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一类序参数守恒的相场模型的弱解存在性

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摘　要

本文将基于序参数守恒情形下的 **Alber-Zhu** 模型展开研究, 该模型序参数的演化由退化的四阶抛物型方程控制, 可用于描述微观尺度下可弹性变形固体中晶界的运动, 这个过程的一个典型例子是烧结, 在烧结过程中会引发致密化及晶粒生长等基本微观组织结构变化. 由于在界面上没有发生原子交换, 因此被界面分开的不同区域的体积是守恒的, 扩散过程仅由体自由能的降低来驱动. 本文考虑简化后的模型, 即忽略弹性效应, 并将方程磨光为一维空间的单个非退化方程, 其中该序参数的边界条件为 **Neumann** 边界条件和无流条件相结合. 运用 **Galerkin** 方法证明了该模型在约化后的初边值问题弱解的存在性. 尽管所考虑的问题是序参数的单一方程, 但由于未知数 S 梯度项的存在，它仍存在固有的困难.

关键词

晶界运动，相场模型，抛物方程，弱解的存在性

Existence of Weak Solutions for a Class of Phase-Field Models with Conservation of Order Parameter

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Abstract

In this paper, our research will be based on the Alber-Zhu model which order parameter is conserved. The evolution equation in this model is a fourth order, nonlinear degenerate partial differential equation of parabolic type. This model is a phase-field model which describes the interface motion by interface diffusion in elastically deformed solids. For example, the basic phenomena occurring during this process, called sintering, are densification and grain growth. Since no atom exchange occurs at the interface, the volumes of the different regions separated by the interface are conserved. We ignore the elastic effect and reduce the original initial boundary value problem to a single non-degenerate equation of 1 dimensional situation, and prove the existence of the weak solution of the reduced initial boundary value problem of this model, where the boundary conditions of the order parameters are a combination of the Neumann boundary conditions and the no-flow conditions. The Galerkin method is used to prove the existence of weak solutions for the reduced initial boundary value problem of this model. Although the problem considered is a single equation of order parameters, it is inherently difficult due to the presence which is the gradient of unknown function S. Hence, we need to mollify the gradient term, which causes a lot of difficulties on the theoretical analysis and numerical simulation.

Keywords

Motion of Grain Boundaries, Phase-Field Model, Parabolic Equation, Existence of Weak Solutions

1. 介绍

Cahn和Hilliard在1958年建立了序参量守恒的相场模型 [1], 这种模型已被材料科学家广泛应用 [2]. 许多学者对Cahn-Hilliard模型的存在性 [3], 解的唯一性 [4], 解的正则性 [5]和渐近性 [6]进行了研究. 除了在材料科学领域, Cahn-Hilliard模型在其他领域也有应用, 如: 种群动态 [7], 肿瘤生长 [8], 细菌膜 [9], 薄膜 [10], 化学 [11]和图像处理 [12,13]. Allen-Cahn模型 [14,15]是材料科学中序参量不守恒的另一种模型. 早在1979年, 为了描述晶体中反相位边界运动, Allen和Cahn建立了这个模型, 并且被广泛的应用于处理各种问题, 例如图像分析 [16,17], 平均曲率-流量 [18], 和晶体生长 [19].

在2007年, Alber和朱佩成一起提出了新的相场模型, 称为Alber-Zhu模型 [20,21], Alber-Zhu模型用一个非光滑梯度项区别于Allen-Cahn模型和Cahn-Hilliard模型. 对于Alber-Zhu 模型, 有学者已经从各个方面进行了理论和数值的研究. 对于非守恒的Alber-Zhu 模型: 在时间范围内全局弱解的存在性 [21], 解的正则性 [22], 行波解 [23], 粘性解 [24,25], 数值模拟 [26]. 对于守恒模型: Alber和朱佩成在 [27]中首次求解了由界面扩散引起的界面运动耦合模型的弱解的存在性. 其中关于存在性的证明, 在 [28]得到了更一般性的结论. 这两种证明都是基于迭代法或局部解延拓法.

2. 模型建立

序参数守恒情形下的Alber-Zhu模型 [27]是一种可用于描述可弹性变形固体中原子沿界面扩散运动的界面相场模型. 首先, 为了建立这个模型, 引入一些符号. 设Ω 表示实体的质点, 并且$\Omega \subset \mathbb{R}^3$ 是一个开集. 序参数$S(t,x) \in \mathbb{R}$ 表示不同的相. 如果序参数的值接近0 或1, 那么在t 时刻，质点$x \in \Omega$, 处于相1 或相2. 设$\mathcal{S}^3$ 表示对称的3×3 矩阵的集合, 其中一个未知量$T(t,x) \in \mathcal{S}^3$ 代表柯西应力张量, $u(t,x) \in \mathbb{R}^3$ 表示位移.

未知函数(u,T,S) 须满足以下准静态方程:

$$-\mathrm{div}_x\, T(t,x) = b(t,x), \tag{2.1}$$

$$T(t,x) = D(\varepsilon(\nabla_x u) - \bar{\varepsilon} S)(t,x), \tag{2.2}$$

$$S_t(t,x) = c\,\mathrm{div}_x(\nabla_x(\psi_S(\varepsilon(\nabla_x u), S) - \nu \Delta_x S)|\nabla_x S|)(t,x), \tag{2.3}$$

其中$(t,x) \in (0,\infty) \times \Omega$. 方程的初边值条件是:

$$u(t,x) = 0, \ (t,x) \in [0,\infty) \times \partial\Omega, \tag{2.4}$$

$$\frac{\partial}{\partial n} S(t,x) = 0, \ (t,x) \in [0,\infty) \times \partial\Omega, \tag{2.5}$$

$$\frac{\partial}{\partial n}(\psi_S(\varepsilon, S) - \nu \Delta_x S)|\nabla_x S|(t,x) = 0, \ (t,x) \in [0,\infty) \times \partial\Omega, \tag{2.6}$$

$$S(0,x) = S_0(x), \ x \in \bar{\Omega}. \tag{2.7}$$

其中, $b : [0,\infty) \times \Omega \to \mathbb{R}^3$ 叫做体积力, $\nabla_x u$ 表示u 的一阶导数是3×3 矩阵, $(\nabla_x u)^T$ 表示位移梯度的转置矩阵, 且

$$\varepsilon(\nabla_x u) = \frac{1}{2}\big(\nabla_x u + (\nabla_x u)^T\big)$$

是应变张量. $\bar{\varepsilon} \in \mathcal{S}^3$ 是一个给定的矩阵, $D : \mathcal{S}^3 \to \mathcal{S}^3$ 是一个线性的, 对称的, 正定的映射, 代表弹性张量. 在自由能中

$$\psi(\varepsilon, S) = \frac{1}{2}\big(D(\varepsilon - \bar{\varepsilon}S)\big) \cdot (\varepsilon - \bar{\varepsilon}S) + \hat{\psi}(S), \tag{2.8}$$

令$\hat{\psi} \in C^2(\mathbb{R}, [0, \infty))$ 表示双势阱函数, 其在$S = 0$ 和$S = 1$ 处取得最小值. 本文取一个特殊的双势阱函数:

$$\hat{\psi}(S) = \big(S(1 - S)\big)^2. \tag{2.9}$$

两个矩阵的标量积为$A \cdot B = \sum a_{ij}b_{ij}$. ψ_S 是关于S 的偏导数, $c > 0$ 是一个常数, ν 一个取得足够小的正常数. 初值数据$S_0 : \Omega \to \mathbb{R}$ 是给定的.

这样就完成了初边值问题的建立. 上述模型是一个包含弹性效应的耦合系统. 在本文中, 忽略了固体的弹性效应, 即$T = 0$, 我们有

$$\begin{aligned}
\psi_S &= -\frac{1}{2}\big(D(\bar{\varepsilon}) \cdot (\varepsilon - \bar{\varepsilon}S) + D(\varepsilon - \bar{\varepsilon}S) \cdot (\bar{\varepsilon})\big) + \hat{\psi}_S \\
&= -D(\varepsilon - \bar{\varepsilon}S) \cdot (\bar{\varepsilon}) + \hat{\psi}_S \\
&= \hat{\psi}_S.
\end{aligned}$$

将这个初边值问题简化为一维的问题(具体推导见 [27]). 该模型关于序参数的原方程是退化的, 其主部含有未知函数 S 的梯度, 而本文我们将梯度项进行磨光处理, 研究磨光后的非退化方程. 我们用非退化方程代替退化抛物方程(2.3) 来表述近似问题

$$S_t = c((\psi_S - \nu S_{xx})_x |S_x|_\kappa)_x + cr|S_x|_\kappa, \tag{2.10}$$

其中

$$|y|_\kappa := \sqrt{|y|^2 + \kappa^2}, \tag{2.11}$$

常数$\kappa \in (0, 1]$, 我们容易得到

$$|y| \le |y|_\kappa \le |y| + \kappa \le |y| + 1. \tag{2.12}$$

该简化的Alber-Zhu模型可以改写为以下形式:

$$S_t = c((\psi_S - \nu S_{xx})_x |S_x|_\kappa)_x + cr|S_x|_\kappa, \tag{2.13}$$

边界条件和初始条件是：

$$S_x = 0, \quad (t,x) \in [0,T_e] \times \partial\Omega, \tag{2.14}$$

$$(\hat{\psi}_S - \nu S_{xx})_x |S_x|_\kappa = 0, \quad (t,x) \in [0,T_e] \times \partial\Omega, \tag{2.15}$$

$$S(0,x) = S_0(x), \quad x \in \Omega. \tag{2.16}$$

设$\Omega = (a,d)$ 是有界开区间且常数$a < d$. 记$Q_T := (0,T_e) \times \Omega$, 且$T_e$ 是一个正常数, 定义

$$(\upsilon, \varphi)_Z = \int_Z \upsilon(y)\varphi(y)dy,$$

当$Z = \Omega$ 或者$Z = Q_{T_e}$.

定义 2.1 *设$r \in L^\infty(0,T_e)$ 且$S_0 \in L^2(\Omega)$. 称函数$S(x,t)$ 为初边值问题(2.13)–(2.16) 的一个弱解, 满足*

$$S \in L^\infty(0,T_e;H^1(\Omega)) \cap L^2(0,T_e;H^3(\Omega)), \tag{2.17}$$

且对于任意的测试函数$\varphi \in C_0^\infty((-\infty,T_e) \times \mathbb{R})$, 满足

$$(S,\varphi_t)_{Q_{T_e}} + c(\nu S_{xxx}|S_x|_\kappa, \varphi_x)_{Q_{T_e}} = c((\hat{\psi}_S)_x|S_x|_\kappa, \varphi_x)_{Q_{T_e}} - c(r|S_x|_\kappa, \varphi)_{Q_{T_e}} - (S_0, \varphi(0))_\Omega. \tag{2.18}$$

定理 2.1 *假设$S_0 \in H^1(\Omega)$, 问题(2.13) – (2.16) 存在弱解S 满足：*

$$S \in L^\infty(0,T_e;H^1(\Omega)) \cap L^2(0,T_e;H^3(\Omega)), \tag{2.19}$$

$$S_t \in L^{\frac{4}{3}}(0,T_e;W^{-1,\frac{4}{3}}(\Omega)). \tag{2.20}$$

3. 构造近似解

为证明问题(2.13) – (2.16) 弱解的存在性, 在本小节中, 利用Galerkin方法构造了该初边值问题的一个近似解, 并证明这个近似解的局部弱解存在.

首先, 取这样一个序列$\omega_1, \cdots, \omega_m, \cdots$, 它满足: $\omega_i \in C^\infty(\forall\, i)$, 并且$\omega_1, \cdots, \omega_m$, 是线性无关的. 对任意的$m$, 序列属于空间$H^1(\Omega)$. 在这里, ω_i 是下列方程的解:

$$\begin{cases} -\frac{\mathrm{d}^2\omega_i}{\mathrm{d}x^2} = \lambda_i\omega_i, \\ \frac{\mathrm{d}\omega_i}{\mathrm{d}x}\Big|_{\partial\Omega} = 0, \end{cases}$$

其中$\omega_1 = 1$. 考虑问题的近似解: $S^m = S^m(t)$, 它有以下形式:

$$S^m(t) = \sum_{i=0}^m g_{im}(t)\omega_i. \tag{3.1}$$

在这里, g_{im} 是由下列常微分方程组所决定的:

$$(S_t^m, \omega_j) = ((c(\hat{\psi}_{S^m}(S^m) - \nu S_{xx}^m)_x |S_x^m|_\kappa)_x + cr|S_x^m|_\kappa, \omega_j), 1 \le j \le m, \tag{3.2}$$

即

$$((S^m)', \omega_j) = c(((\hat{\psi}''(S^m)S_x^m - \nu S_{xxx}^m)|S_x^m|_\kappa)_x, \omega_j) + c(r|S_x^m|_\kappa, \omega_j), 1 \le j \le m, \tag{3.3}$$

根据分部积分公式, 得到

$$(((\hat{\psi}''(S^m)S_x^m - \nu S_{xxx}^m)|S_x^m|_\kappa)_x, \omega_j) = -((\hat{\psi}''(S^m)S_x^m - \nu S_{xxx}^m)|S_x^m|_\kappa, \omega_{jx}), 1 \le j \le m. \tag{3.4}$$

把(3.4) 代入到(3.3) 中, 可以得到

$$((S^m)', \omega_j) + c((\hat{\psi}''(S^m)S_x^m - \nu S_{xxx}^m)|S_x^m|_\kappa, \omega_{jx}) - c(r|S_x^m|_\kappa, \omega_j) = 0, 1 \le j \le m. \tag{3.5}$$

现在, 需要说明方程(3.5) 是一个常微分方程组.

由$(S^m)' = \sum\limits_{i=1}^{m} g'_{im}(t)\omega_i$ 并且$\{\omega_i\}$ 在空间$L^2(\Omega)$ 中是标准正交的, 所以

$$\begin{aligned}\big((S^m)', \omega_j\big) &= (\sum_{i=1}^{m} g'_{im}(t)\omega_i, \omega_j\big) \\ &= \int_\Omega \sum_{i=1}^{m} g'_{im}(t)\omega_i(x)\omega_j(x)dx \\ &= \sum_{i=1}^{m} g'_{im}(t)\int_\Omega \omega_i(x)\omega_j(x)dx = g'_{jm}(t), 1 \le j \le m,\end{aligned} \tag{3.6}$$

只与变量时间t 有关. 同理, 微分方程组(3.5) 只与变量t 有关.

利用双势阱函数(2.9) 可以得到:

$$\begin{aligned}&((\hat{\psi}''(S^m)S_x^m - \nu S_{xxx}^m)|S_x^m|_\kappa, \omega_{j,x}) \\ =&(((12(S^m)^2 - 12S^m + 2)S_x^m - \nu S_{xxx}^m)|S_x^m|_\kappa, \omega_{j,x}) \\ =&12((S^m)^2 S_x^m|S_x^m|_\kappa, \omega_{j,x}) - 12(S^m S_x^m|S_x^m|_\kappa, \omega_{j,x}) \\ &+ 2(S_x^m|S_x^m|_\kappa, \omega_{j,x}) - \nu(S_{xxx}^m|S_x^m|_\kappa, \omega_{j,x}), 1 \le j \le m.\end{aligned} \tag{3.7}$$

由于在Ω 中, $-\frac{\mathrm{d}^2\omega_i}{\mathrm{d}x^2}=\lambda_i\omega_i$

$$\begin{aligned}
&\nu(S^m_{xxx}|S^m_x|_\kappa,\omega_{j,x})\\
=&\nu\int_\Omega\sum_{i=1}^m g_{im}\omega_{i,xxx}\left|\sum_{h=1}^m g_{hm}\omega_{h,x}\right|_\kappa\omega_{j,x}dx\\
=&-\nu\int_\Omega\sum_{i=1}^m g_{im}\lambda_i\omega_{i,x}\left|\sum_{h=1}^m g_{hm}\omega_{h,x}\right|_\kappa\omega_{j,x}dx\\
=&-\nu\sum_{i=1}^m g_{im}\lambda_i\int_\Omega\left|\sum_{h=1}^m g_{hm}\omega_{h,x}\right|_\kappa\omega_{i,x}\omega_{j,x}dx,1\le j\le m,
\end{aligned}\tag{3.8}$$

将(3.8) 代入到(3.7) 中, 可以得到

$$\begin{aligned}
&((\hat{\psi}''(S^m)S^m_x-\nu S^m_{xxx})|S^m_x|_\kappa,\omega_{j,x})\\
=&12\sum_{i=1}^m g_{im}\sum_{k=1}^m g_{km}\sum_{l=1}^m g_{lm}\int_\Omega\left|\sum_{h=1}^m g_{hm}\omega_{h,x}\right|_\kappa\omega_i\omega_k\omega_{l,x}\omega_{j,x}dx\\
&-12\sum_{k=1}^m g_{km}\sum_{l=1}^m g_{lm}\int_\Omega\left|\sum_{h=1}^m g_{hm}\omega_{h,x}\right|_\kappa\omega_k\omega_{l,x}\omega_{j,x}dx\\
&+2\sum_{l=1}^m g_{lm}\int_\Omega\left|\sum_{h=1}^m g_{hm}\omega_{h,x}\right|_\kappa\omega_{l,x}\omega_{j,x}dx\\
&+\nu\sum_{i=1}^m g_{im}\lambda_i\int_\Omega\left|\sum_{h=1}^m g_{hm}\omega_{h,x}\right|_\kappa\omega_{i,x}\omega_{j,x}dx,1\le j\le m,
\end{aligned}\tag{3.9}$$

且

$$c(r|S^m_x|_\kappa,\omega_j)=cr\int_\Omega\left|\sum_{=1}^m g_{hm}\omega_{h,x}\right|_\kappa\omega_j dx,1\le j\le m.\tag{3.10}$$

微分方程组(3.5) 满足初始条件:

$$S^m(0)=S^m_0=\sum_{i=1}^m\alpha_{im}\omega_j\to S_0\in H^1_{\mathrm{per}}(\Omega),m\to\infty,\tag{3.11}$$

这里, $\alpha_{im}=g_{im}(0)$.

最后将(3.6)–(3.10) 代入(3.5), 则原常微分方程组可改写为关于$\{g_{jm}\}_{j=1}^m$ 的常微分方程组, 如下所示:

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}g_{jm} &= F_j(g_{1m},\ldots,g_{mm},t),\\ g_{jm}(0) &= (S_0,\omega_j),\end{cases}\tag{3.12}$$

其中

$$
\begin{aligned}
&F_j(g_{1m},\ldots,g_{mm},t)\\
=&-12\sum_{i=1}^{m}g_{im}\sum_{k=1}^{m}g_{km}\sum_{l=1}^{m}g_{lm}\int_\Omega\left|\sum_{h=1}^{m}g_{hm}\omega_{h,x}\right|_\kappa\omega_i\omega_k\omega_{l,x}\omega_{j,x}dx\\
&+12\sum_{k=1}^{m}g_{km}\sum_{l=1}^{m}g_{lm}\int_\Omega\left|\sum_{h=1}^{m}g_{hm}\omega_{h,x}\right|_\kappa\omega_k\omega_{l,x}\omega_{j,x}dx\\
&-2\sum_{l=1}^{m}g_{lm}\int_\Omega\left|\sum_{h=1}^{m}g_{hm}\omega_{h,x}\right|_\kappa\omega_{l,x}\omega_{j,x}dx\\
&-\nu\sum_{i=1}^{m}g_{im}\lambda_i\int_\Omega\left|\sum_{h=1}^{m}g_{hm}\omega_{h,x}\right|_\kappa\omega_{i,x}\omega_{j,x}dx\\
&+cr\int_\Omega\left|\sum_{h=1}^{m}g_{hm}\omega_{h,x}\right|_\kappa\omega_j dx,
\end{aligned}
\tag{3.13}
$$

其中$j=1,\ldots,m$. 显然, (3.12) – (3.13) 是一个非线性项F 关于未知函数满足局部$Lipschitz$ 连续的非线性常微分方程组, 根据常微分方程组局部解存在性定理, 可知该解存在于$[0,t_m]$. 根据以上结论, 可以得到对于任意给定的m, S^m 为近似问题(3.5) 的一个局部解. 在下一节中, 将推导关于S^m 有关的先验估计, 证明t_m 可以推广到任意给定的正常数T_e.

4. 一致先验估计

在这一节中, 我们致力于推导关于解S^m 的一些先验估计.

引理 4.1 对任何$t\in[0,T_e]$

$$S^m\in L^\infty(0,T_e;L^2(\Omega)),\tag{4.1}$$

$$S_x^m\in L^3(0,T_e;L^3(\Omega)),\tag{4.2}$$

$$\int_{Q_{T_e}}(S^m)^2|S_x^m|^3dxd\tau\le C,\tag{4.3}$$

$$\int_{Q_{T_e}}(S_{xx}^m)^2|S_x^m|dxd\tau\le C,\tag{4.4}$$

$$\kappa\int_{Q_{T_e}}|S_{xx}^m|^2dxd\tau\le C.\tag{4.5}$$

证明. 对(3.5) 的第j 个方程乘以$g_{jm}(t)$, 然后对j 求和, 这样可以得到:

$$((S^m)',S^m)+c((\hat\psi''(S^m)S_x^m-\nu S_{xxx}^m)|S_x^m|_\kappa),S_x^m)-c(r|S_x^m|_\kappa,S^m)=0.\tag{4.6}$$

其中

$$((S^m)',S^m)=\frac{1}{2}\frac{d}{dt}\|S^m\|_{L^2(\Omega)}^2.\tag{4.7}$$

通过对x 进行分部积分, 可以得到

$$(S^m_{xxx}|S^m_x|_\kappa, S^m_x) = -(S^m_{xx}, S^m_{xx}|S^m_x|_\kappa) - (S^m_{xx}, S^m_{xx}(S^m_x)^2(|S^m_x|_\kappa)^{-1}). \tag{4.8}$$

将(4.7) 和(4.8) 代入到(4.6) 中, 得到

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|S^m\|^2_{L^2(\Omega)} + c(\hat{\psi}''(S^m)S^m_x|S^m_x|_\kappa, S^m_x) + c\nu(S^m_{xx}, S^m_{xx}|S^m_x|_\kappa)\\ &+ c\nu(S^m_{xx}, S^m_{xx}(S^m_x)^2(|S^m_x|_\kappa)^{-1}) - c(r|S^m_x|_\kappa, S^m) = 0,\end{aligned} \tag{4.9}$$

这里

$$\hat{\psi}''(S^m) = 12(S^m)^2 - 12S^m + 2. \tag{4.10}$$

将(4.10) 代入到(4.9), 可得

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|S^m\|^2_{L^2(\Omega)} + c\int_\Omega (12(S^m)^2 - 12S^m + 2)|S^m_x|^3_\kappa dx\\ &+ c\nu\int_\Omega (S^m_{xx})^2|S^m_x|_\kappa dx + c\nu\int_\Omega (S^m_{xx})^2(S^m_x)^2(|S^m_x|_\kappa)^{-1}dx - cr\int_\Omega |S^m_x|_\kappa S^m dx = 0,\end{aligned} \tag{4.11}$$

可将(4.11) 改写为

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|S^m\|^2_{L^2(\Omega)} + 12c\int_\Omega (S^m)^2|S^m_x|^2|S^m_x|_\kappa dx + 2c\int_\Omega |S^m_x|^2|S^m_x|_\kappa dx\\ &+ c\nu\int_\Omega (S^m_{xx})^2|S^m_x|_\kappa dx + c\nu\int_\Omega (S^m_{xx})^2(S^m_x)^2(|S^m_x|_\kappa)^{-1}dx\\ =&12c\int_\Omega S^m|S^m_x|^2|S^m_x|_\kappa dx + cr\int_\Omega |S^m_x|_\kappa S^m dx\\ =&: I_1 + I_2.\end{aligned} \tag{4.12}$$

结合(2.11), 应用$Young$ 不等式, 来处理I_1 部分,

$$\begin{aligned}I_1 &= 12c\int_\Omega S^m|S^m_x|^2|S^m_x|_\kappa dx \le 12c\int_\Omega S^m|S^m_x|^3_\kappa dx \le 12c\int_\Omega (\varepsilon(S^m)^2 + C_\varepsilon)|S^m_x|^3_\kappa dx\\ &\le \varepsilon\int_\Omega (S^m)^2|S^m_x|^3_\kappa dx + C\int_\Omega |S^m_x|^3_\kappa dx \le \varepsilon\int_\Omega (S^m)^2(|S^m_x|^3 + C)dx + C\int_\Omega (|S^m_x|^3 + C)dx\\ &\le \varepsilon\int_\Omega (S^m)^2|S^m_x|^3dx + \varepsilon\int_\Omega (S^m)^2dx + C\int_\Omega |S^m_x|^3dx + C,\end{aligned} \tag{4.13}$$

再由分部积分, 得到

$$\int_\Omega |S^m_x|^3dx = \int_\Omega (S^m_x|S^m_x|)S^m_x dx = -2\int_\Omega S^m|S^m_x|S^m_{xx}dx. \tag{4.14}$$

利用$Young$ 不等式可得

$$
\begin{aligned}
\int_\Omega |S_x^m|^3 dx &= \int_\Omega (S_x^m|S_x^m|)S_x^m dx = -2\int_\Omega S^m|S_x^m|S_{xx}^m dx \\
&\le 2\int_\Omega |S^m||S_x^m||S_{xx}^m|dx = 2\int_\Omega |S^m||S_x^m|^{\frac{1}{2}}\cdot|S_x^m|^{\frac{1}{2}}|S_{xx}^m|dx \\
&\le C_\xi\int_\Omega |S^m|^2|S_x^m|dx + \xi\int_\Omega |S_x^m||S_{xx}^m|^2dx \\
&\le C_\xi\int_\Omega |S^m|^2(\eta|S_x^m|^3 + C_\eta)dx + \xi\int_\Omega |S_x^m||S_{xx}^m|^2dx \\
&\le \eta\int_\Omega |S^m|^2|S_x^m|^3dx + C\int_\Omega |S^m|^2dx + \xi\int_\Omega |S_x^m||S_{xx}^m|^2dx.
\end{aligned}
\tag{4.15}
$$

接下来处理I_2 部分,

$$
\begin{aligned}
I_2 &= cr\int_\Omega |S_x^m|_\kappa S^m dx = cr\int_\Omega |S_x^m|_\kappa (S^m)^{\frac{2}{3}}(S^m)^{\frac{1}{3}}dx \\
&\le cr\gamma\int_\Omega |S_x^m|_\kappa^3|S^m|^2dx + crC_\gamma\int_\Omega (S^m)^{\frac{1}{2}}dx \\
&\le cr\gamma\int_\Omega |S_x^m|_\kappa^3|S^m|^2dx + crC_\gamma\int_\Omega (\zeta|S^m|^2 + C_\zeta)dx \\
&\le \gamma\int_\Omega |S_x^m|_\kappa^3|S^m|^2dx + \zeta\int_\Omega |S^m|^2dx + rC \\
&\le \gamma\int_\Omega (|S_x^m|^3 + C)|S^m|^2dx + \zeta\int_\Omega |S^m|^2dx + rC \\
&\le \gamma\int_\Omega |S_x^m|^3|S^m|^2dx + (\gamma+\zeta)\int_\Omega |S^m|^2dx + rC,
\end{aligned}
\tag{4.16}
$$

通过(4.13) – (4.16), 可以得到

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|S^m\|_{L^2(\Omega)}^2 + 12c\int_\Omega (S^m)^2|S_x^m|^2|S_x^m|_\kappa dx + 2c\int_\Omega |S_x^m|^2|S_x^m|_\kappa dx \\
&+ c\nu\int_\Omega (S_{xx}^m)^2|S_x^m|_\kappa dx + c\nu\int_\Omega (S_{xx}^m)^2(S_x^m)^2(|S_x^m|_\kappa)^{-1}dx \\
\le&(\varepsilon+\eta+\gamma)\int_\Omega (S^m)^2|S_x^m|^3dx + (\varepsilon+C+\gamma+\zeta)\int_\Omega |S^m|^2dx + \xi\int_\Omega |S_x^m||S_{xx}^m|^2dx + rC,
\end{aligned}
\tag{4.17}
$$

令$\varepsilon,\xi,\eta,\gamma,\zeta$ 取足够小, 使得

$$
\begin{aligned}
(\varepsilon+\eta+\gamma) \le&\quad c, \\
(\varepsilon+C+\gamma+\zeta) \le&\quad 2C, \\
\xi \le&\quad \frac{c\nu}{2}.
\end{aligned}
\tag{4.18}
$$

然后通过(4.17) 和(4.18), 得到

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|S^m\|^2_{L^2(\Omega)}+11c\int_\Omega (S^m)^2|S^m_x|^2|S^m_x|_\kappa dx+2c\int_\Omega |S^m_x|^2|S^m_x|_\kappa dx\\&+\frac{c\nu}{2}\int_\Omega (S^m_{xx})^2|S^m_x|_\kappa dx+c\nu\int_\Omega (S^m_{xx})^2(S^m_x)^2(|S^m_x|_\kappa)^{-1}dx\\\leq&2C\int_\Omega |S^m|^2dx+Cr.\end{aligned}\tag{4.19}$$

对(4.19) 关于时间t 做积分, 通过应用$Gronwall$ 不等式, 可得

$$\begin{aligned}&\frac{1}{2}\|S^m\|^2_{L^2(\Omega)}+11c\int_o^t\int_\Omega (S^m)^2|S^m_x|^2|S^m_x|_\kappa dxd\tau+2c\int_o^t\int_\Omega |S^m_x|^2|S^m_x|_\kappa dxd\tau\\&+\frac{c\nu}{2}\int_o^t\int_\Omega (S^m_{xx})^2|S^m_x|_\kappa dxd\tau+c\nu\int_o^t\int_\Omega (S^m_{xx})^2(S^m_x)^2(|S^m_x|_\kappa)^{-1}dxd\tau\\\leq&C_t+C_t\|S^m(0)\|^2_{L^2}\leq C_{T_e}.\end{aligned}\tag{4.20}$$

通过(4.20), (2.11) 和(2.12), 可得到引理4.1 的结论. ∎

引理 4.2 对任意的$t\in[0,T_e]$

$$S^m_x\in L^\infty(0,T_e;L^2(\Omega)),\tag{4.21}$$

$$S^m\in L^\infty(0,T_e;H^1(\Omega)),\tag{4.22}$$

$$\int_{Q_t}(S^m_{xxx})^2|S^m_x|_\kappa dxd\tau\leq C,\tag{4.23}$$

$$\kappa\int_{Q_{T_e}}|S^m_{xxx}|^2dxd\tau\leq C.\tag{4.24}$$

证明. 将(3.3) 的第j 个方程乘以$\lambda_j g_{jm}(t)$, 然后对j 从0 到m 求和

$$((S^m)',-S^m_{xx})-c(((\hat\psi''(S^m)S^m_x-\nu S^m_{xxx})|S^m_x|_\kappa)_x,-S^m_{xx})-c(r|S^m_x|_\kappa,-S^m_{xx})=0,\tag{4.25}$$

即

$$((S^m)',-S^m_{xx})-c((\hat\psi''(S^m)S^m_x-\nu S^m_{xxx})|S^m_x|_\kappa,S^m_{xxx})+c(r|S^m_x|_\kappa,S^m_{xx})=0.\tag{4.26}$$

由于$t\in[0,T_e]$, 可以得到

$$((S^m)',-S^m_{xx})=\frac{1}{2}\frac{d}{dt}\|S^m_x\|^2_{L^2(\Omega)},\tag{4.27}$$

且

$$\hat\psi''(S^m)=12(S^m)^2-12S^m+2.\tag{4.28}$$

把(4.27) 代入(4.26), 可以得到

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|S_x^m\|_{L^2(\Omega)}^2+c\nu\int_\Omega (S_{xxx}^m)^2|S_x^m|_\kappa dx\\=&c\int_\Omega \hat{\psi}''(S^m)S_x^mS_{xxx}^m|S_x^m|_\kappa dx-cr\int_\Omega |S_x^m|_\kappa S_{xx}^m dx\\=&0,\end{aligned}\tag{4.29}$$

然后, 通过(4.28) 可以得到

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|S_x^m\|_{L^2(\Omega)}^2+c\nu\int_\Omega (S_{xxx}^m)^2|S_x^m|_\kappa dx\\=&12c\int_\Omega (S^m)^2S_x^mS_{xxx}^m|S_x^m|_\kappa dx-12c\int_\Omega S^mS_x^mS_{xxx}^m|S_x^m|_\kappa dx\\&+2c\int_\Omega S_x^mS_{xxx}^m|S_x^m|_\kappa dx-cr\int_\Omega |S_x^m|_\kappa S_{xx}^m dx\\\le&12c\int_\Omega (S^m)^2S_x^mS_{xxx}^m|S_x^m|_\kappa dx+12c\int_\Omega |S^m||S_x^m|_\kappa^2|S_{xxx}^m|dx\\&+2c\int_\Omega S_x^mS_{xxx}^m|S_x^m|_\kappa dx+cr\int_\Omega |S_x^m|_\kappa|S_{xx}^m|dx\\=:&I_3+I_4+I_5+I_6.\end{aligned}\tag{4.30}$$

结合(2.11), 利用$Young$ 不等式, 可得

$$\begin{aligned}I_3&=12c\int_\Omega (S^m)^2S_x^mS_{xxx}^m|S_x^m|_\kappa dx\\&=12c\int_\Omega (S^m)^2S_x^m|S_x^m|_\kappa^{\frac{1}{2}}\cdot|S_x^m|_\kappa^{\frac{1}{2}}S_{xxx}^m dx\\&\le C\int_\Omega (S^m)^4|S_x^m|_\kappa^3dx+\varepsilon\int_\Omega |S_x^m|_\kappa|S_{xxx}^m|^2dx.\end{aligned}\tag{4.31}$$

由已知的$Sobolev$ 嵌入定理和引理4.1 的结论, 得到

$$\begin{aligned}&\int_\Omega (S^m)^4|S_x^m|_\kappa^3dx\le\int_\Omega (S^m)^2|S_x^m|_\kappa^3dx\cdot\|S^m\|_{L^\infty}^2\\\le&\int_\Omega (S^m)^2|S_x^m|_\kappa^3dx(\|S_x^m\|_{L^2}^2+\|S^m\|_{L^2}^2)\le\int_\Omega (S^m)^2|S_x^m|_\kappa^3dx(\|S_x^m\|_{L^2}^2+C)\\\le&\|S_x^m\|_{L^2}^2\int_\Omega (S^m)^2|S_x^m|_\kappa^3dx+C\int_\Omega (S^m)^2|S_x^m|_\kappa^3dx.\\\le&\|S_x^m\|_{L^2}^2\int_\Omega (S^m)^2(|S_x^m|^3+C)dx+C\int_\Omega (S^m)^2(|S_x^m|^3+C)dx.\\\le&\|S_x^m\|_{L^2}^2\int_\Omega (S^m)^2|S_x^m|^3dx+C\|S_x^m\|_{L^2}^2\int_\Omega (S^m)^2dx+C\int_\Omega (S^m)^2|S_x^m|^3dx+C\int_\Omega (S^m)^2dx,\end{aligned}\tag{4.32}$$

利用$Young$ 不等式得到

$$
\begin{aligned}
I_4 = 12c\int_\Omega |S^m||S_x^m|_\kappa^2|S_{xxx}^m|dx &= 12c\int_\Omega |S^m||S_x^m|_\kappa^{\frac{3}{2}}\cdot|S_x^m|_\kappa^{\frac{1}{2}}|S_{xxx}^m|dx \\
&\le C\int_\Omega (S^m)^2|S_x^m|_\kappa^3dx + \xi\int_\Omega |S_x^m|_\kappa|S_{xxx}^m|^2dx, \\
&\le C\int_\Omega (S^m)^2|S_x^m|^3dx + C\int_\Omega (S^m)^2dx + \xi\int_\Omega |S_x^m|_\kappa|S_{xxx}^m|^2dx,
\end{aligned} \tag{4.33}
$$

$$
\begin{aligned}
I_5 = 2c\int_\Omega S_x^mS_{xxx}^m|S_x^m|_\kappa dx &= 2c\int_\Omega |S_x^m||S_x^m|_\kappa^{\frac{1}{2}}\cdot|S_x^m|_\kappa^{\frac{1}{2}}|S_{xxx}^m|dx \\
&\le C\int_\Omega |S_x^m|_\kappa^3dx + \eta\int_\Omega |S_x^m|_\kappa|S_{xxx}^m|^2dx \\
&\le C\int_\Omega |S_x^m|^3dx + \eta\int_\Omega |S_x^m|_\kappa|S_{xxx}^m|^2dx + C,
\end{aligned} \tag{4.34}
$$

且

$$
\begin{aligned}
I_6 = cr\int_\Omega |S_x^m|_\kappa|S_{xx}^m|dx &= cr\int_\Omega |S_x^m|_\kappa^{\frac{1}{2}}|S_x^m|_\kappa^{\frac{1}{2}}|S_{xx}^m|dx \\
&\le crC_\gamma\int_\Omega |S_x^m|_\kappa dx + cr\gamma\int_\Omega |S_x^m|_\kappa|S_{xx}^m|^2dx \\
&\le crC_\gamma\int_\Omega (\mu|S_x^m|_\kappa^2 + C_\mu)dx + cr\gamma\int_\Omega |S_x^m|_\kappa|S_{xx}^m|^2dx \\
&\le cr\mu C_\gamma\int_\Omega |S_x^m|_\kappa^2dx + crC_\gamma\int_\Omega C_\mu dx + cr\gamma\int_\Omega |S_x^m|_\kappa|S_{xx}^m|^2dx \\
&\le \mu\int_\Omega |S_x^m|_\kappa^2dx + \gamma\int_\Omega |S_x^m|_\kappa|S_{xx}^m|^2dx + rC \\
&\le \mu\int_\Omega (|S_x^m|^2 + C)dx + \gamma\int_\Omega |S_x^m|_\kappa|S_{xx}^m|^2dx + rC. \\
&\le \mu\int_\Omega |S_x^m|^2dx + \gamma\int_\Omega |S_x^m|_\kappa|S_{xx}^m|^2dx + rC.
\end{aligned} \tag{4.35}
$$

结合(4.31) − (4.35), 可得

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|S_x^m\|_{L^2(\Omega)}^2 + c\nu\int_\Omega (S_{xxx}^m)^2|S_x^m|_\kappa dx \\
\le&(\varepsilon+\xi+\eta)\int_\Omega |S_x^m|_\kappa|S_{xxx}^m|^2dx + C\|S_x^m\|_{L^2}^2\int_\Omega (S^m)^2|S_x^m|^3dx \\
&+ C\int_\Omega (S^m)^2|S_x^m|^3dx + C\int_\Omega |S_x^m|^3dx + C\|S_x^m\|_{L^2}^2\int_\Omega (S^m)^2dx + C\int_\Omega (S^m)^2dx \\
&+ \mu\int_\Omega |S_x^m|^2dx + \gamma\int_\Omega |S_x^m|_\kappa|S_{xx}^m|^2dx + rC.
\end{aligned} \tag{4.36}
$$

令$\varepsilon,\xi,\eta,\mu,\gamma$ 足够小, 并且$0<\varepsilon+\xi+\eta<\frac{c\nu}{2}$, 结合引理4.1, 我们有

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|S_x^m\|_{L^2(\Omega)}^2+\frac{c\nu}{2}\int_\Omega(S_{xxx}^m)^2|S_x^m|_\kappa dx\\ \leq& C\|S_x^m\|_{L^2}^2\int_\Omega(S^m)^2|S_x^m|^3dx+C\int_\Omega(S^m)^2|S_x^m|^3dx\\ &+C\int_\Omega|S_x^m|^3dx+C\|S_x^m\|_{L^2}^2\int_\Omega(S^m)^2dx+C\int_\Omega(S^m)^2dx+rC.\end{aligned}\tag{4.37}$$

然后, 根据微分形式的$Gronwall$ 不等式, 以及(4.1)–(4.24), 可以得到

$$\frac{1}{2}\|S_x^m(t)\|_{L^2(\Omega)}^2\leq C+C\|S_x^m(0)\|_{L^2(\Omega)}^2\leq C_{T_e}.\tag{4.38}$$

对(4.37) 关于时间t 积分, 根据(4.38), 以及引理4.1 可以得到

$$\frac{1}{2}\|S_x^m\|_{L^2(\Omega)}^2+\frac{c\nu}{2}\int_0^t\int_\Omega(S_{xxx}^m)^2|S_x^m|_\kappa dxd\tau\quad\leq C_{T_e}.\tag{4.39}$$

至此引理4.2 证明完毕. ▮

引理 4.3 对任意的$t\in[0,T_e]$,

$$\int_0^t\int_\Omega(|S_x^m|_\kappa|S_{xxx}^m|)^{\frac{4}{3}}dxd\tau\quad\leq C,\tag{4.40}$$

$$\||S_x^m|_\kappa S_{xxx}^m\|_{L^{\frac{4}{3}}(Q_{T_e})}\quad\leq C,\tag{4.41}$$

$$\|((\hat{\psi}_{S^m})_x|S_x^m|_\kappa\|_{L^{\frac{3}{2}}(Q_{T_e})}\quad\leq C.\tag{4.42}$$

证明. 通过Hölder不等式, 对于$1\leq p<2$, $q=\frac{2}{p}$, 且$\frac{1}{q}+\frac{1}{q'}=1$, 得到

$$\begin{aligned}&\int_0^t\int_\Omega(|S_x^m|_\kappa|S_{xxx}^m|)^pdxd\tau\\ =&\int_0^t\int_\Omega|S_x^m|_\kappa^{\frac{p}{2}}(|S_x^m|_\kappa^{\frac{p}{2}}|S_{xxx}^m|^p)dxd\tau\\ \leq&\left(\int_0^t\int_\Omega|S_x^m|_\kappa^{\frac{pq'}{2}}dxd\tau\right)^{\frac{1}{q'}}\left(\int_0^t\int_\Omega|S_x^m|_\kappa^{\frac{pq}{2}}|S_{xxx}^m|^{pq}dxd\tau\right)^{\frac{1}{q}}\\ \leq&\left(\int_0^t\int_\Omega|S_x^m|_\kappa^{\frac{p}{2-p}}dxd\tau\right)^{\frac{2-p}{2}}\left(\int_0^t\int_\Omega|S_x^m|_\kappa|S_{xxx}^m|^{pq}dxd\tau\right)^{\frac{p}{2}}.\end{aligned}\tag{4.43}$$

不等式(4.43) 意味着对$\frac{p}{2-p}\leq 2$, 即$p\leq\frac{4}{3}$, 不等式的右边是有界的, 当$p=\frac{4}{3}$ 时, 可得

$$\begin{aligned}&\int_0^t\int_\Omega(|S_x^m|_\kappa|S_{xxx}^m|)^{\frac{4}{3}}dxd\tau\\ \leq&\left(\int_0^t\int_\Omega|S_x^m|_\kappa^2dxd\tau\right)^{\frac{1}{3}}\left(\int_0^t\int_\Omega|S_x^m|_\kappa|S_{xxx}^m|^2dxd\tau\right)^{\frac{2}{3}}.\end{aligned}\tag{4.44}$$

由引理4.1 和引理4.2 的结论可得(4.40), (4.41) 得证.

通过Hölder不等式, 令$p=\frac{3}{2}$, 以及由引理4.1 和引理4.2 的结论可以得出

$$\begin{aligned}&\int_0^t\int_\Omega((\hat{\psi}_{S^m})_x|S_x^m|_\kappa)^p dxd\tau\\ =&\int_0^t\int_\Omega((\hat{\psi}''S_x^m|S_x^m|_\kappa)^p dxd\tau\\ \leq& C\int_0^t\int_\Omega(|S_x^m|_\kappa)^{2p}dxd\tau\leq C.\end{aligned}\tag{4.45}$$

综上引理4.3 证明完成. ∎

引理 4.4 *存在一个常数C, 使得*

$$\|S_t^m\|_{L^{\frac{4}{3}}(0,T_e;W^{-1,\frac{4}{3}}(\Omega))}\quad\leq\quad C.\tag{4.46}$$

证明. 结合引理4.1 和引理4.3 的结论, 我们可以得到, 对任意的$\varphi\in C_0^\infty(Q_{T_e})$,

$$\begin{aligned}|(S_t^m,\varphi)_{Q_{T_e}}|\quad&=\quad|-c((\hat{\psi}_{S^m}-\nu S_{xx}^m)_x|S_x^m|_\kappa,\varphi_x)_{Q_{T_e}}+c(r|S_x^m|,\varphi)_{Q_{T_e}}|\\ &\leq\quad c\|(\hat{\psi}_{S^m}-\nu S_{xx}^m)_x|S_x^m|_\kappa\|_{L^{\frac{4}{3}}(Q_{T_e})}\|\varphi_x\|_{L^4(Q_{T_e})}\\ &\quad\quad+c\bar{r}\||S_x^m|\|_{L^{\frac{4}{3}}(Q_{T_e})}\|\varphi\|_{L^4(Q_{T_e})}\\ &\leq\quad C\|\varphi\|_{L^4(0,T_e;W_0^{1,4}(\Omega))}.\end{aligned}\tag{4.47}$$

引理4.4 得证. ∎

5. 极限过程

在上一部分中, 建立了一系列的先验估计. 在本节中, 利用上一节中建立的先验估计来研究序列S^m 当$m\to\infty$ 时的收敛性. 本节将证明存在收敛于初边值问题弱解的子序列.

引理 5.1 (Aubin – Lions) *[29] 设B_0 和B_1 是自反的, 假设B 是一个Banach 空间, 使得B_0 紧嵌入到B 中, B 被嵌入到B_1 中. 对于$1\leq p_0,p_1\leq+\infty$, 定义*

$$W=\{f\mid f\in L^{p_0}(0,T;B_0),f'=\frac{\mathrm{d}f}{\mathrm{d}t}\in L^{p_1}(0,T;B_1)\}.$$

(i) 如果$p_0<+\infty$, 则W 紧嵌入到$L^{p_0}(0,T;B)$ 中.
(ii) 如果$p_0=+\infty$ 和$p_1>1$, 则W 紧嵌入到$C([0,T];B)$ 中.

这个引理的证明, 参考 [30]中的第57页. 以下是本节的主要结果.

定理2.1的证明: 在这一部分, 我们证明当$m \to \infty$ 时, S 是问题(2.13) – (2.16) 的弱解. 由估计(4.22), (4.24), (4.46) 及弱紧性定理, 我们可得存在子序列S^{m_n}, 这里还用S^m 表示, 满足:

$$S^m \rightharpoonup S, \quad 弱 * 收敛于 L^\infty(0, T_e; H^1(\Omega)), \tag{5.1}$$
$$S^m \rightharpoonup S, \quad 弱收敛于 L^2(0, T_e; H^3(\Omega)), \tag{5.2}$$
$$S_t^m \rightharpoonup S_t, \quad 弱收敛于 L^{\frac{4}{3}}(0, T_e; W^{-1,\frac{4}{3}}(\Omega)). \tag{5.3}$$

因此, 由(5.1), (5.2) 可推出(2.19), 且(5.3) 可证得结论(2.20) 成立. 最后, 我们研究(2.18) 的收敛性. 根据$Aubin - Lions$ 引理, 我们取:

$$p_0 = 2, p_1 = \tfrac{4}{3},$$
$$B_0 = H^2(\Omega), B = C^{1+\alpha}(\bar{\Omega}), B_1 = W^{-1,\frac{4}{3}}(\Omega),$$

则存在序列S^m, 使得

$$\|S^m - S\|_{L^2(0,T_e;C^{1+\alpha}(\bar{\Omega}))} \to 0, \quad m \to \infty. \tag{5.4}$$

由(5.4) 我们容易得出:

$$\|S_x^m - S_x\|_{L^2(0,T_e;C^{\alpha}(\bar{\Omega}))} \to 0, \quad m \to \infty. \tag{5.5}$$
$$\||S_x|_\kappa^m - |S_x|_\kappa\|_{L^2(0,T_e;L^2(\Omega))} \to 0, \quad m \to \infty. \tag{5.6}$$

从而, 我们可通过(5.2), (5.6) 得到

$$|S_x^m|_\kappa S_{xxx}^m \rightharpoonup |S_x|_\kappa S_{xxx}, \ 弱收敛于 L^1(Q_{T_e}). \tag{5.7}$$

此外, 通过(4.22), (5.4) 和(5.5) 不难得到

$$(\hat{\psi}_{S^m})_x = \hat{\psi}''(S^m)S_x^m \rightharpoonup \hat{\psi}''(S)S_x, \ 弱收敛于 L^2(Q_{T_e}). \tag{5.8}$$

即

$$(\hat{\psi}_{S^m})_x|S_x^m|_\kappa \rightharpoonup (\hat{\psi}_S)_x|S_x|_\kappa, \ 弱收敛于 L^1(Q_{T_e}). \tag{5.9}$$

通过(5.6) – (5.9) 可得对任意的$\varphi \in C_0^\infty((-\infty, T_e) \times \mathbb{R})$ 成立

$$(S^m, \varphi_t)_{Q_{T_e}} \quad \to \quad (S, \varphi_t)_{Q_{T_e}}, \tag{5.10}$$
$$c(r|S_x^m|_\kappa, \varphi)_{Q_{T_e}} \quad \to \quad c(r|S_x|_\kappa, \varphi)_{Q_{T_e}}, \tag{5.11}$$
$$(|S_x^m|_\kappa S_{xxx}^m, \varphi_x)_{Q_{T_e}} \quad \to \quad (|S_x|_\kappa S_{xxx}, \varphi_x)_{Q_{T_e}}, \tag{5.12}$$
$$((\hat{\psi}_{S^m})_x|S_x^m|_\kappa, \varphi_x)_{Q_{T_e}} \quad \to \quad ((\hat{\psi}_S)_x|S_x|_\kappa, \varphi_x)_{Q_{T_e}}. \tag{5.13}$$

即得S 满足弱形式(2.18). 这意味着S 是问题(2.13) – (2.16) 的一个弱解, 定理2.1 得证.

6. 总结

本文研究的是一类序参数守恒的相场模型的弱解存在性, 在Alber-Zhu序参数守恒模型的基础上, 忽略弹性效应, 化为一维空间非退化的单个方程. 运用Galerkin方法构造了该初边值问题的一个近似解并证明其局部弱解存在, 通过一致先验估计及标准的极限过程, 得到该约化问题整体弱解的存在性.

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