{\lambda_2+\delta+\beta}{c}(t-u)}\right)\left(\sigma_{1,\delta}(t)-\sigma_{2,\delta}(t)\right)\mathrm{d}t$$

$$D^2\xi(u)=\int_u^{\infty}\left(\left(\frac{\lambda_1+\delta+\beta}{c}\right)^2 \mathrm{e}^{-\frac{\lambda_1+\delta+\beta}{c}(t-u)}-\left(\frac{\lambda_2+\delta+\beta}{c}\right)^2 \mathrm{e}^{-\frac{\lambda_2+\delta+\beta}{c}(t-u)}\right)\left(\sigma_{1,\delta}(t)-\sigma_{2,\delta}(t)\right)\mathrm{d}t$$
$$-\frac{\lambda_1-\lambda_2}{c}\left(\sigma_{1,\delta}(t)-\sigma_{2,\delta}(t)\right)$$

可以得到 $\beta(D)\xi(u)=\frac{c}{\lambda}\left(\sigma_{1,\delta}(t)-\sigma_{2,\delta}(t)\right)$

$$D\eta(u)=\int_u^{\infty}\frac{\lambda_1+\delta}{c}\left(\mathrm{e}^{-\frac{\lambda_1+\delta}{c}(t-u)}-\frac{\lambda_2+\delta}{c}\mathrm{e}^{-\frac{\lambda_2+\delta}{c}(t-u)}\right)\sigma_{2,\delta}(t)\mathrm{d}t$$

$$D^2\eta(u)=\int_u^{\infty}\left(\frac{\lambda_1+\delta}{c}\right)^2\left(\mathrm{e}^{-\frac{\lambda_1+\delta}{c}(t-u)}-\left(\frac{\lambda_2+\delta}{c}\right)^2\mathrm{e}^{-\frac{\lambda_2+\delta}{c}(t-u)}\right)\sigma_{2,\delta}(t)\mathrm{d}t-\frac{\lambda_1-\lambda_2}{c}\sigma_{2,\delta}(u)$$

$$D^3\eta(u)=\int_u^{\infty}\left(\frac{\lambda_1+\delta}{c}\right)^3\left(\mathrm{e}^{-\frac{\lambda_1+\delta}{c}(t-u)}-\left(\frac{\lambda_2+\delta}{c}\right)^3\mathrm{e}^{-\frac{\lambda_2+\delta}{c}(t-u)}\right)\sigma_{2,\delta}(t)\mathrm{d}t$$
$$-\left[\left(\frac{\lambda_1+\delta}{c}\right)^2-\left(\frac{\lambda_2+\delta}{c}\right)^2\right]\sigma_{2,\delta}(u)-\frac{\lambda_1-\lambda_2}{c}D\sigma_{2,\delta}(u)$$

由上式可以得到 $\alpha(D)\eta(u)=\frac{c}{\lambda}\gamma(D)\sigma_2(u)$

$$\alpha(D)\beta(D)m_\delta(u)=\alpha(D)\beta(D)\left[\frac{\lambda}{c}\xi(u)+\frac{\lambda}{c}\eta(u)\right]=\alpha(D)\left(\sigma_1(u)-\sigma_2(u)\right)+\beta(D)\sigma_2(u)$$

证毕。

4. 林德伯格等式

令 $U_0=u$，在 k 次索赔之后的盈余则为：

$$U_k=u+\sum_{i=1}^{k}\left(cW_i-X_i\right),k=1,2,\cdots,$$

当 $s>0$，过程 $\left\{\mathrm{e}^{-\delta\sum_{i=1}^{k}W_i+sU_k},k=0,1,2,\cdots\right\}$ 是一个鞅，当且仅当

$$E\left[\mathrm{e}^{-\delta W}\mathrm{e}^{s(cW-X)}\right]=1,$$

这个等式被称为林德伯格等式，计算整理可得

$$\begin{aligned}E\left[\mathrm{e}^{-\delta W}\mathrm{e}^{-S(X-CW)}\right]&=E\left[\mathrm{e}^{-(\delta-cs)W}E\left[\mathrm{e}^{-sX}\mid W\right]\right]\\&=E\left[\mathrm{e}^{-(\delta-cs)W}\left(\mathrm{e}^{-\beta W}\hat{f}_1(x)+\left(1-\mathrm{e}^{-\beta W}\right)\hat{f}_2(x)\right)\right]\\&=E\left[\mathrm{e}^{-(\beta+\delta-cs)W}\right]\hat{f}_1(x)+E\left[\mathrm{e}^{-(\delta-cs)W}\right]\hat{f}_2(x)-E\left[\mathrm{e}^{-(\beta+\delta-cs)W}\right]\hat{f}_2(x)\\&=\frac{\lambda_1\lambda_2}{(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)}\hat{f}_1(x)\\&\quad+\left(\frac{\lambda_1\lambda_2}{(\delta+\lambda_1-cs)(\delta+\lambda_2-cs)}-\frac{\lambda_1\lambda_2}{(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)}\right)\hat{f}_2(x)\\&=\lambda_1\lambda_2\frac{(\delta+\lambda_1-cs)(\delta+\lambda_2-cs)\hat{f}_1(x)+\beta(\beta+2\delta+\lambda_1+\lambda_2-2cs)\hat{f}_2(x)}{(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)(\delta+\lambda_1-cs)(\delta+\lambda_2-cs)}\end{aligned}\tag{9}$$

即

$$\frac{\lambda_1\lambda_2}{c^2}\frac{\left(\frac{\delta+\lambda_1}{c}-s\right)\left(\frac{\delta+\lambda_2}{c}-s\right)\hat{f}_1(x)+\left[\left(\frac{\beta+\delta+\lambda_1}{c}-s\right)\left(\frac{\beta+\delta+\lambda_2}{c}-s\right)-\left(\frac{\delta+\lambda_1}{c}-s\right)\left(\frac{\delta+\lambda_2}{c}-s\right)\right]\hat{f}_2(x)}{\left(\frac{\beta+\delta+\lambda_1}{c}-s\right)\left(\frac{\beta+\delta+\lambda_2}{c}-s\right)\left(\frac{\delta+\lambda_1}{c}-s\right)\left(\frac{\delta+\lambda_2}{c}-s\right)}=1$$

整理可得：

$$\frac{\lambda_1\lambda_2}{(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)}=\left(\frac{\beta+\delta+\lambda_1}{\lambda_1}-\frac{c}{\lambda_1}s\right)^{-1}\left(\frac{\beta+\delta+\lambda_2}{\lambda_2}-\frac{c}{\lambda_2}s\right)^{-1}=\frac{1}{\beta(s)}$$

$$\begin{aligned}&\frac{\lambda_1\lambda_2\beta(\beta+2\delta+\lambda_1+\lambda_2-2cs)}{(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)(\delta+\lambda_1-cs)(\delta+\lambda_2-cs)}\\&=\frac{\lambda_1\lambda_2}{(\delta+\lambda_1-cs)(\delta+\lambda_2-cs)}-\frac{\lambda_1\lambda_2}{(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)}=\frac{1}{\alpha(s)}-\frac{1}{\beta(s)}\end{aligned}$$

上式整理后得 $\hat{f}_1(s)\frac{1}{\beta(s)}+\hat{f}_2(s)\left(\frac{1}{\alpha(s)}-\frac{1}{\beta(s)}\right)=1$ 即

$$\alpha(s)\beta(s)-\alpha(s)\hat{f}_1(s)-(\beta(s)-\alpha(s))\hat{f}_2(s)=0 \tag{10}$$

下面我们将应用 Rouché’s 定理来求解林德伯格等式根。

定理 2 对于 $\delta>0$，我们要证明林德伯格等式有四个根，这四个根分别记为 $\rho_1(\delta)$，$\rho_2(\delta)$，$\rho_3(\delta)$，$\rho_4(\delta)$，$Re(\rho_i(\delta))>0, i=1,2,3,4$。

证明：林德伯格等式可以写成

$$\begin{aligned}&(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)(\delta+\lambda_1-cs)(\delta+\lambda_2-cs)\\&=\lambda_1\lambda_2(\delta+\lambda_1-cs)(\delta+\lambda_2-cs)\hat{f}_1(s)+\left[(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)-(\delta+\lambda_1-cs)(\delta+\lambda_2-cs)\right]\hat{f}_2(s)\end{aligned}$$

可以看出上面的等式有四个实部大于零的根，令 $r>0$，C_r 表示 Contour 集，包含 $-ir$ 到 ir 的虚轴和一个半径为 r 的半圈，顺时针方向从 $-ir$ 到 ir，即 $C_r=\{s\in C:|s|=r, Re(s)\geq 0, r>0\}$。令 $r\to\infty$，用 C 表示有限 Contour 集。我们在封闭 Contour 集 C 上应用 Rouché’s 定理证明结果。

1) 当 $Re(s)>0$，即当 s 在半圆上时，当 $r\to\infty$，我们有 $|\beta+\delta+\lambda_1-cs|\to\infty$，$|\beta+\delta+\lambda_2-cs|\to\infty$，$|\delta+\lambda_1-cs|\to\infty$，$|\delta+\lambda_2-cs|\to\infty$，因此在 C 上有

$$\begin{aligned}&\left|\frac{\lambda_1\lambda_2}{(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)}\hat{f}_1(s)+\left(\frac{\lambda_1\lambda_2}{(\delta+\lambda_1-cs)(\delta+\lambda_2-cs)}\right.\right.\\&\left.\left.-\frac{\lambda_1\lambda_2}{(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)}\right)\hat{f}_2(s)\right|\\&\leq\left|\hat{f}_1(s)\right|\left|\frac{\lambda_1\lambda_2}{(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)}\right|\\&\quad+\left|\hat{f}_2(s)\right|\left|\frac{\lambda_1\lambda_2}{(\delta+\lambda_1-cs)(\delta+\lambda_2-cs)}-\frac{\lambda_1\lambda_2}{(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)}\right|\to 0\end{aligned} \tag{11}$$

因此 $\left|\frac{\lambda_1\lambda_2}{(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)}\hat{f}_1(s)+\left(\frac{\lambda_1\lambda_2}{(\delta+\lambda_1-cs)(\delta+\lambda_2-cs)}-\frac{\lambda_1\lambda_2}{(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)}\right)\hat{f}_2(s)\right|<1$ 在 C 上成立

2) $Re(s)=0$，即 s 在虚轴上时，对 $\delta>0$，令

$$\hat{d}_\delta(s)=\frac{\lambda_1\lambda_2}{(\delta+\lambda_1-cs)(\delta+\lambda_2-cs)}-\frac{\lambda_1\lambda_2}{(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)}$$

$$\begin{aligned}\left|\hat{d}_\delta(s)\right|&=\left|\frac{\lambda_1\lambda_2}{(\delta+\lambda_1-cs)(\delta+\lambda_2-cs)}-\frac{\lambda_1\lambda_2}{(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)}\right|\\&=\lambda_1\lambda_2\left|\frac{\beta(\beta+2\delta+\lambda_1+\lambda_2-2cs)}{(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)(\delta+\lambda_1-cs)(\delta+\lambda_2-cs)}\right|\\&\le\lambda_1\lambda_2\beta\left|\frac{(\beta+2\delta+\lambda_1+\lambda_2)^2+(2cs)^2}{(\beta+\delta+\lambda_1)(\beta+\delta+\lambda_2)(\delta+\lambda_1)(\delta+\lambda_2)(\beta+2\delta+\lambda_1+\lambda_2)}\right|\\&\le\lambda_1\lambda_2\beta\left|\frac{(\beta+2\delta+\lambda_1+\lambda_2)^2}{(\beta+\delta+\lambda_1)(\beta+\delta+\lambda_2)(\delta+\lambda_1)(\delta+\lambda_2)(\beta+2\delta+\lambda_1+\lambda_2)}\right|=\hat{d}_\delta(0)\end{aligned}\tag{12}$$

此时

$$\begin{aligned}&\left|\frac{\lambda_1\lambda_2}{(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)}\hat{f}_1(s)+\left(\frac{\lambda_1\lambda_2}{(\delta+\lambda_1-cs)(\delta+\lambda_2-cs)}-\frac{\lambda_1\lambda_2}{(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)}\right)\hat{f}_2(s)\right|\\&=\left|\frac{\lambda_1\lambda_2}{(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)}\hat{f}_1(s)+\hat{f}_2(s)\hat{d}_\delta(s)\right|\\&\le\left|\frac{\lambda_1\lambda_2}{(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)}\right|+\left|\hat{d}_\delta(s)\right|\\&\le\left|\frac{\lambda_1\lambda_2}{(\beta+\delta+\lambda_1)(\beta+\delta+\lambda_2)}\right|+\left|\hat{d}_\delta(0)\right|\end{aligned}$$

因为 $\delta>0,\hat{d}_\delta(0)>0$，所以 $\frac{\lambda_1\lambda_2}{(\delta+\lambda_1)(\delta+\lambda_2)}-\frac{\lambda_1\lambda_2}{(\beta+\delta+\lambda_1)(\beta+\delta+\lambda_2)}>0$，因此，当 s 在虚轴上时，$\delta>0$ 等式变为

$$\begin{aligned}&\left|\frac{\lambda_1\lambda_2}{(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)}\hat{f}_1(s)+\left(\frac{\lambda_1\lambda_2}{(\delta+\lambda_1-cs)(\delta+\lambda_2-cs)}-\frac{\lambda_1\lambda_2}{(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)}\right)\hat{f}_2(s)\right|\\&\le\left|\frac{\lambda_1\lambda_2}{(\beta+\delta+\lambda_1)(\beta+\delta+\lambda_2)}\right|+\left|\hat{d}_\delta(0)\right|<1\end{aligned}$$

综上所述，我们证明出在这两种情况下不等式

$$\left|\lambda_1\lambda_2(\delta+\lambda_1-cs)(\delta+\lambda_2-cs)\hat{f}_1(s)+\left[(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)-(\delta+\lambda_1-cs)(\delta+\lambda_2-cs)\right]\hat{f}_2(s)\right|$$
$$<(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)(\delta+\lambda_1-cs)(\delta+\lambda_2-cs)$$

应用 Rouché's 定理，可知等式与 $(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)(\delta+\lambda_1-cs)(\delta+\lambda_2-cs)=0$ 在 C_r 内有相同个数的根，因此林德伯格等式在 C_r 内也有四个实部为正数的根，证毕。

当 $\delta=0$，由于

$$\begin{aligned}&\left|\frac{\lambda_1\lambda_2}{(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)}\hat{f}_1(s)+\left(\frac{\lambda_1\lambda_2}{(\delta+\lambda_1-cs)(\delta+\lambda_2-cs)}\right.\right.\\&\left.\left.-\frac{\lambda_1\lambda_2}{(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)}\right)\hat{f}_2(s)\right|\\&=\left|\frac{\lambda_1\lambda_2}{(\beta+\lambda_1)(\beta+\lambda_2)}+1-\frac{\lambda_1\lambda_2}{(\beta+\lambda_1)(\beta+\lambda_2)}\right|=1\end{aligned}\tag{13}$$

Rouché's 定理不再满足，可以利用 Klimenok (2007)中的方法来求解林德伯格方程中根的个数。

定理 3 当 $\delta=0$ 时，林德伯格等式有三个正实部的根，记为 $\rho_1(0),\rho_2(0),\rho_3(0)$，还有一个根为 0。

证明：定义 Contour 集 $D_k=s:|z|=1$，令 $z=1-\frac{s}{k}$。根据 s，Contour 集 D_k 是一个圆心为 k，半径为 k 的圆。与定理 2 类似，此时令 $k\to\infty$，记 D 是有限 Contour 集，函数 $\lambda_1\lambda_2(\delta+\lambda_1-cs)(\delta+\lambda_2-cs)\hat{f}_1(s)+\left[(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)-(\delta+\lambda_1-cs)(\delta+\lambda_2-cs)\right]\hat{f}_2(s)$ 和 $(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)(\delta+\lambda_1-cs)(\delta+\lambda_2-cs)$ 在 D 上都连续。根据 Klimenok (2001)中的定理 1，需证：

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}z}\left\{1-\frac{\lambda_1\lambda_2}{(\beta+\lambda_1-ck(1-z))(\beta+\lambda_2-ck(1-z))}\hat{f}_1(k-kz)\left[\frac{\lambda_1\lambda_2}{(\lambda_1-ck(1-z))(\lambda_2-ck(1-z))}\right.\right.\\&\left.\left.\left.-\frac{\lambda_1\lambda_2}{(\beta+\lambda_1-ck(1-z))(\beta+\lambda_2-ck(1-z))}\right]\hat{f}_2(k-kz)\right\}\right|_{z=1}>0\\&=\frac{\mathrm{d}}{\mathrm{d}z}\left\{1-E\left[\mathrm{e}^{(k-kz)(cW-X)}\right]\right\}\bigg|_{z=1}=kE(cW-X)>0\end{aligned}$$

由于 $E(cW-X)>0$ 成立，因此上述证明成立。由参考文献 Klimenok (2001)可推出在 D 中等式根有 3 个，也就是 $(\beta+\delta+\lambda_1-cs)(\beta+\delta+\lambda_2-cs)(\delta+\lambda_1-cs)(\delta+\lambda_2-cs)$ 根的个数减少一个，且知有一根为 0。

5. 惩罚函数的拉普拉斯变换

设惩罚函数的拉普拉斯变换为 $\hat{m}_\delta(s)$，即 $\hat{m}_\delta(s)=\int_0^\infty \mathrm{e}^{-su}m_\delta(u)\mathrm{d}u$ 对于 $u\ge 0$

$$cm_\delta(u)=\lambda\left[\int_u^\infty\left(\mathrm{e}^{-\frac{\lambda_1+\delta+\beta}{c}(t-u)}-\mathrm{e}^{-\frac{\lambda_2+\delta+\beta}{c}(t-u)}\right)(\sigma_{1,\delta}(t)-\sigma_{2,\delta}(t))\mathrm{d}t+\int_u^\infty\left(\mathrm{e}^{-\frac{\lambda_1+\delta}{c}(t-u)}-\mathrm{e}^{-\frac{\lambda_2+\delta}{c}(t-u)}\right)\sigma_{2,\delta}(t)\mathrm{d}t\right]$$

其拉普拉斯变换为

$$
\begin{aligned}
c\hat{m}_\delta(s) &= \lambda\int_0^\infty e^{-\frac{\lambda_1+\delta+\beta}{c}t}\left(\sigma_{1,\delta}(t)-\sigma_{2,\delta}(t)\right)\int_0^t e^{-\left(s-\frac{\lambda_1+\delta+\beta}{c}\right)u}\mathrm{d}u\mathrm{d}t \\
&\quad -\lambda\int_0^\infty e^{-\frac{\lambda_2+\delta+\beta}{c}t}\left(\sigma_{1,\delta}(t)-\sigma_{2,\delta}(t)\right)\int_0^t e^{-\left(s-\frac{\lambda_2+\delta+\beta}{c}\right)u}\mathrm{d}u\mathrm{d}t \\
&\quad +\lambda\int_0^\infty e^{-\frac{\lambda_1+\delta}{c}t}\sigma_{2,\delta}(t)\int_0^t e^{-\left(s-\frac{\lambda_1+\delta}{c}\right)u}\mathrm{d}u\mathrm{d}t-\lambda\int_0^\infty e^{-\frac{\lambda_2+\delta}{c}t}\sigma_{2,\delta}(t)\int_0^t e^{-\left(s-\frac{\lambda_2+\delta}{c}\right)u}\mathrm{d}u\mathrm{d}t
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
c\hat{m}_\delta(s) &= \lambda\left[\frac{1}{-\left(s-\frac{\lambda_1+\delta+\beta}{c}\right)}-\frac{1}{-\left(s-\frac{\lambda_2+\delta+\beta}{c}\right)}\right]\left(\hat{\sigma}_{1,\delta}(s)-\hat{\sigma}_{2,\delta}(s)\right) \\
&\quad +\lambda\frac{1}{s-\frac{\lambda_1+\delta+\beta}{c}}\int_0^\infty e^{-\frac{\lambda_1+\delta+\beta}{c}t}\left(\sigma_{1,\delta}(t)-\sigma_{2,\delta}(t)\right)\mathrm{d}t \\
&\quad -\lambda\frac{1}{s-\frac{\lambda_2+\delta+\beta}{c}}\int_0^\infty e^{-\frac{\lambda_2+\delta+\beta}{c}t}\left(\sigma_{1,\delta}(t)-\sigma_{2,\delta}(t)\right)\mathrm{d}t \\
&\quad +\lambda\left[\frac{1}{-\left(s-\frac{\lambda_1+\delta}{c}\right)}-\frac{1}{-\left(s-\frac{\lambda_2+\delta}{c}\right)}\right]\hat{\sigma}_{2,\delta}(s) \\
&\quad +\lambda\frac{1}{s-\frac{\lambda_1+\delta}{c}}\int_0^\infty e^{-\frac{\lambda_1+\delta}{c}t}\sigma_{2,\delta}(t)\mathrm{d}t-\lambda\frac{1}{s-\frac{\lambda_2+\delta}{c}}\int_0^\infty e^{-\frac{\lambda_2+\delta}{c}t}\sigma_{2,\delta}(t)\mathrm{d}t
\end{aligned}
\tag{15}
$$

令

$$
\begin{aligned}
\hat{B}_\delta(s) &= \lambda\int_0^\infty\left(\sigma_{1,\delta}(t)-\sigma_{2,\delta}(t)\right)\left[\frac{1}{s-\frac{\lambda_1+\delta+\beta}{c}}e^{-\frac{\lambda_1+\delta+\beta}{c}t}-\frac{1}{s-\frac{\lambda_2+\delta+\beta}{c}}e^{-\frac{\lambda_2+\delta+\beta}{c}t}\right]\mathrm{d}t \\
&\quad +\lambda\int_0^\infty\sigma_{2,\delta}(t)\left[\frac{1}{s-\frac{\lambda_1+\delta}{c}}e^{-\frac{\lambda_1+\delta}{c}t}-\frac{1}{s-\frac{\lambda_2+\delta}{c}}e^{-\frac{\lambda_2+\delta}{c}t}\right]\mathrm{d}t
\end{aligned}
$$

其中

$$
\hat{\gamma}_i(s)=\int_0^\infty e^{-su}\gamma_i(u)\mathrm{d}u,\ i=1,2. \tag{16}
$$

$$
\hat{\sigma}_{i,\delta}(s)=\hat{m}_\delta(s)\hat{f}_i(s)+\hat{\gamma}_i(s),\ i=1,2. \tag{17}
$$

因此

$$
\begin{aligned}
c^2\hat{m}_\delta(s) &= \lambda_1\lambda_2\frac{1}{\left(s-\frac{\lambda_1+\delta+\beta}{c}\right)\left(s-\frac{\lambda_2+\delta+\beta}{c}\right)}\left(\hat{\sigma}_{1,\delta}(s)-\hat{\sigma}_{2,\delta}(s)\right) \\
&\quad +\lambda_1\lambda_2\frac{1}{\left(s-\frac{\lambda_1+\delta}{c}\right)\left(s-\frac{\lambda_2+\delta}{c}\right)}\hat{\sigma}_{2,\delta}(s)+\hat{B}_\delta(s)
\end{aligned}
$$

$$c^2\hat{m}_\delta(s)=\lambda_1\lambda_2\frac{1}{\left(s-\frac{\lambda_1+\delta+\beta}{c}\right)\left(s-\frac{\lambda_2+\delta+\beta}{c}\right)}\left\{\hat{m}_\delta(s)\left[\hat{f}_1(s)-\hat{f}_2(s)\right]+\hat{\gamma}_1(s)-\hat{\gamma}_2(s)\right\}$$
$$+\lambda_1\lambda_2\frac{1}{\left(s-\frac{\lambda_1+\delta}{c}\right)\left(s-\frac{\lambda_2+\delta}{c}\right)}\left[\hat{m}_\delta(s)\hat{f}_2(s)+\hat{\gamma}_2(s)\right]+\hat{B}_\delta(s)$$

整理可得

$$\hat{m}_\delta(s)\left\{c^2-\lambda_1\lambda_2\frac{1}{\left(s-\frac{\lambda_1+\delta+\beta}{c}\right)\left(s-\frac{\lambda_2+\delta+\beta}{c}\right)}\left[\hat{f}_1(s)-\hat{f}_2(s)\right]-\lambda_1\lambda_2\frac{1}{\left(s-\frac{\lambda_1+\delta}{c}\right)\left(s-\frac{\lambda_2+\delta}{c}\right)}\hat{f}_2(s)\right\}$$
$$=\lambda_1\lambda_2\frac{1}{\left(s-\frac{\lambda_1+\delta+\beta}{c}\right)\left(s-\frac{\lambda_2+\delta+\beta}{c}\right)}\left[\hat{\gamma}_1(s)-\hat{\gamma}_2(s)\right]+\lambda_1\lambda_2\frac{1}{\left(s-\frac{\lambda_1+\delta}{c}\right)\left(s-\frac{\lambda_2+\delta}{c}\right)}\hat{\gamma}_2(s)+\hat{B}_\delta(s)$$

上式两边同乘 $\frac{1}{c^2}\left(s-\frac{\lambda_1+\delta+\beta}{c}\right)\left(s-\frac{\lambda_2+\delta+\beta}{c}\right)\left(s-\frac{\lambda_1+\delta}{c}\right)\left(s-\frac{\lambda_2+\delta}{c}\right)$ 等式变为

$$\begin{aligned}&\hat{m}_\delta(s)\left\{\left(s-\frac{\lambda_1+\delta+\beta}{c}\right)\left(s-\frac{\lambda_2+\delta+\beta}{c}\right)\left(s-\frac{\lambda_1+\delta}{c}\right)\left(s-\frac{\lambda_2+\delta}{c}\right)\right.\\&\left.-\frac{\lambda_1\lambda_2}{c^2}\left(s-\frac{\lambda_1+\delta}{c}\right)\left(s-\frac{\lambda_2+\delta}{c}\right)\left[\hat{f}_1(s)-\hat{f}_2(s)\right]-\frac{\lambda_1\lambda_2}{c^2}\left(s-\frac{\lambda_1+\delta+\beta}{c}\right)\left(s-\frac{\lambda_2+\delta+\beta}{c}\right)\hat{f}_2(s)\right\}\\&=\frac{\lambda_1\lambda_2}{c^2}\left(s-\frac{\lambda_1+\delta}{c}\right)\left(s-\frac{\lambda_2+\delta}{c}\right)\left[\hat{\gamma}_1(s)-\hat{\gamma}_2(s)\right]+\frac{\lambda_1\lambda_2}{c^2}\left(s-\frac{\lambda_1+\delta+\beta}{c}\right)\left(s-\frac{\lambda_2+\delta+\beta}{c}\right)\hat{\gamma}_2(s)\\&+\frac{1}{c^2}\left(s-\frac{\lambda_1+\delta+\beta}{c}\right)\left(s-\frac{\lambda_2+\delta+\beta}{c}\right)\left(s-\frac{\lambda_1+\delta}{c}\right)\left(s-\frac{\lambda_2+\delta}{c}\right)\hat{B}_\delta(s)\end{aligned} \tag{18}$$

为书写美观我们令

$$A=\left(s-\frac{\lambda_1+\delta+\beta}{c}\right)\left(s-\frac{\lambda_2+\delta+\beta}{c}\right),B=\left(s-\frac{\lambda_1+\delta}{c}\right)\left(s-\frac{\lambda_2+\delta}{c}\right)$$

将上式整理后我们可以得到

$$\hat{m}_\delta(s)=\frac{\frac{\lambda_1\lambda_2}{c^2}B\left[\hat{\gamma}_1(s)-\hat{\gamma}_2(s)\right]+\frac{\lambda_1\lambda_2}{c^2}A\hat{\gamma}_2(s)+\frac{1}{c^2}AB\hat{B}_\delta(s)}{AB-\frac{\lambda_1\lambda_2}{c^2}B\left[\hat{f}_1(s)-\hat{f}_2(s)\right]-\frac{\lambda_1\lambda_2}{c^2}A\hat{f}_2(s)}$$

即得到了惩罚函数拉普拉斯变换的表达式。

定理 4 在此 Erlang(2)相依结构风险模型中，Gerber-Shiu 惩罚函数的拉普拉斯变换 $\hat{m}_\delta(s)$ 的表达式为：

$$\hat{m}_\delta(s)=\frac{\hat{\beta}_{1,\delta}(s)+\hat{\beta}_{2,\delta}(s)}{\hat{h}_{1,\delta}(s)-\hat{h}_{2,\delta}(s)},$$

其中

$$\hat{h}_{1,\delta}(s)=\left(s-\frac{\lambda_1+\delta+\beta}{c}\right)\left(s-\frac{\lambda_2+\delta+\beta}{c}\right)\left(s-\frac{\lambda_1+\delta}{c}\right)\left(s-\frac{\lambda_2+\delta}{c}\right)$$

$$\hat{h}_{2,\delta}(s)=\frac{\lambda_1\lambda_2}{c^2}\left(s-\frac{\lambda_1+\delta}{c}\right)\left(s-\frac{\lambda_2+\delta}{c}\right)\left[\hat{f}_1(s)-\hat{f}_2(s)\right]+\frac{\lambda_1\lambda_2}{c^2}\left(s-\frac{\lambda_1+\delta+\beta}{c}\right)\left(s-\frac{\lambda_2+\delta+\beta}{c}\right)\hat{f}_2(s)$$

$$\hat{\beta}_{1,\delta}(s)=\frac{\lambda_1\lambda_2}{c^2}\left(s-\frac{\lambda_1+\delta}{c}\right)\left(s-\frac{\lambda_2+\delta}{c}\right)\left[\hat{\gamma}_1(s)-\hat{\gamma}_2(s)\right]+\frac{\lambda_1\lambda_2}{c^2}\left(s-\frac{\lambda_1+\delta+\beta}{c}\right)\left(s-\frac{\lambda_2+\delta+\beta}{c}\right)\hat{\gamma}_2(s)$$

$$\hat{\beta}_{2,\delta}(s)=-\sum_{j=1}^{4}\hat{\beta}_{1,\delta}(\rho_j)\prod_{k=1,k\neq j}^{4}\frac{s-\rho_k}{\rho_j-\rho_k}.$$

证明：根据 $\hat{m}_\delta(s)$ 的表达式我们可以得到上式表达，其中

$$\begin{aligned}
\hat{\beta}_{2,\delta}(s)&=\frac{1}{c^2}\hat{h}_{1,\delta}(s)\hat{B}_\delta(s)\\
&=\frac{1}{c^2}\left(s-\frac{\lambda_1+\delta+\beta}{c}\right)\left(s-\frac{\lambda_2+\delta+\beta}{c}\right)\left(s-\frac{\lambda_1+\delta}{c}\right)\left(s-\frac{\lambda_2+\delta}{c}\right)\frac{\lambda}{s-\frac{\lambda_1+\delta+\beta}{c}}\int_0^\infty\left(\sigma_{1,\delta}(t)-\sigma_{2,\delta}(t)\right)e^{-\frac{\lambda_1+\delta+\beta}{c}t}dt\\
&-\frac{1}{c^2}\left(s-\frac{\lambda_1+\delta+\beta}{c}\right)\left(s-\frac{\lambda_2+\delta+\beta}{c}\right)\left(s-\frac{\lambda_1+\delta}{c}\right)\left(s-\frac{\lambda_2+\delta}{c}\right)\frac{\lambda}{s-\frac{\lambda_2+\delta+\beta}{c}}\int_0^\infty\left(\sigma_{1,\delta}(t)-\sigma_{2,\delta}(t)\right)e^{-\frac{\lambda_2+\delta+\beta}{c}t}dt\\
&+\frac{1}{c^2}\left(s-\frac{\lambda_1+\delta+\beta}{c}\right)\left(s-\frac{\lambda_2+\delta+\beta}{c}\right)\left(s-\frac{\lambda_1+\delta}{c}\right)\left(s-\frac{\lambda_2+\delta}{c}\right)\frac{\lambda}{s-\frac{\lambda_1+\delta}{c}}\int_0^\infty\left(\sigma_{1,\delta}(t)-\sigma_{2,\delta}(t)\right)e^{-\frac{\lambda_1+\delta}{c}t}dt\\
&-\frac{1}{c^2}\left(s-\frac{\lambda_1+\delta+\beta}{c}\right)\left(s-\frac{\lambda_2+\delta+\beta}{c}\right)\left(s-\frac{\lambda_1+\delta}{c}\right)\left(s-\frac{\lambda_2+\delta}{c}\right)\frac{\lambda}{s-\frac{\lambda_2+\delta}{c}}\int_0^\infty\left(\sigma_{1,\delta}(t)-\sigma_{2,\delta}(t)\right)e^{-\frac{\lambda_2+\delta}{c}t}dt\\
&=\frac{1}{c^2}\left(s-\frac{\lambda_1+\delta+\beta}{c}\right)\left(s-\frac{\lambda_2+\delta+\beta}{c}\right)\left(s-\frac{\lambda_1+\delta}{c}\right)\left(s-\frac{\lambda_2+\delta}{c}\right)\left[\hat{\mu}_1\left(\frac{\lambda_1+\delta+\beta}{c}\right)-\hat{\mu}_2\left(\frac{\lambda_2+\delta+\beta}{c}\right)\right]\\
&+\frac{1}{c^2}\left(s-\frac{\lambda_1+\delta+\beta}{c}\right)\left(s-\frac{\lambda_2+\delta+\beta}{c}\right)\left(s-\frac{\lambda_1+\delta}{c}\right)\left(s-\frac{\lambda_2+\delta}{c}\right)\left[\hat{\delta}_1\left(\frac{\lambda_1+\delta}{c}\right)-\hat{\delta}_2\left(\frac{\lambda_2+\delta}{c}\right)\right]
\end{aligned}$$

$$\hat{\mu}_j\left(\frac{\lambda_j+\delta+\beta}{c}\right)=\frac{\lambda}{s-\frac{\lambda_j+\delta+\beta}{c}}\int_0^\infty\left(\sigma_{1,\delta}(t)-\sigma_{2,\delta}(t)\right)e^{-\frac{\lambda_j+\delta+\beta}{c}t}dt$$

$$\hat{\delta}_j\left(\frac{\lambda_j+\delta}{c}\right)=\frac{\lambda}{s-\frac{\lambda_j+\delta}{c}}\int_0^\infty\left(\sigma_{1,\delta}(t)-\sigma_{2,\delta}(t)\right)e^{-\frac{\lambda_j+\delta}{c}t}dt\qquad(j=1,2)$$

林德伯格等式等价于 $\hat{h}_{1,\delta}(s)-\hat{h}_{2,\delta}(s)=0$，而 $\rho_j's, j=1,2,3,4$ 是等式 $\hat{h}_{1,\delta}(s)-\hat{h}_{2,\delta}(s)=0$ 中分母的根。因为是在 $Re(s)>0$ 的情况下分析的 $\hat{m}_\delta(s)$ 即 $\rho_j's, j=1,2,3,4$ 也是等式 $\hat{m}_\delta(s)=\frac{\hat{\beta}_{1,\delta}(s)+\hat{\beta}_{2,\delta}(s)}{\hat{h}_{1,\delta}(s)-\hat{h}_{2,\delta}(s)}$ 的根，因此就有 $\hat{\beta}_{2,\delta}(\rho_j)=-\hat{\beta}_{1,\delta}(\rho_j), j=1,2,3,4$。又 $\hat{\beta}_{2,\delta}(s)$ 是一个三次多项式，通过在 $\rho_1,\rho_2,\rho_3,\rho_4$ 四点处的拉格朗日插值定理，得到

$$\hat{\beta}_{2,\delta}(s)=\sum_{j=1}^{4}\hat{\beta}_{2,\delta}(\rho_j)\prod_{k=1,k\neq j}^{4}\frac{s-\rho_k}{\rho_j-\rho_k}=-\sum_{j=1}^{4}\hat{\beta}_{1,\delta}(\rho_j)\prod_{k=1,k\neq j}^{4}\frac{s-\rho_k}{\rho_j-\rho_k},\tag{19}$$

证毕。

定理 5 Gerber-Shiu 惩罚函数 $m_\delta(u)$ 的拉普拉斯变换另一表达式为:

$$\hat{m}_\delta(s)=\frac{T_sT_{\rho_1}\cdots T_{\rho_4}\beta_{1,\delta}(0)}{1-T_sT_{\rho_1}\cdots T_{\rho_4}h_{2,\delta}(0)}. \tag{20}$$

证明. 根据 Dickson-Hipp operator 的性质，即：

$$T_sT_{\rho_1}\cdots T_{\rho_j}f(0)=(-1)^j\left[\frac{\hat{f}(s)}{\pi_j(s)}-\sum_{i=1}^{j}\frac{\hat{f}(\rho_i)}{(s-\rho_i)\pi_j'(\rho_i)}\right],$$

其中 $T_sf(0)=\hat{f}(s),\pi_j(s)=\prod_{i=1}^{j}(s-\rho_i)$。

使用拉格朗日插值公式，可得

$$\hat{\beta}_{1,\delta}(s)+\hat{\beta}_{2,\delta}(s)=\hat{\beta}_{1,\delta}(s)-\sum_{j=1}^{4}\hat{\beta}_{1,\delta}(\rho_j)\prod_{k=1,k\neq j}^{4}\frac{s-\rho_k}{\rho_j-\rho_k},$$

令 $\pi_4(s)=\prod_{j=1}^{4}(s-\rho_j)$，则有

$$\hat{\beta}_{1,\delta}(s)-\sum_{j=1}^{4}\hat{\beta}_{1,\delta}(\rho_j)\prod_{k=1,k\neq j}^{4}\frac{s-\rho_k}{\rho_j-\rho_k}=\pi_4(s)\left\{\frac{\hat{\beta}_{1,\delta}(s)}{\pi_4(s)}-\sum_{j=1}^{4}\frac{\hat{\beta}_{1,\delta}(\rho_j)}{(s-\rho_j)\pi'(\rho_j)}\right\},$$

应用 Dickson-Hipp operator 的性质就可以得到：

$$\hat{\beta}_{1,\delta}(s)+\hat{\beta}_{2,\delta}(s)=\pi_4(s)\left\{\frac{\hat{\beta}_{1,\delta}(s)}{\pi_4(s)}-\sum_{j=1}^{4}\frac{\hat{\beta}_{1,\delta}(\rho_j)}{(s-\rho_j)\pi'(\rho_j)}\right\}=\pi_4(s)T_sT_{\rho_1}\cdots T_{\rho_4}\beta_{1,\delta}(0). \tag{21}$$

再次应用拉格朗日插值公式，则有

$$\hat{h}_{1,\delta}(s)-\hat{h}_{2,\delta}(s)=\pi_4(s)\left[\frac{\hat{h}_{1,\delta}(0)}{\pi_4(0)}-\sum_{j=1}^{4}\frac{\hat{h}_{2,\delta}(\rho_j)}{(-\rho_j)\pi_4'(\rho_j)}+\sum_{j=1}^{4}\frac{\hat{h}_{2,\delta}(\rho_j)}{(s-\rho_j)\pi_4'(\rho_j)}-\frac{\hat{h}_{2,\delta}(s)}{\pi_4(s)}\right],$$

因此当 $s=0$ 时，有 $\hat{h}_{2,\delta}(\rho_j)=\hat{h}_{1,\delta}(\rho_j),\ j=1,2,3,4$，可以得到

$$\begin{aligned}
&\frac{\hat{h}_{1,\delta}(0)}{\pi(0)}+\sum_{j=1}^{4}\frac{\hat{h}_{2,\delta}(\rho_j)}{\rho_j\pi'(\rho_j)}\\
&=\frac{\left(\frac{\lambda_1+\delta+\beta}{c}\right)\left(\frac{\lambda_2+\delta+\beta}{c}\right)\left(\frac{\lambda_1+\delta}{c}\right)\left(\frac{\lambda_2+\delta}{c}\right)}{\prod_{i=1}^{4}(-\rho_j)}\\
&\quad+\sum_{j=1}^{4}\frac{\left(\frac{\lambda_1+\delta+\beta}{c}-\rho_j\right)\left(\frac{\lambda_2+\delta+\beta}{c}-\rho_j\right)\left(\frac{\lambda_1+\delta}{c}-\rho_j\right)\left(\frac{\lambda_2+\delta}{c}-\rho_j\right)}{\rho_j\prod_{k=1,k\neq j}^{4}(\rho_j-\rho_k)}\\
&=\frac{(\lambda_1+\delta+\beta)(\lambda_2+\delta+\beta)(\lambda_1+\delta)(\lambda_2+\delta)}{c^4\prod_{i=1}^{4}(-\rho_j)}+\left[1-\frac{(\lambda_1+\delta+\beta)(\lambda_2+\delta+\beta)(\lambda_1+\delta)(\lambda_2+\delta)}{c^4\prod_{i=1}^{4}\rho_j}\right]\\
&=1.
\end{aligned}$$

那么就变换为

$$\hat{h}_{1,\delta}(s)-\hat{h}_{2,\delta}(s)=\pi_4(s)\left[1-T_sT_{\rho_1}\cdots T_{\rho_4}h_{2,\delta}(0)\right]. \tag{22}$$

最后，根据等式(21)和(22)，我们就可以得到等式(20)。

6. 缺陷更新方程

定理 6 Gerber-Shiu 折现惩罚函数 $m_\delta(u)$ 满足的瑕疵更新方程为：

$$m_\delta(u)=\int_0^u m_\delta(u-y)\zeta_\delta(y)\mathrm{d}y+G_\delta(u),\ u\ge 0 \tag{23}$$

其中，

$$\zeta_\delta(y)=T_{\rho_1}\cdots T_{\rho_4}h_{2,\delta}(u),$$
$$G_\delta(u)=T_{\rho_1}\cdots T_{\rho_4}\beta_{1,\delta}(u).$$

并且，$m_\delta(u)$ 还可以写为：

$$m_\delta(u)=\frac{1}{1+\kappa_\delta}\int_0^u m_\delta(u-y)\theta_\delta(y)\mathrm{d}y+\frac{1}{1+\kappa_\delta}\Lambda_\delta(u),\ u\ge 0 \tag{24}$$

其中 κ_δ 被定义为 $\frac{1}{1+\kappa_\delta}=T_0T_{\rho_1}\cdots T_{\rho_4}h_{2,\delta}(0)$。还有

$$\Lambda_\delta(u)=(1+\kappa_\delta)G_\delta(u),$$
$$\theta_\delta(y)=(1+\kappa_\delta)\zeta_\delta(y).$$

证明：由 Dickson-Hipp operator 的性质，即 $T_0f(0)=\int_0^\infty f(u)\mathrm{d}u$，和

$$T_sT_{\rho_1}\cdots T_{\rho_j}f(0)=(-1)^j\left[\frac{\hat{f}(s)}{\pi_j(s)}-\sum_{i=1}^{j}\frac{\hat{f}(\rho_i)}{(s-\rho_i)\pi_j'(\rho_i)}\right],$$

其中 $\pi_j(s)=\prod_{i=1}^{j}(s-\rho_i)$，可得：

$$\int_0^\infty\zeta_\delta(y)\mathrm{d}y=T_0T_{\rho_1}T_{\rho_2}T_{\rho_3}T_{\rho_4}h_{2,\delta}(0)=\frac{\hat{h}_{2,\delta}(0)}{\pi(0)}+\sum_{j=1}^{4}\frac{\hat{h}_{2,\delta}(\rho_j)}{\rho_j\pi'(\rho_j)}.$$

$$\int_0^\infty\zeta_\delta(y)\mathrm{d}y=1-\frac{\hat{h}_{1,\delta}(0)}{\pi(0)}+\frac{\hat{h}_{2,\delta}(0)}{\pi(0)}=1-\frac{\delta(\lambda_1+\lambda_2+\delta)(\lambda_1+\delta+\beta)(\lambda_2+\delta+\beta)}{c^4\rho_1\rho_2\rho_3\rho_4}. \tag{25}$$

由于 $\frac{\delta(\lambda_1+\lambda_2+\delta)(\lambda_1+\delta+\beta)(\lambda_2+\delta+\beta)}{c^4\rho_1\rho_2\rho_3\rho_4}<1$，则 $\frac{1}{1+\kappa_\delta}=\int_0^\infty\zeta_\delta(y)\mathrm{d}y<1$。

证毕。

7. 数值分析

假设随机变量 X 代表每次索赔金额，其服从一个混合指数分布，参数为 μ_1,μ_2，密度函数为

$$f_X(t)=\mathrm{e}^{-\beta x}f_1(x)+\left(1-\mathrm{e}^{-\beta x}\right)f_2(x),\ x>0, \tag{26}$$

其中 $f_1(x)=\mu_1\mathrm{e}^{-\mu_1 x},f_2(x)=\mu_2\mathrm{e}^{-\mu_2 x}$，拉普拉斯变换为：$\hat{f}_1(s)=\frac{\mu_1}{\mu_1+s},\hat{f}_2(s)=\frac{\mu_2}{\mu_2+s}$。

将等式两边进行拉普拉斯变换，可以得到一个显式表达式为：

$$\hat{m}_\tau(s)=\frac{m_\tau(0)-\hat{\zeta}_\delta(s)}{s\left[1-\hat{\zeta}_\delta(s)\right]}=\frac{1-\hat{\zeta}_\delta(s)-\left[1-m_\tau(0)\right]}{s\left[1-\hat{\zeta}_\delta(s)\right]}.$$

可得

$$\hat{h}_{1,\delta}(s)-\hat{h}_{2,\delta}(s)=\left[1-\hat{\zeta}_{\delta}(s)\right]\prod_{i=1}^{4}(\rho_i-s),$$

因此得到

$$\hat{m}_{\tau}(s)=\frac{\hat{h}_{1,\delta}(s)-\hat{h}_{2,\delta}(s)-\left[1-m_{\tau}(0)\right]\prod_{i=1}^{4}(\rho_i-s)}{s\left[\hat{h}_{1,\delta}(s)-\hat{h}_{2,\delta}(s)\right]}.$$

整理得到

$$\begin{aligned}
&\hat{h}_{1,\delta}(s)-\hat{h}_{2,\delta}(s)\\
&=\frac{1}{c^4}(\lambda_1+\delta+\beta-cs)(\lambda_2+\delta+\beta-cs)(\lambda_1+\delta-cs)(\lambda_2+\delta-cs)\\
&\quad-\frac{\lambda_1\lambda_2}{c^4}(\lambda_1+\delta-cs)(\lambda_2+\delta-cs)\left(\frac{\mu_1}{\mu_1+s}-\frac{\mu_2}{\mu_2+s}\right)-\frac{\lambda_1\lambda_2}{c^4}(\lambda_1+\delta+\beta-cs)(\lambda_2+\delta+\beta-cs)\frac{\mu_2}{\mu_2+s}\\
&=\frac{Q_{4,\delta}(s)}{c^4(\mu_1+s)(\mu_2+s)}
\end{aligned}\tag{27}$$

其中

$$\begin{aligned}
Q_{4,\delta}(s)=&(\mu_1+s)(\mu_2+s)(\lambda_1+\delta+\beta-cs)(\lambda_2+\delta+\beta-cs)(\lambda_1+\delta-cs)(\lambda_2+\delta-cs)\\
&-\lambda_1\lambda_2\mu_1(\mu_2+s)(\lambda_1+\delta-cs)(\lambda_2+\delta-cs)-\lambda_1\lambda_2\mu_2(\mu_1+s)\left(-2cs\beta+\beta^2+\beta(\lambda_1+\lambda_2+\delta)\right).
\end{aligned}$$

$Q_{4,\delta}(s)$ 是一个 4 次多项式，因此当 $Q_{4,\delta}(s)=0$ 时，有 4 个根。从命题 1 和等式可以知道，$Q_{4,\delta}(s)=0$ 有 4 个正实部的根为：$\rho_1,\rho_2,\rho_3,\rho_4$ 和两个虚根，记为：$-R_i=-R_i(\delta)$，其中 $Re(R_i)>0, i=1,2$。因此，可将重新 $Q_{4,\delta}(s)$ 写为：

$$Q_{4,\delta}(s)=c^4(s+R_1)(s+R_2)\prod_{i=1}^{4}(\rho_i-s).$$

$\hat{m}_{\tau}(s)$ 变为

$$\hat{m}_{\tau}(s)=\frac{\left(1-\frac{R_1R_2}{\mu_1\mu_2}\right)s+R_1+R_2-\frac{R_1R_2(\mu_1+\mu_2)}{\mu_1\mu_2}}{(s+R_1)(s+R_2)}.\tag{28}$$

计算可得 $1-m_{\tau}(0)=\frac{R_1R_2}{\mu_1\mu_2}$。

假设 R_1,R_2 不同，则可以得到

$$\hat{m}_{\tau}(s)=\sum_{j=1}^{2}\frac{\xi_{i,\delta}}{s+R_j},$$

其中，

$$\begin{aligned}
\xi_{1,\delta}&=\frac{R_2}{R_2-R_1}\left(1-\frac{R_1(\mu_1+\mu_2)}{\mu_1\mu_2}+\frac{R_1^2}{\mu_1\mu_2}\right),\\
\xi_{2,\delta}&=\frac{R_1}{R_2-R_1}\left(1-\frac{R_2(\mu_1+\mu_2)}{\mu_1\mu_2}+\frac{R_2^2}{\mu_1\mu_2}\right).
\end{aligned}\tag{29}$$

那么，最后有

$$m_\tau(u) = \xi_{1,\delta} e^{-R_1 u} + \xi_{2,\delta} e^{-R_2 u}, u \ge 0,$$

令 $\delta \to 0$，就可以得到破产概率 $\Psi(u)$。

相依参数 β 的影响

在这一节中将会讨论在两个指数函数和相依模型中，相依参数 β 对破产概率的具体影响。根据上一节中的相关计算，我们可以绘制出在不同相依参数 β 下，破产概率随初始盈余变化的图像。

我们令 $\lambda_1 = 0.5, \lambda_2 = 1, c = 1.5, \mu_1 = 1, \mu_2 = 3$

$$\beta = 0.5$$

$$\psi(u) = 0.0770863654 e^{-0.923017747u} - 0.00269254831 e^{-2.99091027u}$$

$$\beta = 0.75$$

$$\psi(u) = 0.0645201875 e^{-0.935605619u} - 0.00388226563 e^{-2.9871482u}$$

$$\beta = 1$$

$$\psi(u) = 0.0548836375 e^{-0.94525333u} - 0.00496318778 e^{-2.98381337u}$$

$$\beta = 2$$

$$\psi(u) = 0.0321485575 e^{-0.967987219u} - 0.0083708274 e^{-2.97363621u}$$

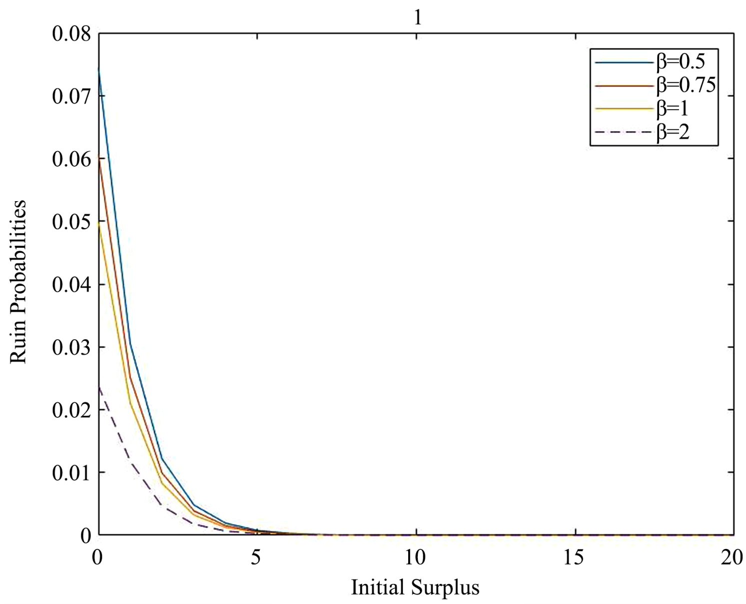

Figure 1. Ruin probability image
图 1. 破产概率图像

通过观察图 1 可以看出，相依参数 β 对破产概率的影响为：当相依 β 参数越大时，破产概率越小。

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