

微分差分方程亚纯解的性质

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收稿日期: 2023年11月27日; 录用日期: 2023年12月13日; 发布日期: 2024年1月26日

摘要

本文研究了一类微分差分方程

$$f^n + L_d(z, f) = b_1 e^{\omega_1 z} + \dots + b_m e^{\omega_m z}$$

的亚纯解的性质, 其中正整数 $n \geq m$, $L_d(z, f)$ 为 f 的微差分多项式, 且次数 $d = \deg(L_d) \leq n-1$, b_1, \dots, b_m 为非零常数, $\omega_1, \dots, \omega_m$ 为不同的非零常数。特别地, 在某些特定条件下给出了方程亚纯解的表达式。

关键词

亚纯函数, 微分差分方程, 值分布, 非线性

On the Properties of Meromorphic Solutions to Differential-Difference Equations

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Received: Nov. 27th, 2023; accepted: Dec. 13th, 2023; published: Jan. 26th, 2024

Abstract

The aim of this paper is to investigate the properties of meromorphic solutions to the differential-difference equation

$$f^n + L_d(z, f) = b_1 e^{\omega_1 z} + \dots + b_m e^{\omega_m z},$$

where $n, m \in \mathbb{N}^+$, $n \geq m$, $L_d(z, f)$ is a differential-difference polynomial in f of degree $d = \deg(L_d) \leq n-1$, b_1, \dots, b_m are nonzero constants, $\omega_1, \dots, \omega_m$ are distinct nonzero constants. In

particular, we give the exact form of meromorphic solutions of the above equation under certain conditions.

Keywords

Meromorphic Function, Differential-Difference Equation, Value Distribution, Nonlinear

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1. 引言

Nevanlinna 值分布理论是研究复域方程解的性质的重要工具,我们假定读者已熟悉常用的 Nevanlinna 理论的标准符号,如 $T(r, f)$, $m(r, f)$, $N(r, f)$ 等, 见文[1] [2], 为了简便,我们在本文用 $N_{10}(r, f)$ 表示 f 的单极点的计数函数, $N_{20}(r, f)$ 表示 f 的多重极点的计数函数.一般地,我们用 $S(r, f)$ 表示 $o(T(r, f))$, $r \rightarrow \infty$ 可能除去一个 r 的有限测度集合.若亚纯函数 $a(z)$ 满足 $T(r, a) = S(r, f)$, 称 a 为 f 的小函数.此外,定义 f 的级 $\rho(f)$, 零点收敛指数 $\lambda(f)$ 如下

$$\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ T(r, f)}{\log r}, \lambda(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ N\left(r, \frac{1}{f}\right)}{\log r}.$$

1964 年, Hayman [1] 将 Tumura-Clunie 定理 [3] [4] 推广到了只含有一个主项 f^n 的微分方程 $f^n(z) + P_d(z, f) = h(z)$, 其中 $P_d(z, f)$ 是 f 及其导数的微分多项式, 其次数为 d , 系数为 f 的小函数, $h(z)$ 为给定的整函数或亚纯函数. 2004 年, Yang-Li [5] 研究了一类更为具体的微分方程 $4f^3 + 3f'' = -\sin 3z$, 指出此方程具有三个非零整函数解 $f_1(z) = \sin z$, $f_2(z) = (\sqrt{3} \cos z - \sin z)/2$, $f_3(z) = (-\sqrt{3} \cos z - \sin z)/2$. 由此, 借助于三角函数与指数函数的关系, 在过去二十年, 许多学者开始广泛研究了以下形式的 Tumura-Clunie 型微分方程 [6] [7] [8] [9]

$$f^n(z) + P_d(z, f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}, \quad (1)$$

其中 $p_1(z)$, $p_2(z)$ 为非零有理函数且 $\alpha_1(z)$, $\alpha_2(z)$ 为非常数多项式.

2013 年, Liao-Yang-Zhang 推广了 Li [10] 的结论得到:

定理 A ([11]) 设整数 $n \geq 3$, 如果 $d \leq n-2$ 且方程(1)存在仅有有限个极点的亚纯解 f , 则 $\frac{\alpha'_1}{\alpha'_2}$ 为有理函数, 进一步地, 有下列情况之一成立:

1) $f(z) = q(z)e^{P(z)}$, $\frac{\alpha'_1}{\alpha'_2} = 1$, 其中 $q(z)$ 为有理函数, $P(z)$ 为多项式且 $nP' = \alpha'_1 = \alpha'_2$;

2) $f(z) = q(z)e^{P(z)}$, $\frac{\alpha'_1}{\alpha'_2} = \frac{n}{k}$ 或 $\frac{k}{n}$, 其中 $q(z)$ 为有理函数, k 为整数且 $1 \leq k \leq d$, $P(z)$ 为多项式且 $nP' = \alpha'_1$ 或 α'_2 ;

3) f 满足 $f' = \left(\frac{1}{n} \frac{p'_1}{p_1} + \frac{1}{n} \alpha'_1\right) f + \varphi$, $\frac{\alpha'_1}{\alpha'_2} = \frac{n}{n-1}$, 或 $f' = \left(\frac{1}{n} \frac{p'_2}{p_2} + \frac{1}{n} \alpha'_2\right) f + \varphi$, $\frac{\alpha'_1}{\alpha'_2} = \frac{n-1}{n}$, 其中 φ 为有

理函数:

$$4) f(z) = c_1(z)e^{\beta(z)} + c_2(z)e^{-\beta(z)}, \frac{\alpha'_1}{\alpha'_2} = -1, \text{ 其中 } c_1(z), c_2(z) \text{ 为有理函数, } \beta(z) \text{ 为多项式且 } n\beta' = \alpha'_1$$

或 α'_2 。

2018 年, Zhang 得到了如下结论:

定理 B ([12]) 设整数 $n \geq 4$, $d \leq n-3$, 若方程(1)存在仅有有限个极点的超越亚纯解 f , 则 $\frac{\alpha'_1}{\alpha'_2}$ 为有理

数且 $f(z) = q(z)e^{P(z)}$, 其中 $q(z)$ 为非零有理函数, $P(z)$ 为非常数多项式, 进一步有下列情况之一成立:

$$1) \frac{\alpha'_1}{\alpha'_2} = 1, P_d(z, f) \equiv 0 \text{ 且 } nP' = \alpha'_1 = \alpha'_2;$$

$$2) \frac{\alpha'_1}{\alpha'_2} = \frac{t}{n}, \text{ 其中 } t \text{ 为整数且 } 1 \leq t < d, P_d(z, f) = p_1(z)e^{\alpha_1(z)}, nP' = \alpha'_2, \text{ 或者 } \frac{\alpha'_1}{\alpha'_2} = \frac{n}{t}, \text{ 其中 } t \text{ 为整数}$$

且 $1 \leq t < d, P_d(z, f) = p_2(z)e^{\alpha_2(z)}, nP' = \alpha'_1$ 。

我们注意到(1)的右端是两个指数项, 2020 年, Chen-Lian 研究了右端有三个指数项的情况:

定理 C ([13]) 设整数 $n \geq 5$, $P_d(z, f)$ 为 f 的次数 $d \leq n-4$ 且系数为有理函数的微分多项式, 设 $p_j(z) (j=1,2,3)$ 为非零有理函数, $\alpha_j(z) (j=1,2,3)$ 为使得 $\alpha'_j(z) (j=1,2,3)$ 互异的非常数多项式, 若 $d \leq n-4$ 且微分方程

$$f^n(z) + P_d(z, f) = \sum_{j=1}^3 p_j e^{\alpha_j(z)} \tag{2}$$

存在仅有有限多个极点的超越亚纯解 f , 则 $\frac{\alpha'_1}{\alpha'_2}, \frac{\alpha'_2}{\alpha'_3}$ 为有理数且 $f(z) = q(z)e^{P(z)}$, 其中 $q(z)$ 为非零有理函数,

$P(z)$ 为非常数多项式。进一步地, 存在正整数 l_0, l_1, l_2 满足 $l_0, l_1, l_2 = 1, 2, 3$ 和不同的整数 k_1, k_2 满足 $1 \leq k_1, k_2 \leq d$, 使得 $\alpha'_0 : \alpha'_1 : \alpha'_2 = n : k_1 : k_2, nP' = \alpha'_0$, 且 $P_d(z, f) = p_{l_1}(z)e^{\alpha_{k_1}(z)} + p_{l_2}(z)e^{\alpha_{k_2}(z)}$ 。

受此启发, 我们自然想知道如果把方程(2)右端指数项扩充成 m 项会有什么样的结果? 如果减弱定理 C 中的条件 $n \geq 5, d \leq n-4$ 是否会有类似的结果出现呢?

定理 1 设正整数 $n \geq m, L_d(z, f)$ 为 f 及其导数的微差分多项式满足

$$L_d(z, f) = \sum_{j=0}^l a_j \prod_{k=0}^s (f^{(k)}(z + \eta_k))^{n_{k,j}},$$

且 $d = \deg(L_d) \leq n-1$, 其中 $l, s, n_{k,j} \in \mathbb{N}^+, \eta_k (0 \leq k \leq s)$ 为不同的复常数, $a_j (0 \leq j \leq l)$ 为常数。设 b_1, \dots, b_m 为非零常数, $\omega_1, \dots, \omega_m$ 为不同的非零常数。若方程

$$f^n + L_d(z, f) = b_1 e^{\omega_1 z} + \dots + b_m e^{\omega_m z} \tag{3}$$

存在有限级亚纯解 f 且 $N(r, f) = S(r, f)$, 则 $\rho(f) = 1$ 且有以下两种可能:

$$1) N\left(r, \frac{1}{f}\right) = S(r, f) \text{ 且 } f(z) = \tau_j e^{\frac{\omega_j z}{n}} (1 \leq j \leq m), \tau_j^n = b_j (1 \leq j \leq m), \text{ 且}$$

$$\omega_j : \omega_1 : \dots : \omega_{j-1} : \omega_{j+1} : \dots : \omega_m = n : n_1 : \dots : n_{m-1}, n_i (1 \leq i \leq m-1) \in \{1, \dots, d\}.$$

$$2) N\left(r, \frac{1}{f}\right) \neq S(r, f) \text{ 且 } T(r, f) \leq n\bar{N}(r, 1/f) + S(r, f). \text{ 特别地, 当 } m=3 \text{ 并且 } \omega_1 + \omega_2 + \omega_3 \neq 0,$$

$\omega_1 \omega_2 + \omega_1 \omega_3 + \omega_2 \omega_3 \neq 0$ 时, 令

$$\begin{aligned} \phi = & \omega_1\omega_2\omega_3f^3 - n(\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3)f^2f' + n(n-1)(\omega_1 + \omega_2 + \omega_3)f(f')^2 \\ & + n(\omega_1 + \omega_2 + \omega_3)f^2f'' - n(n-1)(n-2)(f')^3 - 3n(n-1)ff'f'' - nf^2f''' , \end{aligned}$$

则有 $\phi \neq 0$ ，并且有以下式子成立：

$$N\left(r, \frac{1}{f}\right) \leq N_1\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{\phi}\right) + S(r, f),$$

$$T\left(r, \frac{1}{f}\right) \leq N_1\left(r, \frac{1}{f}\right) + \frac{1}{3}T\left(r, \frac{1}{\phi}\right) + \frac{2}{3}N\left(r, \frac{1}{\phi}\right) + S(r, f).$$

(i) $\phi = c \neq 0$, $f = c_1 e^{\frac{\omega_1 + \omega_2 + \omega_3}{3(n-1)}z} + c_2$, 其中 c_1, c_2 为非零常数且 $\frac{n(\omega_1 + \omega_2 + \omega_3)}{3(n-1)} = \omega_j \in \{\omega_1, \omega_2, \omega_3\}$, $c_1^n = b_j$;

(ii) 若 ϕ 为非常数亚纯函数则 $T(r, \phi) \neq S(r, f)$ 。特别地, 设 $n=3$, 且 $\phi = \sum_{i=1}^s h_i e^{Q_i(z)}$, 其中 h_i 为常数, Q_i 为多项式, 则存在 $Q_j (1 \leq j \leq s)$, 使得 $\deg(Q_j) = 1$ 且 $f^3 = \sum_{i=1}^3 \tilde{c}_i e^{\omega_i z} - \sum_{j=1}^t H_j e^{a_j z} - H_0$, 其中, $\tilde{c}_i (1 \leq i \leq 3)$, $H_j (0 \leq j \leq t)$ 为常数。

下面的例子表明定理 1 的结果是精确的。

例 1 方程 $f^4(z) - (f'(z + 2\pi))^2 + (f''(z + \pi))^3 + f(z) = 2e^{iz} + 4e^{2iz} + 8e^{3iz} + 16e^{4iz}$ 有解 $f(z) = 2e^{iz}$ 。

此时 $N\left(r, \frac{1}{f}\right) = S(r, f)$, $\omega_4 = 4i$, $\frac{\omega_4 i}{4} = i$, $2^4 = 16 = b_4$ 。满足定理 1 中的(1)。

例 2 方程 $f^3(z) - (f''(z + 2\pi i))^2 + 3f'(z + \pi i) - f(z) = 8e^{2z} - 13e^{4z} + e^{6z}$ 有解 $f(z) = e^{2z} + 1$ 。

此时 $N\left(r, \frac{1}{f}\right) \neq S(r, f)$, $\phi = 48$, $\frac{n(\omega_1 + \omega_2 + \omega_3)}{3(n-1)} = 6$, $c_1^n = 1$ 。满足定理 1 中的(2) (i)。

例 3 方程 $f^3 - 3(f'')^2 - f - 5f' = e^{3z} + e^{-3z} + 10e^{-z}$ 有解 $f(z) = e^{-z} + e^z + 1$ 。

此时 $N\left(r, \frac{1}{f}\right) \neq S(r, f)$, $\phi = 45e^{2z} - 15e^{-2z} + 96e^z + 63$, $f^3 = 6e^{-z} + 3e^{-2z} + 3e^{2z} + e^{-3z} + e^{3z} + 6e^z + 7$ 。

满足定理 1 中的(2) (ii)。

2. 主要引理

引理 1 [4] 假设 $f(z)$ 是满足 $f^n P(f) = Q(f)$ 的超越亚纯函数, 其中 $P(f)$ 和 $Q(f)$ 是关于 f 及其导数的多项式, 系数为亚纯函数, 记为 $\{a_\lambda | \lambda \in I, I \text{ 为指标集}\}$, 对任意 $\lambda \in I$, 有 $m(r, a_\lambda) = S(r, f)$, 且 $Q(f)$ 的次数至多为 n , 则有

$$m(r, P(f)) = S(r, f).$$

引理 2 [14] 设 $m, q \in \mathbb{N}^+$, $\alpha_1, \dots, \alpha_m$ 为不同的非零复常数, $A_0(z), \dots, A_m(z)$ 为满足 $A_i \neq 0 (1 \leq i \leq m)$ 的增长级小于 q 的亚纯函数。记 $\varphi(z) = A_0(z) + \sum_{i=1}^m A_i(z) e^{\alpha_i z^q}$, 则存在两个正常数 $d_1 < d_2$, 使得对充分大的 r 有 $d_1 r^q \leq T(r, \varphi) \leq d_2 r^q$ 。

引理 3 [15] 设 $f_i(z) (i=1, 2, \dots, n (n \geq 2))$ 为亚纯函数, $g_i(z) (i=1, 2, \dots, n)$ 为整函数且满足

$$1) \sum_{i=1}^n f_i(z) e^{g_i(z)} \equiv 0;$$

2) 对于 $1 \leq j < m \leq n$, $g_j(z) - g_m(z)$ 不为常数;

3) 对于 $1 \leq i \leq n, 1 \leq t < k \leq n, T(r, f_i) = o(T(r, e^{g_t - g_k}))(r \rightarrow \infty, r \notin E)$;

则 $f_i(z) \equiv 0 (i = 1, \dots, n)$ 。

引理 4 [16] 设 $n \in \mathbb{N}^+, n \geq 2, \alpha_1, \alpha_2$ 为不同的非零常数, p_1, p_2 为非零的亚纯函数。则方程

$$f^n(z) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}$$

不存在任何满足 $T(r, p_j) = S(r, f) (j = 1, 2)$ 的亚纯解 f 。

引理 5 设 $n, m \in \mathbb{N}^+, n \geq m, m > 2, a_j (j = 1, 2, \dots, m)$ 为不同的非零常数, $b_j (j = 1, 2, \dots, m)$ 为非零的亚纯函数。则方程

$$f^n(z) = b_1 e^{a_1 z} + \dots + b_m e^{a_m z} \tag{4}$$

不存在任何满足 $T(r, b_j) = S(r, f) (j = 1, 2, \dots, m)$ 的亚纯解 f 。

证明: $m = 3$ 时, Chen-Chen [17] 已证, 下面我们考虑 $m > 3$ 的情况。

设(4)存在一个亚纯解 $f(z)$ 满足 $T(r, b_j) = S(r, f) (j = 1, 2, \dots, m)$, 则

$$T(r, f) = \frac{1}{n} T(r, f^n) \leq O(T(r, e^z)) + S(r, f),$$

从而有 $S(r, f) = o(r)$ 。对(4)改写有:

$$f^n e^{-a_1 z} = \left(f e^{-\frac{a_1 z}{n}} \right)^n = b_1 + \sum_{j=2}^m b_j e^{(a_j - a_1)z}.$$

记 $g = f e^{-\frac{a_1 z}{n}}, \alpha_j = a_j - a_1 (2 \leq j \leq m)$, 显然 α_j 互异, 从而

$$g^n(z) = b_1 + \sum_{j=2}^m b_j e^{\alpha_j z}. \tag{5}$$

注意到

$$\bar{N}\left(r, \frac{1}{g^n}\right) = \bar{N}\left(r, \frac{1}{g}\right) \leq \frac{1}{n} T(r, g^n) + O(1),$$

所以有

$$\Theta(g^n, 0) = 1 - \limsup_{r \rightarrow \infty} \frac{\bar{N}\left(r, \frac{1}{g^n}\right)}{T(r, g^n)} \geq 1 - \frac{1}{n}. \tag{6}$$

我们对(5)应用引理 2, 存在 $d_1 < d_2$, 使得当 r 充分大时, $d_1 r \leq T(r, g^n) \leq d_2 r$, 从而结合 $S(r, f) = o(r)$ 有

$$\Theta(g^n, \infty) \geq 1 - \limsup_{r \rightarrow \infty} \frac{\sum_{j=1}^m N(r, b_j)}{d_1 r} = 1. \tag{7}$$

又由引理 3, 有

$$\Theta(g^n, b_1) \geq 1 - \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, b_1)}{d_1 r} = 1. \tag{8}$$

由(6), (7)及(8)和 Nevanlinna 亏值定理[1]有

$$3 - \frac{1}{n} \leq \Theta(g^n, 0) + \Theta(g^n, \infty) + \Theta(g^n, b_1) \leq 2,$$

这说明 $n=1$ ，与 $n \geq m > 3$ 矛盾。

3. 定理 1 的证明

先证 $\rho(f)=1$ 。根据对数导数引理[1]和差分对数导数引理[18]我们有

$$m(r, L_d(f)) \leq dm(r, f) + S(r, f).$$

又因为 $N(r, f) = S(r, f)$ ，我们有

$$\begin{aligned} nT(r, f) &= T(r, f^n) = T\left(r, \sum_{i=1}^m b_i e^{\omega_i z} - L_d(f)\right) \\ &\leq m\left(r, \sum_{i=1}^m b_i e^{\omega_i z}\right) + dm(r, f) + S(r, f) \\ &\leq T\left(r, \sum_{i=1}^m b_i e^{\omega_i z}\right) + dT(r, f) + S(r, f), \end{aligned} \tag{9}$$

由于 $d+1 \leq n$ ，结合上式有

$$T(r, f) \leq O(r). \tag{10}$$

另一方面，我们根据(3)有

$$T\left(r, \sum_{i=1}^m b_i e^{\omega_i z}\right) \leq T(r, P_d(f)) + nT(r, f),$$

即

$$O(r) \leq dT(r, f) + S(r, f) + nT(r, f). \tag{11}$$

从而根据(10)和(11)我们可以得到 $\rho(f)=1$ 。

记 $F_1 = f^n$ ， $H_1 = L_d(z, f)$ ，改写(3)为

$$F_1 + H_1 = \sum_{j=1}^m b_j e^{\omega_j z}, \tag{12}$$

对上式进行微分有

$$F_1' + H_1' = \sum_{j=1}^m \omega_j b_j e^{\omega_j z}. \tag{13}$$

由(12)及(13)消去 $e^{\omega_1 z}$ 有

$$F_2 + H_2 = \sum_{j=2}^m (\omega_1 - \omega_j) b_j e^{\omega_j z},$$

其中 $F_2 = \omega_1 F_1 - F_1'$ ， $H_2 = \omega_1 H_1 - H_1'$ 。类似地，我们依次消去 $e^{\omega_2 z}, \dots, e^{\omega_{m-1} z}$ 有

$$F_{m+1} + H_{m+1} = \omega_m F_m - F_m' + \omega_m H_m - H_m' = 0. \tag{14}$$

对于(14)我们分以下两种情况讨论：

情况 1. $\omega_m F_m - F_m' \equiv 0$ 。由 $F_j = \omega_{j-1} F_{j-1} - F_{j-1}'$ ($2 \leq j \leq m$)，代入假设有

$$\begin{aligned} \omega_m F_m - F_m' &= \prod_{j=1}^m \omega_j F_1 + (-1) \sum_{j=1}^m \prod_{i=1, i \neq j}^m \omega_i F_1' + (-1)^2 \frac{1}{2} \sum_{j=1}^m \sum_{k=1, k \neq j}^m \prod_{i=1, i \neq j, k}^m \omega_i F_1'' \\ &\quad + \dots + (-1)^{m-1} \sum_{j=1}^m \omega_j F_1^{(m-1)} + (-1)^m F_1^{(m)} \\ &= 0, \end{aligned} \tag{15}$$

解得(15)的一般解为:

$$F_1 = \bar{c}_1 e^{\omega_1 z} + \bar{c}_2 e^{\omega_2 z} + \cdots + \bar{c}_m e^{\omega_m z}, \quad (16)$$

其中 $\bar{c}_1, \dots, \bar{c}_m$ 为常数。由 $F_1 = f^n$, 引理 4 和引理 5 有 $f(z) = \tau_j e^{\frac{\omega_j z}{n}}$ ($1 \leq j \leq m$), $\tau_j^n = \bar{c}_j$ 。显然满足 $N\left(r, \frac{1}{f}\right) = S(r, f)$ 。

不妨设

$$f(z) = \tau_1 e^{\frac{\omega_1 z}{n}}, \quad \tau_1^n = \bar{c}_1, \quad (17)$$

从而

$$\left(f^{(k)}(z + \eta_k)\right)^{n_{k,i}} = \tilde{\tau}_1 e^{\frac{n_{k,i} \omega_1 z}{n}}, \quad (18)$$

记 $t_i = \sum_{k=0}^s n_{k,i}$, 从而 $0 \leq t_i \leq d$ ($0 \leq i \leq l$)。则(3)可改写为

$$\tau_1^n e^{\omega_1 z} + \sum_{s=0}^d \beta_s e^{\frac{s \omega_1 z}{n}} = \sum_{j=1}^m b_j e^{\omega_j z}, \quad (19)$$

其中 β_s ($0 \leq s \leq d$) 为常数。

由 $n \geq d+1$, $\tau_1^n = b_1$, 对(19)应用引理 3 有 $\tau_1^n = b_1, \beta_0 = 0$, 存在 $m-1$ 个常数 $n_i \in \{1, \dots, d\}$, 使得 $\omega_1 : \omega_2 : \dots : \omega_m = n : n_1 : \dots : n_{m-1}$ 。否则由引理 3, 至少存在一个 $b_j = 0$ ($2 \leq j \leq m$), 这与定理的假设相矛盾。

同理有下式成立: $f(z) = \tau_j e^{\frac{\omega_j z}{n}}$ ($2 \leq j \leq m$), $\tau_j^n = b_j$, 且

$$\omega_j : \omega_1 : \dots : \omega_{j-1} : \omega_{j+1} : \dots : \omega_m = n : n_1 : \dots : n_{m-1}, n_i \in \{1, \dots, d\} (2 \leq j \leq m, 1 \leq i \leq m-1).$$

情况 2. $\omega_m F_m - F'_m \neq 0$ 。由 $F_j = \omega_{j-1} F_{j-1} - F'_{j-1}$ ($2 \leq j \leq m$) 及 $F_1 = f^n$,

$$\begin{aligned} \omega_m F_m - F'_m &= \prod_{j=1}^m \omega_j f^n + (-1) \sum_{j=1}^m \prod_{i=1, i \neq j}^m \omega_i (n f^{n-1} f') \\ &\quad + (-1)^2 \frac{1}{2} \sum_{j=1}^m \sum_{k=1, k \neq j}^m \prod_{i=1, i \neq j, k}^m \omega_i (n(n-1) f^{n-2} (f')^2 + n f^{n-1} f'') \\ &\quad + \cdots + (-1)^{m-1} \sum_{j=1}^m \omega_j (f^n)^{(m-1)} + (-1)^m (f^n)^{(m)} \\ &= - \left\{ \prod_{j=1}^m \omega_j H_1 + (-1) \sum_{j=1}^m \prod_{i=1, i \neq j}^m \omega_i H'_1 + (-1)^2 \frac{1}{2} \sum_{j=1}^m \sum_{k=1, k \neq j}^m \prod_{i=1, i \neq j, k}^m \omega_i H''_1 \right. \\ &\quad \left. + \cdots + (-1)^{m-1} \sum_{j=1}^m \omega_j (H_1)^{(m-1)} + (-1)^m (H_1)^{(m)} \right\}, \end{aligned}$$

为了方便我们改写上式为

$$\begin{aligned} f^{n-1} G &= - \left\{ \prod_{j=1}^m \omega_j H_1 + (-1) \sum_{j=1}^m \prod_{i=1, i \neq j}^m \omega_i H'_1 + (-1)^2 \frac{1}{2} \sum_{j=1}^m \sum_{k=1, k \neq j}^m \prod_{i=1, i \neq j, k}^m \omega_i H''_1 \right. \\ &\quad \left. + \cdots + (-1)^{m-1} \sum_{j=1}^m \omega_j (H_1)^{(m-1)} + (-1)^m (H_1)^{(m)} \right\}, \quad (20) \end{aligned}$$

其中

$$G = \prod_{j=1}^m \omega_j f + (-1) \sum_{j=1}^m \prod_{i=1, i \neq j}^m \omega_i (nf') + (-1)^2 \frac{1}{2} \sum_{j=1}^m \sum_{k=1, k \neq j}^m \prod_{i=1, i \neq j, k}^m \omega_i \left(n(n-1) \frac{f'}{f} f' + nf'' \right) + \dots + (-1)^m \frac{(f^n)^{(m)}}{f^{n-1}}. \tag{21}$$

由(20), (21)及引理 1 有

$$m(r, G) = S(r, f), \tag{22}$$

注意到 G/f 的项由以下式子线性组成:

$$\frac{f^{(t)}}{f}, \frac{f^{(i_1)} f^{(i_2)}}{f f}, \frac{f^{(i_1)} f^{(j_2)} f^{(j_3)}}{f f f}, \dots, \left(\frac{f'}{f} \right)^t,$$

其中 $i_1 \leq i_2, i_1 + i_2 = t, j_1 \leq j_2 \leq j_3, j_1 + j_2 + j_3 = t, 0 \leq t \leq m$ 。从而

$$m\left(r, \frac{G}{f}\right) = S(r, f), N\left(r, \frac{G}{f}\right) \leq m\bar{N}\left(r, \frac{1}{f}\right). \tag{23}$$

由(21)及(22)有

$$\begin{aligned} T(r, f) &= m\left(r, \frac{f}{G} \cdot G\right) + S(r, f) \\ &\leq m\left(r, \frac{G}{f}\right) + N\left(r, \frac{G}{f}\right) + S(r, f) \\ &\leq m\bar{N}\left(r, \frac{1}{f}\right) + S(r, f), \end{aligned} \tag{24}$$

显然当 $N(r, 1/f) = S(r, f)$ 时, 上式不成立。

接下来, 我们在 $N(r, 1/f) \neq S(r, f)$ 的条件下进一步考虑, 当 $m = 3$ 时, 有

$$f^n + L_d(z, f) = b_1 e^{\omega_1 z} + b_2 e^{\omega_2 z} + b_3 e^{\omega_3 z}, \tag{25}$$

与(12)~(14)类似消去 $e^{\omega_1 z}, e^{\omega_2 z}, e^{\omega_3 z}$ 有:

$$f^{n-3} \phi = L''' - (\omega_1 + \omega_2 + \omega_3) L'' + (\omega_1 \omega_2 + \omega_1 \omega_3 + \omega_2 \omega_3) L' - \omega_1 \omega_2 \omega_3 L, \tag{26}$$

其中

$$\begin{aligned} \phi &= \omega_1 \omega_2 \omega_3 f^3 - n(\omega_1 \omega_2 + \omega_1 \omega_3 + \omega_2 \omega_3) f^2 f' + n(n-1)(\omega_1 + \omega_2 + \omega_3) f (f')^2 \\ &\quad + n(\omega_1 + \omega_2 + \omega_3) f^2 f'' - n(n-1)(n-2)(f')^3 - 3n(n-1) f f' f'' - n f^2 f'''. \end{aligned} \tag{27}$$

情况 2.1. $\phi \equiv 0$ 。从而结合(27)有:

$$\begin{aligned} \omega_1 \omega_2 \omega_3 f^3 &= n(\omega_1 \omega_2 + \omega_1 \omega_3 + \omega_2 \omega_3) f^2 f' - n(n-1)(\omega_1 + \omega_2 + \omega_3) f (f')^2 \\ &\quad - n(\omega_1 + \omega_2 + \omega_3) f^2 f'' + n(n-1)(n-2)(f')^3 + 3n(n-1) f f' f'' + n f^2 f'''. \end{aligned} \tag{28}$$

由 $N(r, 1/f) \neq S(r, f)$, 设 z_0 为 f 的 k 重零点。由(27)及假设, 可以得到 $k \geq 2$, 在 z_0 的邻域有: $f(z) = c_k (z - z_0)^k + \dots, (c_k \neq 0)$ 。对于(28), z_0 为等式左端的 $3k$ 重零点, 而等式右端 $(z - z_0)^{3k-3}$ 的系数为 $nkc_k^3 \left((n^2 + 1)k^2 - 3nk + 3 \right)$, 显然 $(z - z_0)^{3k-3}$ 的系数不为零, 矛盾。从而有 $\phi \neq 0$ 。

根据(27)及对数导数引理有

$$m\left(r, \frac{\phi}{f^3}\right) = S(r, f).$$

从而

$$\begin{aligned} 3m\left(r, \frac{1}{f}\right) &= m\left(r, \frac{1}{f^3}\right) \leq m\left(r, \frac{\phi}{f^3}\right) + m\left(r, \frac{1}{\phi}\right) \\ &= m\left(r, \frac{1}{\phi}\right) + S(r, f). \end{aligned} \tag{29}$$

由(27)及(29)有

$$\begin{aligned} T\left(r, \frac{1}{f}\right) &= m\left(r, \frac{1}{f}\right) + N_{(1)}\left(r, \frac{1}{f}\right) + N_{(2)}\left(r, \frac{1}{f}\right) \\ &\leq m\left(r, \frac{1}{f}\right) + N_{(1)}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{\phi}\right) + S(r, f) \\ &\leq N_{(1)}\left(r, \frac{1}{f}\right) + \frac{1}{3}T\left(r, \frac{1}{\phi}\right) + \frac{2}{3}N\left(r, \frac{1}{\phi}\right) + S(r, f). \end{aligned} \tag{30}$$

情况 2.2. $\phi \equiv c \neq 0$ 。由于 $N(r, 1/f) \neq S(r, f)$ ，设 z_1 为 f 的 k 重零点，则可以得到 $\phi(z_1) = -n(n-1)(n-2)(f')^3(z_1) \neq 0$ ，所以 $k=1$ ，即 f 的所有零点都为单零点，由此我们有

$$N\left(r, \frac{1}{f}\right) = N_{(1)}\left(r, \frac{1}{f}\right) + S(r, f). \tag{31}$$

由于 $\phi \equiv c \neq 0$ ，有 $\phi' = 0$ ，即

$$\begin{aligned} \phi' &= 3\omega_1\omega_2\omega_3f^2f' - 2n(\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3)f(f')^2 - n(\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3)f^2f'' \\ &\quad + n(n-1)(\omega_1 + \omega_2 + \omega_3)(f')^3 + 2n^2(\omega_1 + \omega_2 + \omega_3)ff'f'' - 3n(n-1)^2(f')^2f'' \\ &\quad + n(\omega_1 + \omega_2 + \omega_3)f^2f''' - 3n(n-1)f(f'')^2 - (3n^2 - n)ff'f''' - nf^2f^{(4)}. \end{aligned} \tag{32}$$

由于 z_1 为 f 的单零点，所以 $(\omega_1 + \omega_2 + \omega_3)f'(z_1) - 3(n-1)f''(z_1) = 0$ 。

令

$$g_1 = \frac{(\omega_1 + \omega_2 + \omega_3)f' - 3(n-1)f''}{f}. \tag{33}$$

情况 2.2.1. $g_1 \equiv 0$ 。在此假设条件下，由 g_1 的定义式(33)可得 $(\omega_1 + \omega_2 + \omega_3)f' - 3(n-1)f'' = 0$ ，从而

$$f = c_1 e^{\frac{\omega_1 + \omega_2 + \omega_3}{3(n-1)}z} + c_2, \tag{34}$$

其中 c_1, c_2 为常数，显然它们非零，否则 $N(r, f) = S(r, f)$ 。将(34)代入(25)有：

$$\sum_{i=0}^n \bar{\beta}_i e^{\frac{i(\omega_1 + \omega_2 + \omega_3)}{3(n-1)}z} + \sum_{i=0}^d \tilde{\beta}_i e^{\frac{i(\omega_1 + \omega_2 + \omega_3)}{3(n-1)}z} = b_1 e^{\omega_1 z} + b_2 e^{\omega_2 z} + b_3 e^{\omega_3 z}, \tag{35}$$

其中 $\bar{\beta}_i, \tilde{\beta}_j, 0 \leq i \leq n, 0 \leq j \leq d$ 为常数。由引理3，有 $1 \leq n-d \leq 3$ ， $\frac{n(\omega_1 + \omega_2 + \omega_3)}{3(n-1)} = \omega_j \in \{\omega_1, \omega_2, \omega_3\}$ ， $c_1^n = b_j$ 。

情况 2.2.2. $g_1 \neq 0$ 。由对数导数引理有

$$m(r, g_1) = S(r, f). \tag{36}$$

由于 f 的零点只能为单零点, 从而 g_1 的极点只能为 f 的极点, 结合(36)有

$$T(r, g_1) = S(r, f). \quad (37)$$

改写(33)有

$$f'' = A_1 f' + A_2 f, \quad (38)$$

其中 $A_1 = \frac{\omega_1 + \omega_2 + \omega_3}{3(n-1)}$, $A_2 = \frac{-g_1}{3(n-1)}$ 。对上式微分有

$$f''' = (A_1^2 + A_2) f' + (A_1 A_2 + A_2') f, \quad (39)$$

$$f^{(4)} = (A_1^3 + 2A_1 A_2 + 2A_2') f' + (A_1^2 A_2 + A_2^2 + A_1 A_2' + A_2'') f, \quad (40)$$

将(38), (39), (40)代入(32)有

$$\phi' = \tilde{A}_1 f^3 + \tilde{A}_2 f^2 f' + \tilde{A}_3 f (f')^2 + \tilde{A}_4 (f')^3 \equiv 0, \quad (41)$$

其中

$$\begin{aligned} \tilde{A}_1 = & (-n(\omega_1 \omega_2 + \omega_1 \omega_3 + \omega_2 \omega_3) + A_1 n(\omega_1 + \omega_2 + \omega_3) - A_1^2 n) A_2 + (-3n^2 + 2n) A_2^2 \\ & + (n(\omega_1 + \omega_2 + \omega_3) - n A_1) A_2' + (-n) A_2'', \end{aligned}$$

$$\begin{aligned} \tilde{A}_2 = & (3\omega_1 \omega_2 \omega_3 - A_1 n(\omega_1 \omega_2 + \omega_1 \omega_3 + \omega_2 \omega_3) + A_1^2 n(\omega_1 + \omega_2 + \omega_3) - n A_1^3) \\ & + ((n + 2n^2)(\omega_1 + \omega_2 + \omega_3) - A_1(9n^2 - 5n)) A_2 + (-3n^2 - n) A_2', \end{aligned}$$

$$\begin{aligned} \tilde{A}_3 = & (-2n(\omega_1 \omega_2 + \omega_1 \omega_3 + \omega_2 \omega_3) + A_1 2n^2(\omega_1 + \omega_2 + \omega_3) - A_1^2(6n^2 - 4n)) \\ & + (-3n^3 + 3n^2 - 2n) A_2, \end{aligned}$$

$$\tilde{A}_4 = 0.$$

由于 $f \neq 0$ 及(41)有

$$\tilde{A}_1 f^2 + \tilde{A}_2 f f' + \tilde{A}_3 (f')^2 = 0. \quad (42)$$

假设 $\tilde{A}_1 \neq 0$, 改写(42)有

$$f^2 = -\frac{\tilde{A}_2}{\tilde{A}_1} f f' - \frac{\tilde{A}_3}{\tilde{A}_1} (f')^2. \quad (43)$$

从而由(31)及(37)有

$$\begin{aligned} N\left(r, \frac{1}{f}\right) &= N_1\left(r, \frac{1}{f}\right) + S(r, f) \\ &\leq N\left(r, \frac{1}{\tilde{A}_3}\right) + N(r, \tilde{A}_1) + S(r, f) \\ &\leq S(r, f). \end{aligned}$$

这与 $N(r, 1/f) \neq S(r, f)$ 矛盾, 故 $\tilde{A}_1 \equiv 0$ 。所以 $\tilde{A}_2 f f' + \tilde{A}_3 (f')^2 = 0$ 。假设 $\tilde{A}_2 \neq 0$, 同理得到矛盾。结合(42)有 $\tilde{A}_1 = \tilde{A}_2 = \tilde{A}_3 \equiv 0$, 在定理条件下无解。

情况 2.3. ϕ 是 f 的小函数且 ϕ 不为常数。由(27)及(32)有

$$\begin{aligned}
0 = & \omega_1 \omega_2 \omega_3 \phi' f^3 - (n(\omega_1 \omega_2 + \omega_1 \omega_3 + \omega_2 \omega_3) \phi' + 3\omega_1 \omega_2 \omega_3 \phi) f^2 f' + (n(n-1)(\omega_1 + \omega_2 + \omega_3) \phi' \\
& + 2n(\omega_1 \omega_2 + \omega_1 \omega_3 + \omega_2 \omega_3) \phi) f (f')^2 + (n(\omega_1 + \omega_2 + \omega_3) \phi' + n(\omega_1 \omega_2 + \omega_1 \omega_3 + \omega_2 \omega_3) \phi) f^2 f'' \\
& - (n(n-1)(n-2) \phi' + n(n-1)(\omega_1 + \omega_2 + \omega_3) \phi) (f')^3 - (3n(n-1) \phi' + 2n^2(\omega_1 + \omega_2 + \omega_3) \phi) f f' f'' \quad (44) \\
& - (n \phi' + n(\omega_1 + \omega_2 + \omega_3) \phi) f^2 f''' + 3n(n-1)^2 \phi (f')^2 f'' \\
& + 3n(n-1) \phi f (f'')^2 + (3n^2 - n) \phi f f' f''' + n \phi f^2 f^{(4)}.
\end{aligned}$$

由于 $N(r, 1/f) \neq S(r, f)$ 以及 $T(r, \phi) = S(r, f)$, 所以可以令 z_2 为 f 的 l 重零点, 并且它既不是 ϕ 的零点, 也不是(44)的系数的极点, 从而由 ϕ 的定义式有 $l=1$. 即 f 的零点都是单零点. 同时又由(44), z_2 也为 $-((n-2)\phi' + (\omega_1 + \omega_2 + \omega_3)\phi)f' + 3(n-1)\phi f''$ 的零点.

令

$$g_2 = \frac{3(n-1)\phi f'' - ((n-2)\phi' + (\omega_1 + \omega_2 + \omega_3)\phi)f'}{f}. \quad (45)$$

情况 2.3.1. $g_2 \neq 0$. 由对数导数引理有 $m(r, g_2) = S(r, f)$, 又因为 $T(r, \phi) = S(r, f)$ 有

$$T(r, g_2) = m(r, g_2) + N(r, g_2) = S(r, f). \quad (46)$$

根据(46)有

$$f'' = B_1 f' + B_2 f, \quad (47)$$

其中

$$B_1 = \frac{n-2}{3(n-1)} \frac{\phi'}{\phi} + \frac{\omega_1 + \omega_2 + \omega_3}{3(n-1)}, \quad B_2 = \frac{g_2}{3(n-1)\phi}.$$

对(47)微分得

$$f''' = (B_1^2 + B_1' + B_2) f' + (B_1 B_2 + B_2') f = B_3 f' + B_4 f, \quad (48)$$

$$\begin{aligned}
f^{(4)} = & (B_1^3 + 3B_1 B_1' + 2B_1 B_2 + B_1'' + 2B_2') f' + (B_1^2 B_2 + 2B_1' B_2 + B_2^2 + B_1 B_2' + B_2'') f \\
= & B_5 f' + B_6 f.
\end{aligned} \quad (49)$$

将(47)和(48)代入(44)化简后有

$$\widetilde{B}_1 f^2 + \widetilde{B}_2 f f' + \widetilde{B}_3 (f')^2 = 0, \quad (50)$$

其中

$$\begin{aligned}
\widetilde{B}_1 = & 3n(n-1)B_2^2 + \left(n(\omega_1 + \omega_2 + \omega_3) \frac{\phi'}{\phi} + n(\omega_1 \omega_2 + \omega_1 \omega_3 + \omega_2 \omega_3) \right) B_2 \\
& - B_4 \left(n \frac{\phi'}{\phi} + n(\omega_1 + \omega_2 + \omega_3) \right) + B_6 n + \omega_1 \omega_2 \omega_3 \frac{\phi'}{\phi}, \\
\widetilde{B}_2 = & B_1 \left(n(\omega_1 + \omega_2 + \omega_3) \frac{\phi'}{\phi} + n(\omega_1 \omega_2 + \omega_1 \omega_3 + \omega_2 \omega_3) \right) \\
& - B_2 \left(3n(n-1) \frac{\phi'}{\phi} + 2n^2(\omega_1 + \omega_2 + \omega_3) \right) - B_3 \left(n \frac{\phi'}{\phi} + n(\omega_1 + \omega_2 + \omega_3) \right) \\
& - B_4 (n - 3n^2) + B_5 n - n(\omega_1 \omega_2 + \omega_1 \omega_3 + \omega_2 \omega_3) \frac{\phi'}{\phi} - 3\omega_1 \omega_2 \omega_3 + 6B_1 B_2 n(n-1),
\end{aligned}$$

$$\begin{aligned} \widetilde{B}_3 = & 2n(\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3) - B_3(n - 3n^2) - B_1\left(3n(n-1)\frac{\phi'}{\phi} + 2n^2(\omega_1 + \omega_2 + \omega_3)\right) \\ & + n(n-1)(\omega_1 + \omega_2 + \omega_3)\frac{\phi'}{\phi} + 3B_1^2n(n-1) + 3B_2n(n-1)^2. \end{aligned}$$

假设 $\widetilde{B}_3 \neq 0$ ，改写(50)有

$$\frac{\widetilde{B}_1}{\widetilde{B}_3}f^2 = -\frac{\widetilde{B}_2}{\widetilde{B}_3}ff' - (f')^2, \tag{51}$$

由于 $N\left(r, \frac{1}{f}\right) \neq S(r, f)$ ， $T(r, \phi) = S(r, f)$ 以及 $T(r, g_2) = S(r, f)$ ，可假设 z_3 为 f 的 k 重零点，并且它不是 $\widetilde{B}_1, \widetilde{B}_2, \widetilde{B}_3$ 的零点和极点，从而由(51)，有 $k \geq 2$ 。另一方面我们注意到 z_3 为(51)左端 $2k$ 重零点，右端 $2k - 2$ 重零点，这显然是矛盾的。所以 $\widetilde{B}_3 = 0$ 。同理有 $\widetilde{B}_2 = 0$ 。从而 $\widetilde{B}_1 = \widetilde{B}_2 = \widetilde{B}_3 = 0$ ，这在定理条件下无解。

情况 2.3.2. $g_2 \equiv 0$ 。使用与情况 2.3.1 类似的方法可以推得矛盾，在此简述方便理解。由假设及(46)有

$$f'' = D_1f', \tag{52}$$

其中 $D_1 = \frac{n-2}{3(n-1)}\frac{\phi'}{\phi} + \frac{\omega_1 + \omega_2 + \omega_3}{3(n-1)}$ 。对(47)微分有

$$f''' = (D_1^2 + D_1')f', \tag{53}$$

$$f^{(4)} = (D_1^3 + 3D_1D_1' + D_1'')f', \tag{54}$$

将(52)，(53)及(54)代入(44)有

$$\widetilde{D}_1f^3 + \widetilde{D}_2f^2f' + \widetilde{D}_3f(f')^2 + \widetilde{D}_4(f')^3 = 0, \tag{55}$$

其中

$$\begin{aligned} \widetilde{D}_1 &= \omega_1\omega_2\omega_3\phi', \\ \widetilde{D}_2 &= D_1(n(\omega_1 + \omega_2 + \omega_3)\phi' + n(\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3)\phi) - D_2(n\phi' + n(\omega_1 + \omega_2 + \omega_3)\phi) \\ &\quad + D_3n\phi - n(\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3)\phi' - 3\omega_1\omega_2\omega_3\phi, \\ \widetilde{D}_3 &= 3n(n-1)\phi D_1^2 + D_1(-3n(n-1)\phi' - 2n^2(\omega_1 + \omega_2 + \omega_3)\phi) \\ &\quad - D_2(n - 3n^2)\phi + 2n(\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3)\phi + n(n-1)(n-2)\phi', \\ \widetilde{D}_4 &= 3D_1\phi n(n-1)^2 - n(n-1)(\omega_1 + \omega_2 + \omega_3)\phi - n(n-1)(n-2)\phi' = 0. \end{aligned} \tag{56}$$

从而有

$$\widetilde{D}_1f^2 + \widetilde{D}_2ff' + \widetilde{D}_3(f')^2 = 0.$$

注意到 f 只有单零点， $T(r, g_2) = S(r, f)$ ， $T(r, \phi) = S(r, f)$ ，与对(50)的分析同理有 $\widetilde{D}_1 = \widetilde{D}_2 = \widetilde{D}_3 \equiv 0$ ，从而 $\phi' = 0$ ，这与 ϕ 不为常数矛盾。

情况 2.4. $n = 3$ ， $\phi = \sum_{i=1}^s h_i e^{Q_i(z)}$ 。由 ϕ 的定义及 $\rho(f) = 1$ ， h_i 为常数， Q_i 为多项式，显然 Q_i 不全为常数，设 Q 为 Q_i 中不为常数的多项式。因为 $\rho(\phi) \leq \rho(f) = 1$ ， $\deg(Q) \geq 1$ ，所以 $\deg(Q) = 1$ 。不妨设前 t ($1 \leq t \leq s$) 个是非常数多项式，记 $Q_i = a_i z + b_i$ ，则 $\phi = \sum_{i=1}^t \widetilde{h}_i e^{a_i z} + \widetilde{h}_0$ ，代入(26)有

$$\sum_{i=1}^t \tilde{h}_i e^{a_i z} + \tilde{h}_0 = L''' - (\omega_1 + \omega_2 + \omega_3) L'' + (\omega_1 \omega_2 + \omega_1 \omega_3 + \omega_2 \omega_3) L' - \omega_1 \omega_2 \omega_3 L, \quad (57)$$

解(57)得

$$L^* = c_1 e^{\omega_1 z} + c_2 e^{\omega_2 z} + c_3 e^{\omega_3 z} + \sum_{i=1}^t H_i e^{a_i z} + H_0,$$

其中 $c_i, H_j, 1 \leq i \leq 3, 0 \leq j \leq t$ 为常数。从而结合(25)有

$$f^3 = \sum_{i=1}^3 \tilde{c}_i e^{\omega_i z} - \sum_{i=1}^t H_i e^{a_i z} - H_0,$$

其中 $\tilde{c}_i = b_i - c_i (1 \leq i \leq 3)$ 。从而完成定理的证明。

4. 展望

1) 是否可在本文的基础上更深入地研究方程 $f^n + L_d(z, f) = b_1 e^{\omega_1 z} + \dots + b_m e^{\omega_m z}$, 在 $T(r, \phi) \neq S(r, f)$ 时给出更为具体的亚纯解的表达式。

2) $f^n + L_d(z, f) = b_1 e^{\omega_1 z} + \dots + b_m e^{\omega_m z}$, 考虑改变方程左边的主要部分, 用 $f^n + f^{n-1} f'$ 替换原方程的 f^n , 研究方程的亚纯解情况。

基金项目

国家自然科学基金资助项目(12171050, 12071047), 中央高校基本科研基金(500421126)。

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