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Spatial Decay Estimates for a Class of Thermo-Diffusion Equations

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Abstract

In this paper, we study the spatial decay estimates for a class of thermo-diffusion equations. Using the technique of a first-order differential inequality, the exponential decay estimates for the linear differential equations of thermodiffusion in a semi-infinite pipe were established. To make the estimate explicit, the bound for the total energy was also derived.

Keywords

Thermo-Diffusion Equations, Spatial Decay Estimates, Sait-Venent Principle

一类热扩散方程组的空间衰减估计

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摘 要

本文研究了一类热扩散方程组的空间衰减估计，应用一阶微分不等式的方法，建立了半无限管内热扩散线性微分方程组的解的空间指数衰减估计。为了使估计明确，还导出了总能量的界。该结果可看作Saint-Venant原则在热扩散方程组中的应用。

关键词

热扩散微分方程，空间衰减估计，Saint-Venant原则

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1. 前言

在 20 世纪 70 年代，W. Nowacki 在他的论文[1]中给出了一维空间中热扩散的微分方程，参考文献中的许多论文研究了这一系统。例如，[2]和[3]利用不同的参数研究了热扩散线性系统的初边界值问题。通过[4]得到了相关线性柯西问题解的 $L^p - L^q$ 时间衰减估计。然而，在本文中，我们考虑了三维热扩散的微分方程，并研究了系统解的空间衰减估计。实际上，很多文献都是关于各种微分方程系统的空间衰减估计问题，为了查阅关于圣维南原理的这类工作，可以参考[5]-[13]以及其中引用的论文。

我们假设瞬态流体占据了边界 ∂R 的半无限圆柱管 R 的内部。管道的截面用 D 表示，截面的边界用 ∂D 表示，管道 R 平行于 x_3 轴。我们定义：

$$R_z = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in D, x_3 > z \ge 0\},$$

$$D_z = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in D, x_3 = z \ge 0\},$$

其中 z 是沿 x_3 轴的变量。显然，$R_0 = R$ 和 $D_0 = D$。设 u_i、T 和 C 分别表示位移、温度和化学势为独立场。这些依赖于空间变量 (x_1, x_2, x_3) 和时间变量 t，并满足以下方程组：

$$\rho \ddot{u}_i - v\Delta u_i - (\lambda + v) u_{j,ji} + \gamma_1 T_{,i} + \gamma_2 C_{,i} = 0, \quad R \times \{t \ge 0\}, \tag{1.1}$$

$$c\dot{T} - K\Delta T + \gamma_1 \dot{u}_{i,i} + d\dot{C} = 0, \quad R \times \{t \ge 0\}, \tag{1.2}$$

$$n\dot{C} - M\Delta C + \gamma_2 \dot{u}_{i,i} + d\dot{T} = 0, \text{ in } R \times \{t \ge 0\}, \tag{1.3}$$

在初始边界条件下

$$u_i = 0, T = C = 0 \quad \text{on } \partial D \times \{t \ge 0\}, \tag{1.4}$$

$$u_i = \dot{u} = 0, T = C = 0 \quad \text{in } R \times \{t = 0\}. \tag{1.5}$$

$$u_i = f_i(x_1, x_2, t), T = F(x_1, x_2, t), C = G(x_1, x_2, t) \quad \text{on } D_0 \times \{t \ge 0\}, \tag{1.6}$$

$$u_i, u_{i,j}, T, T_i, C, C_{,i}, p = o\left(x_3^{-1}\right) x_1, x_2, t \quad \text{as } x_3 \to \infty. \tag{1.7}$$

在方程(1.1)~(1.3)中，Δ 是拉普拉斯算子；ρ 表示密度；γ_1 和 γ_2 是热和扩散扩张的系数；λ 和 v 是材料系数；K 是导热系数；M 是扩散系数。n, c, d 是热扩散的系数。以上常数均为正，满足

$$cn - d^2 > 0 \tag{1.8}$$

这意味着(1.1)~(1.3)是一个偏微分方程组的双曲分解系统。在下面的几节中，我们可以使用下面的不等式。设 D 为带 t 的平面域 D，他的边界为 ∂D。如果 w 在 ∂D 上等于 0，那么

$$\int_D w_{,\alpha} w_{,\alpha} \mathrm{d}A \ge \lambda_1 \int_D w^2 \mathrm{d}x, \tag{1.9}$$

其中 λ_1 是问题的最小特征值

$$\Delta \varphi + \lambda \varphi = 0 \quad \text{in } D,$$

$$\varphi = 0 \quad \text{on } \partial D$$

这种不平等现象已经得到了很好的研究(见[14] [15])。在本文中，通常的求和约定是使用重复的拉丁下标从 1 到 3，并重复希腊字母下标从 1 到 2。逗号用来表示部分区分，例如：

$$u_{i,j}=\frac{\partial u_i}{\partial x_i},\varphi_{\alpha,\alpha}=\sum_{\alpha=1}^{2}\frac{\partial\varphi_\alpha}{\partial x_\alpha},\dot{u}_i=\frac{\partial u_i}{\partial_t}.$$

2. 能量衰减

在这一部分中，我们导出了问题(1.3)~(1.6)的主要指数衰减结果。应用方程(1.1)和(1.4)~(1.6)，得到

$$\begin{aligned}0&=\int_0^t\int_{R_z}\left[\rho\ddot{u}_i-v\Delta u_i-(\lambda+v)u_{j,ji}+\gamma_1T_{,i}+\gamma_2C_{,i}\right]\dot{u}_i\mathrm{d}x\mathrm{d}\eta\\&=\frac{1}{2}\rho\int_{R_z}\dot{u}_i\dot{u}_i\mathrm{d}x+v\int_0^t\int_{D_z}u_{i,3}\dot{u}_i\mathrm{d}A\mathrm{d}\eta+\frac{v}{2}\int_{R_z}u_{i,j}u_{i,j}\mathrm{d}x+(\lambda+v)\int_0^t\int_{D_z}u_{j,j}\dot{u}_3\mathrm{d}A\mathrm{d}\eta\\&\quad+\frac{\lambda+v}{2}\int_{R_z}u_{i,i}^2\mathrm{d}x+\gamma_1\int_0^t\int_{R_z}T_{,i}\dot{u}_i\mathrm{d}x\mathrm{d}\eta+\gamma_2\int_0^t\int_{R_z}C_{,i}\dot{u}_i\mathrm{d}x\mathrm{d}\eta.\end{aligned}\tag{2.1}$$

我们用 T 乘以(1.2)，并积分得

$$\begin{aligned}0&=\int_0^t\int_{R_z}\left[c\dot{T}-K\Delta T+\gamma_1\dot{u}_{i,i}+d\dot{C}\right]T\mathrm{d}x\mathrm{d}\eta\\&=\frac{c}{2}\int_{R_z}T^2\mathrm{d}x\Big|_{\eta=t}+K\int_0^t\int_{D_z}T_{,3}T\mathrm{d}A\mathrm{d}\eta+K\int_0^t\int_{R_z}T_{,j}T_{,j}\mathrm{d}x\mathrm{d}\eta\\&\quad-\gamma_1\int_0^t\int_{D_z}T\dot{u}_3\mathrm{d}A\mathrm{d}\eta-\gamma_1\int_0^t\int_{R_z}T_{,i}\dot{u}_i\mathrm{d}x\mathrm{d}\eta+d\int_0^t\int_{R_z}C\dot{T}\mathrm{d}x\mathrm{d}\eta.\end{aligned}\tag{2.2}$$

以同样的方式，我们也可得

$$\begin{aligned}0&=\int_0^t\int_{R_z}\left[n\dot{C}-M\Delta C+\gamma_2\dot{u}_{i,i}+d\dot{T}\right]C\mathrm{d}x\mathrm{d}\eta\\&=\frac{n}{2}\int_{R_z}C^2\mathrm{d}x\Big|_{\eta=t}+M\int_0^t\int_{D_z}C_{,3}C\mathrm{d}A\mathrm{d}\eta+M\int_0^t\int_{R_z}C_{,j}C_{,j}\mathrm{d}x\mathrm{d}\eta\\&\quad-\gamma_2\int_0^t\int_{D_z}C\dot{u}_3\mathrm{d}A\mathrm{d}\eta-\gamma_2\int_0^t\int_{R_z}C_{,i}\dot{u}_i\mathrm{d}x\mathrm{d}\eta+d\int_0^t\int_{R_z}C\dot{T}\mathrm{d}x\mathrm{d}\eta\end{aligned}\tag{2.3}$$

现在，我们定义一个函数

$$\begin{aligned}E(z,t)&=\frac{1}{2}\rho\int_{R_z}\dot{u}_i\dot{u}_i\mathrm{d}x+\frac{v}{2}\int_{R_z}u_{i,j}u_{i,j}\mathrm{d}x+\frac{\lambda+v}{2}\int_{R_z}u_{i,i}^2\mathrm{d}x+K\int_0^t\int_{R_z}T_{,j}T_{,j}\mathrm{d}x\mathrm{d}\eta\\&\quad+M\int_0^t\int_{R_z}C_{,j}C_{,j}\mathrm{d}x\mathrm{d}\eta+\int_{R_z}\left[\frac{c}{2}T^2+dCT+\frac{n}{2}C^2\right]\mathrm{d}x\end{aligned}\tag{2.4}$$

然后得

$$\begin{aligned}-\frac{\partial E(z,t)}{\partial z}&=\frac{1}{2}\rho\int_{D_z}\dot{u}_i\dot{u}_i\mathrm{d}A+\frac{v}{2}\int_{D_z}u_{i,j}u_{i,j}\mathrm{d}A+\frac{\lambda+v}{2}\int_{D_z}u_{i,i}^2\mathrm{d}A+K\int_0^t\int_{D_z}T_{,j}T_{,j}\mathrm{d}A\mathrm{d}\eta\\&\quad+M\int_0^t\int_{D_z}C_{,j}C_{,j}\mathrm{d}A\mathrm{d}\eta+\int_{D_z}\left[\frac{c}{2}T^2+dCT+\frac{n}{2}C^2\right]\mathrm{d}A.\end{aligned}\tag{2.5}$$

很明显，(2.5)的最后一项是正的，因为 $cn-d^2>0$。从(2.1)~(2.3)开始，我们有

$$\begin{aligned}E(z,t)&=-v\int_0^t\int_{D_z}u_{i,3}\dot{u}_i\mathrm{d}A\mathrm{d}\eta-(\lambda+v)\int_0^t\int_{D_z}u_{j,j}\dot{u}_3\mathrm{d}A\mathrm{d}\eta+\gamma_1\int_0^t\int_{D_z}T\dot{u}_3\mathrm{d}A\mathrm{d}\eta\\&\quad+\gamma_2\int_0^t\int_{D_z}C\dot{u}_3\mathrm{d}A\mathrm{d}\eta-K\int_0^t\int_{D_z}T_{,3}T\mathrm{d}A\mathrm{d}\eta-M\int_0^t\int_{D_z}C_{,3}C\mathrm{d}A\mathrm{d}\eta.\end{aligned}\tag{2.6}$$

利用 Schwarz，Poincar’e 定理，以及 AG 的平均不等式，我们得到了

$$\begin{aligned}
v\int_0^t\int_{D_z} u_{i,3}\dot{u}_i \mathrm{d}A\mathrm{d}\eta &\le v\left(\int_0^t\int_{D_z} \dot{u}_i\dot{u}_i \mathrm{d}A\mathrm{d}\eta\right)^{\frac{1}{2}}\left(\int_0^t\int_{D_z} u_{i,3}u_{i,3} \mathrm{d}A\mathrm{d}\eta\right)^{\frac{1}{2}} \\
&\le v\left(\int_0^t\int_{D_z} \dot{u}_i\dot{u}_i \mathrm{d}A\mathrm{d}\eta\right)^{\frac{1}{2}}\left(\int_0^t\int_{D_z} u_{i,3}u_{i,3} \mathrm{d}A\mathrm{d}\eta\right)^{\frac{1}{2}} \\
&\le \frac{\sqrt{vt}}{\sqrt{\rho}}\left(-\frac{\partial E(z,t)}{\partial z}\right),
\end{aligned} \tag{2.7}$$

和

$$\begin{aligned}
(\lambda+v)\int_0^t\int_{D_z} u_{j,j}\dot{u} \mathrm{d}A\mathrm{d}\eta &\le (\lambda+v)\left(\int_0^t\int_{D_z} \dot{u}_i\dot{u}_i \mathrm{d}A\mathrm{d}\eta\right)^{\frac{1}{2}}\left(\int_0^t\int_{D_z} u_{j,j}^2 \mathrm{d}A\mathrm{d}\eta\right)^{\frac{1}{2}} \\
&\le \frac{\sqrt{\lambda+vt}}{\sqrt{\rho}}\left(-\frac{\partial E(z,t)}{\partial z}\right),
\end{aligned} \tag{2.8}$$

还有

$$\begin{aligned}
&\gamma_1\int_0^t\int_{D_z} T\dot{u}_3 \mathrm{d}A\mathrm{d}\eta + \gamma_2\int_0^t\int_{D_z} C\dot{u}_3 \mathrm{d}A\mathrm{d}\eta \\
&\le \gamma_1\left(\int_0^t\int_{D_z} T^2 \mathrm{d}A\mathrm{d}\eta\right)^{\frac{1}{2}}\left(\int_0^t\int_{D_z} \dot{u}_3^2 \mathrm{d}A\mathrm{d}\eta\right)^{\frac{1}{2}} + \gamma_2\left(\int_0^t\int_{D_z} C^2 \mathrm{d}A\mathrm{d}\eta\right)^{\frac{1}{2}}\left(\int_0^t\int_{D_z} \dot{u}_3^2 \mathrm{d}A\mathrm{d}\eta\right)^{\frac{1}{2}} \\
&\le \frac{\gamma_1}{\sqrt{\lambda_1}}\left(\int_0^t\int_{D_z} T_{,\alpha}T_{,\alpha} \mathrm{d}A\mathrm{d}\eta\right)^{\frac{1}{2}}\left(\int_0^t\int_{D_z} \dot{u}_3^2 \mathrm{d}A\mathrm{d}\eta\right)^{\frac{1}{2}} + \frac{\gamma_2}{\sqrt{\lambda_1}}\left(\int_0^t\int_{D_z} C_{,\alpha}C_{,\alpha} \mathrm{d}A\mathrm{d}\eta\right)^{\frac{1}{2}}\left(\int_0^t\int_{D_z} \dot{u}_3^2 \mathrm{d}A\mathrm{d}\eta\right)^{\frac{1}{2}} \\
&\le \left[\frac{\gamma_1\sqrt{t}}{2\sqrt{\rho\lambda_1 K}} + \frac{\gamma_2\sqrt{t}}{2\sqrt{\rho\lambda_1 M}}\right]\left(-\frac{\partial E(z,t)}{\partial z}\right).
\end{aligned} \tag{2.9}$$

利用 Schwarz 定理，(1.9)和 AG 平均不等式，我们得到了

$$\begin{aligned}
&-K\int_0^t\int_{D_z} T_{,3}T \mathrm{d}A\mathrm{d}\eta - M\int_0^t\int_{D_z} C_{,3}C \mathrm{d}A\mathrm{d}\eta \\
&\le K\left(\int_0^t\int_{D_z} T_{,3}^2 \mathrm{d}A\mathrm{d}\eta\right)^{\frac{1}{2}}\left(\int_0^t\int_{D_z} T^2 \mathrm{d}A\mathrm{d}\eta\right)^{\frac{1}{2}} + M\left(\int_0^t\int_{D_z} C_{,3}^2 \mathrm{d}A\mathrm{d}\eta\right)^{\frac{1}{2}}\left(\int_0^t\int_{D_z} C^2 \mathrm{d}A\mathrm{d}\eta\right)^{\frac{1}{2}} \\
&\le \frac{K}{\sqrt{\lambda_1}}\left(\int_0^t\int_{D_z} T_{,3}^2 \mathrm{d}A\mathrm{d}\eta\right)^{\frac{1}{2}}\left(\int_0^t\int_{D_z} T_{,\alpha}T_{,\alpha} \mathrm{d}A\mathrm{d}\eta\right)^{\frac{1}{2}} + \frac{M}{\sqrt{\lambda_1}}\left(\int_0^t\int_{D_z} C_{,3}^2 \mathrm{d}A\mathrm{d}\eta\right)^{\frac{1}{2}}\left(\int_0^t\int_{D_z} C_{,\alpha}C_{,\alpha} \mathrm{d}A\mathrm{d}\eta\right)^{\frac{1}{2}} \\
&\le \frac{1}{2\sqrt{\lambda_1}}\left(-\frac{\partial E(z,t)}{\partial z}\right).
\end{aligned} \tag{2.10}$$

将(2.7)~(2.10)代入(2.6)，我们有

$$E(z,t) \le m_1(t)\left(-\frac{\partial E(z,t)}{\partial z}\right), \tag{2.11}$$

其中

$$m_1(t) = \frac{\sqrt{vt}}{\sqrt{\rho}} + \frac{\sqrt{\lambda+vt}}{\sqrt{\rho}} + \frac{\gamma_1\sqrt{t}}{2\sqrt{\rho\lambda_1 K}} + \frac{\gamma_2\sqrt{t}}{2\sqrt{\rho\lambda_1 M}} + \frac{1}{2\sqrt{\lambda_1}} \tag{2.12}$$

我们可以知道，(2.11)得出的结论是：

$$E(z,t) \le E(0,t)\mathrm{e}^{-\frac{1}{m_1(t)}z}. \tag{2.13}$$

不等式(2.13)就是我们所寻求的空间衰减结果。

3. 总能量 $E(0, t)$的边界

为了使我们的衰变结果在第 3 节中明确化，我们根据已知的数据导出了 $E(0,t)$ 的界。首先，我们将等式(2.4)和(2.6)代入 $z=0$ 的时候，即，

$$\begin{aligned} E(0,t) &= \frac{1}{2}\rho\int_R \dot{u}_i\dot{u}_i\mathrm{d}x + \frac{v}{2}\int_R u_{i,j}u_{i,j}\mathrm{d}x + \frac{\lambda+v}{2}\int_R u_{i,i}^2\mathrm{d}x + K\int_0^t\int_R T_{,j}T_{,j}\mathrm{d}x\mathrm{d}\eta \\ &\quad + M\int_0^t\int_R C_{,j}C_{,j}\mathrm{d}x\mathrm{d}\eta + \int_R\left[\frac{c}{2}T^2 + dCT + \frac{n}{2}C^2\right]\mathrm{d}x \\ &= -v\int_0^t\int_{D_0} u_{i,3}\dot{u}_i\mathrm{d}A\mathrm{d}\eta - (\lambda+v)\int_0^t\int_{D_0} u_{j,j}\dot{u}_3\mathrm{d}A\mathrm{d}\eta + \gamma_1\int_0^t\int_{D_0} T\dot{u}_3\mathrm{d}A\mathrm{d}\eta \\ &\quad + \gamma_2\int_0^t\int_{D_0} C\dot{u}_3\mathrm{d}A\mathrm{d}\eta - K\int_0^t\int_{D_0} T_{,3}T\mathrm{d}A\mathrm{d}\eta - M\int_0^t\int_{D_0} C_{,3}C\mathrm{d}A\mathrm{d}\eta. \end{aligned} \tag{3.1}$$

为了得到 $E(0,t)$ 的界，我们现在介绍函数

$$\Psi_i(x_1,x_2,x_3,t) = f_i\mathrm{e}^{-\delta_1x_3}, \xi(x_1,x_2,x_3,t) = F\mathrm{e}^{-\delta_2x_3}, \tau(x_1,x_2,x_3,t) = G\mathrm{e}^{-\delta_3x_3} \tag{3.2}$$

其中 δ_1,δ_2 和 δ_3 是待定的正常数。所以，我们可得

$$\begin{aligned} E(0,t) &= -v\int_0^t\int_{D_0} u_{i,3}\dot{\Psi}_i\mathrm{d}A\mathrm{d}\eta - (\lambda+v)\int_0^t\int_{D_0} u_{j,j}\dot{\Psi}_3\mathrm{d}A\mathrm{d}\eta + \gamma_1\int_0^t\int_{D_0}\xi\dot{\Psi}_3\mathrm{d}A\mathrm{d}\eta \\ &\quad + \gamma_2\int_0^t\int_{D_0}\tau\dot{\Psi}_3\mathrm{d}A\mathrm{d}\eta - K\int_0^t\int_{D_0} T_{,3}\xi\mathrm{d}A\mathrm{d}\eta - M\int_0^t\int_{D_0} C_{,3}\tau\mathrm{d}A\mathrm{d}\eta. \end{aligned} \tag{3.3}$$

根据散度定理可知

$$\begin{aligned} E(0,t) &= v\int_0^t\int_R \Delta u_i\dot{\Psi}_i\mathrm{d}x\mathrm{d}\eta + v\int_0^t\int_R u_{i,j}\dot{\Psi}_{i,j}\mathrm{d}x\mathrm{d}\eta + (\lambda+v)\int_0^t\int_R u_{i,ji}\dot{\Psi}_i\mathrm{d}x\mathrm{d}\eta \\ &\quad + (\lambda+v)\int_0^t\int_R u_{i,j}\dot{\Psi}_{i,i}\mathrm{d}x\mathrm{d}\eta + \gamma_1\int_0^t\int_{D_0}\xi\dot{\Psi}_3\mathrm{d}A\mathrm{d}\eta + \gamma_2\int_0^t\int_{D_0}\tau\dot{\Psi}_3\mathrm{d}A\mathrm{d}\eta \\ &\quad + K\int_0^t\int_R \Delta T\xi\mathrm{d}x\mathrm{d}\eta + K\int_0^t\int_R T_{,i}\xi_{,i}\mathrm{d}x\mathrm{d}\eta + M\int_0^t\int_R \Delta C\tau\mathrm{d}A\mathrm{d}\eta + M\int_0^t\int_R C_{,i}\tau_{,i}\mathrm{d}x\mathrm{d}\eta \\ &= \int_0^t\int_R\left[\rho\ddot{u}_i + \gamma_1T_{,i} + \gamma_2C_{,i}\right]\dot{\Psi}_i\mathrm{d}x\mathrm{d}\eta + v\int_0^t\int_R u_{i,j}\dot{\Psi}_{i,j}\mathrm{d}x\mathrm{d}\eta + (\lambda+v)\int_0^t\int_R u_{i,j}\dot{\Psi}_{i,i}\mathrm{d}x\mathrm{d}\eta \\ &\quad + \gamma_1\int_0^t\int_{D_0}\xi\dot{\Psi}_3\mathrm{d}A\mathrm{d}\eta + \gamma_2\int_0^t\int_{D_0}\tau\dot{\Psi}_3\mathrm{d}A\mathrm{d}\eta + \int_0^t\int_R\left[c\dot{T} + \gamma_1\dot{u}_{i,i} + d\dot{C}\right]\xi\mathrm{d}x\mathrm{d}\eta \\ &\quad + K\int_0^t\int_R T_{,i}\xi_{,i}\mathrm{d}x\mathrm{d}\eta + \int_0^t\int_R\left[n\dot{C} + \gamma_2 + \dot{u}_{i,i} + d\dot{T}\right]\tau\mathrm{d}A\mathrm{d}\eta + M\int_0^t\int_R C_{,i}\tau_{,i}\mathrm{d}x\mathrm{d}\eta \\ &= \rho\int_R \dot{u}_i\dot{\Psi}_i\mathrm{d}x - \rho\int_0^t\int_R \dot{u}_i\ddot{\Psi}_i\mathrm{d}x\mathrm{d}\eta + \gamma_1\int_0^t\int_R T_{,i}\dot{\Psi}_i\mathrm{d}x\mathrm{d}\eta + \gamma_2\int_0^t\int_R C_{,i}\dot{\Psi}\mathrm{d}x\mathrm{d}\eta \\ &\quad + v\int_0^t\int_R u_{i,j}\dot{\Psi}_{i,j}\mathrm{d}x\mathrm{d}\eta + (\lambda+v)\int_0^t\int_R u_{j,j}\dot{\Psi}_{i,i}\mathrm{d}x\mathrm{d}\eta + c\int_R T\xi\mathrm{d}x\mathrm{d}\eta - c\int_0^t\int_R T\xi\mathrm{d}x\mathrm{d}\eta \\ &\quad - \gamma_1\int_0^t\int_R \dot{u}_i\xi_{,i}\mathrm{d}x\mathrm{d}\eta + d\int_R C\xi\mathrm{d}x - d\int_0^t\int_R C\xi\mathrm{d}x\mathrm{d}\eta + K\int_0^t\int_R T_{,i}\xi_{,i}\mathrm{d}x\mathrm{d}\eta + n\int_R C\tau\mathrm{d}x \\ &\quad - n\int_0^t\int_R C\dot{\tau}\mathrm{d}x\mathrm{d}\eta - \gamma_2\int_0^t\int_R \dot{u}_i\tau_{,i}\mathrm{d}x\mathrm{d}\eta + d\int_R T\tau\mathrm{d}A - d\int_0^t\int_R T\dot{\tau}\mathrm{d}x\mathrm{d}\eta + M\int_0^t\int_R C_{,i}\tau_{,i}\mathrm{d}x\mathrm{d}\eta \end{aligned} \tag{3.4}$$

由 Schwarz 的不等式和 AG 的平均不等式，从(3.4)我们得到了

$$\begin{aligned} E(0,t) &\le \frac{\rho\varepsilon_1}{2}\int_R \dot{u}_i\dot{u}_i\mathrm{d}x + \frac{\rho}{2\varepsilon_1}\int_R \dot{\Psi}_i\dot{\Psi}_i\mathrm{d}x + \frac{\rho\varepsilon_2t}{2}\int_R \dot{u}_i\dot{u}_i\mathrm{d}x + \frac{\rho}{2\varepsilon_2}\int_0^t\int_R \ddot{\Psi}_i\ddot{\Psi}_i\mathrm{d}x\mathrm{d}\eta \\ &\quad + \frac{\gamma_1\varepsilon_3}{2}\int_0^t\int_R T_{,i}T_{,i}\mathrm{d}x\mathrm{d}\eta + \frac{\gamma_1}{2\varepsilon_3}\int_0^t\int_R \dot{\Psi}_i\dot{\Psi}_i\mathrm{d}x\mathrm{d}\eta + \frac{\gamma_2\varepsilon_4}{2}\int_0^t\int_R C_{,i}C_{,i}\mathrm{d}x\mathrm{d}\eta \\ &\quad + \frac{\gamma_2}{2\varepsilon_4}\int_0^t\int_R \dot{\Psi}_i\dot{\Psi}_i\mathrm{d}x\mathrm{d}\eta + \frac{v\varepsilon_5t}{2}\int_R u_{i,j}u_{i,j}\mathrm{d}x + \frac{v}{2\varepsilon_5}\int_0^t\int_R \dot{\Psi}_{i,j}\dot{\Psi}_{i,j}\mathrm{d}x\mathrm{d}\eta \end{aligned}$$

$$
\begin{aligned}
&+\frac{(\lambda+v)\varepsilon_6 t}{2}\int_R u_{j,j}^2\mathrm{d}x+\frac{(\lambda+v)}{2\varepsilon_6}\int_0^t\int_R\ddot{\Psi}_{i,i}^2\mathrm{d}x\mathrm{d}\eta+\frac{c\varepsilon_7}{2}\int_R T^2\mathrm{d}x+\frac{c}{2\varepsilon_7}\int_R\xi^2\mathrm{d}x\\
&+\frac{c\varepsilon_8}{2\lambda_1}\int_0^t\int_R T_{,i}T_{,i}\mathrm{d}x\mathrm{d}\eta+\frac{c}{2\varepsilon_8}\int_0^t\int_R\ddot{\Psi}_{i,i}^2\mathrm{d}x\mathrm{d}\eta+\frac{\gamma_1\varepsilon_9 t}{2}\int_R\dot{u}_i\dot{u}_i\mathrm{d}x+\frac{\gamma_1}{2\varepsilon_9}\int_0^t\int_R\xi_{,i}\xi_{,i}\mathrm{d}x\mathrm{d}\eta\\
&+\frac{d\varepsilon_{10}}{2}\int_R C^2\mathrm{d}x+\frac{d}{2\varepsilon_{10}}\int_R\xi^2\mathrm{d}x+\frac{d\varepsilon_{11}}{2\lambda_1}\int_0^t\int_R C_{,i}C_{,i}\mathrm{d}x\mathrm{d}\eta+\frac{d}{2\varepsilon_{11}}\int_0^t\int_R\dot{\xi}^2\mathrm{d}x\mathrm{d}\eta\\
&+\frac{K\varepsilon_{12}}{2}\int_0^t\int_R T_{,i}T_{,i}\mathrm{d}x\mathrm{d}\eta+\frac{K}{2\varepsilon_{12}}\int_0^t\int_R\xi_{,i}\xi_{,i}\mathrm{d}x\mathrm{d}\eta+\frac{n\varepsilon_{13}}{2}\int_R C^2\mathrm{d}x+\frac{n}{2\varepsilon_{13}}\int_R\tau^2\mathrm{d}x\\
&+\frac{n\varepsilon_{14}}{2\lambda_1}\int_0^t\int_R C_{,i}C_{,i}\mathrm{d}x\mathrm{d}\eta+\frac{n}{2\varepsilon_{14}}\int_0^t\int_R\dot{\tau}^2\mathrm{d}x\mathrm{d}\eta+\frac{\gamma_2\varepsilon_{15}t}{2\lambda_1}\int_R\dot{u}_i\dot{u}_i\mathrm{d}x\\
&+\frac{\gamma_2}{2\varepsilon_{15}}\int_0^t\int_R\tau_{,i}\tau_{,i}\mathrm{d}x\mathrm{d}\eta+\frac{d\varepsilon_{16}}{2}\int_R T^2\mathrm{d}x+\frac{d}{2\varepsilon_{16}}\int_R\tau^2\mathrm{d}x+\frac{d\varepsilon_{17}}{2\lambda_1}\int_0^t\int_R T_{,i}T_{,i}\mathrm{d}x\mathrm{d}\eta\\
&+\frac{d}{2\varepsilon_{17}}\int_0^t\int_R\dot{\tau}^2\mathrm{d}x\mathrm{d}\eta+\frac{M\varepsilon_{18}}{2}\int_0^t\int_R C_{,i}C_{,i}\mathrm{d}x\mathrm{d}\eta+\frac{M}{2\varepsilon_{18}}\int_0^t\int_R\tau_{,i}\tau_{,i}\mathrm{d}x\mathrm{d}\eta
\end{aligned}\tag{3.5}
$$

对于 $\varepsilon_i>0, i=(1,2,3,\cdots,18)$。挑选 $\varepsilon_1=\frac{1}{8}$，$\varepsilon_2=\frac{1}{8t}$，$\varepsilon_3=\frac{K}{4\gamma_1}$，$\varepsilon_4=\frac{M}{4\gamma_2}$，$\varepsilon_5=\frac{1}{2t}$，$\varepsilon_6=\frac{1}{2t}$，$\varepsilon_7=\frac{n(cn-d^2)}{4c(cn+d^2)}$，$\varepsilon_8=\frac{\lambda_1 K}{4c}$，$\varepsilon_9=\frac{\rho\lambda_1}{8t\gamma_1}$，$\varepsilon_{10}=\frac{cn-d^2}{8d}$，$\varepsilon_{11}=\frac{\lambda_1 M}{4d}$，$\varepsilon_{12}=\frac{1}{4}$，$\varepsilon_{13}=\frac{cn-d^2}{8n^2}$，$\varepsilon_{14}=\frac{\lambda_1 M}{4nd}$，$\varepsilon_{15}=\frac{\rho\lambda_1}{8t\gamma_2}$，$\varepsilon_{16}=\frac{n(cn-d^2)}{2d(cn+d^2)}$，$\varepsilon_{17}=\frac{\lambda_1 K}{4d}$，$\varepsilon_{18}=\frac{1}{4}$ 结合(3.1)和(3.5)，我们有

$$
E(0,t)\le\frac{1}{2}E(0,t)+\frac{1}{2}m_2(t),\tag{3.6}
$$

因此，

$$
E(0,t)\le m_2(t),\tag{3.7}
$$

其中

$$
\begin{aligned}
m_2(t)=&\frac{\rho}{\varepsilon_1}\int_R\dot{\Psi}_i\dot{\Psi}_i\mathrm{d}x+\frac{\rho}{\varepsilon_2}\int_0^t\int_R\ddot{\Psi}_i\ddot{\Psi}_i\mathrm{d}x\mathrm{d}\eta+\frac{\gamma_1}{\varepsilon_3}\int_0^t\int_R\dot{\Psi}_i\dot{\Psi}_i\mathrm{d}x\mathrm{d}\eta+\frac{\gamma_2}{\varepsilon_4}\int_0^t\int_R\dot{\Psi}_i\dot{\Psi}_i\mathrm{d}x\mathrm{d}\eta\\
&+\frac{v}{\varepsilon_5}\int_0^t\int_R\dot{\Psi}_{i,j}\dot{\Psi}_{i,j}\mathrm{d}x\mathrm{d}\eta+\frac{(\lambda+v)}{\varepsilon_6}\int_0^t\int_R\dot{\Psi}_{i,i}^2\mathrm{d}x\mathrm{d}\eta+\frac{c}{\varepsilon_7}\int_R\xi^2\mathrm{d}x+\frac{c}{\varepsilon_8}\int_0^t\int_R\dot{\xi}^2\mathrm{d}x\mathrm{d}\eta\\
&+\frac{\gamma_1}{\varepsilon_9}\int_0^t\int_R\xi_{,i}\xi_{,i}\mathrm{d}x\mathrm{d}\eta+\frac{d}{\varepsilon_{10}}\int_R\xi^2\mathrm{d}x+\frac{d}{\varepsilon_{11}}\int_0^t\int_R\dot{\xi}^2\mathrm{d}x\mathrm{d}\eta+\frac{K}{\varepsilon_{12}}\int_0^t\int_R\xi_{,i}\xi_{,i}\mathrm{d}x\mathrm{d}\eta\\
&+\frac{n}{\varepsilon_{13}}\int_R\tau^2\mathrm{d}x+\frac{n}{\varepsilon_{14}}\int_0^t\int_R\dot{\tau}^2\mathrm{d}x\mathrm{d}\eta+\frac{\gamma_2}{\varepsilon_{15}}\int_0^t\int_R\tau_{,i}\tau_{,i}\mathrm{d}x\mathrm{d}\eta+\frac{d}{\varepsilon_{16}}\int_R\tau^2\mathrm{d}x\\
&+\frac{d}{\varepsilon_{17}}\int_0^t\int_R\dot{\tau}^2\mathrm{d}x\mathrm{d}\eta+\frac{M}{\varepsilon_{18}}\int_0^t\int_R\tau_{,i}\tau_{,i}\mathrm{d}x\mathrm{d}\eta.
\end{aligned}\tag{3.8}
$$

回顾(3.2)中 Ψ_i,ξ 和 τ 的定义，我们得出结论，我们通过选择适当的 δ_1,δ_2 和 δ_3，$E(0,t)$ 的界，不等式(2.13)就可以明确表示。

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