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# A Lattice Runge-Kutta-Boltzmann Model for the Reaction Diffusion Equation

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## Abstract

**A lattice Runge-Kutta-Boltzmann model for the reaction diffusion equations is constructed in this paper. By using the classical Runge-Kutta formula, we obtain four-order accuracy of truncation error. The Chapman-Enskog expansion and multi-scale technique are employed in order to obtain a series of equations in different time scales and modify partial differential equations of the reaction diffusion equations. Numerical tests show that the scheme can be used to simulate the reaction diffusion equations.**

## Keywords

**Latctice Boltzmann Model, Runge-Kutta Scheme, Reaction Diffusion Equation**

# 一种用于反应扩散方程的格子Runge-Kutta-Boltzmann模型

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## 摘　要

**本文构建了一个求解反应扩散方程的格子Runge-Kutta-Boltzmann模型。通过使用经典的Runge-Kutta**

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公式，得到了四阶截断误差。通过Chapmann-Enskog展开和多尺度展开技术，获得了不同时间尺度的系列偏微分方程和修正的反应扩散方程。数值结果表明，本文的模型可以用来求解反应扩散方程。

## 关键词

**格子Boltzmann模型，Runge-Kutta公式，反应扩散方程**



## 1. 引言

格子Boltzmann方法(LBM)起源于格子气自动机(LGA) [1]，并逐渐发展成为计算流体力学的重要数值模拟方法[2] [3] [4]。目前，LBM已广泛用于模拟单组分水动力学问题[2]，多相和多组分流动，包括悬浮粒子流[5]，磁流体[6]，多孔介质流[7]以及反应扩散系统[8]等。此外，LBM在求解非线性物理方程领域也有出色的表现，如波动方程[9]、KDV方程[10]、Burgers方程[11] [12]、非线性Schrödinger方程[13] [14] [15]和Poisson方程[16] [17]等。本文使用经典的Runge-Kutta公式[18]构建了具有高阶精度的格子Runge-Kutta-Boltzmann模型。通过Chapmann-Enskog展开和多尺度展开技术，获得了不同时间尺度的系列偏微分方程和修正的反应扩散方程。数值结果表明，本文所提出的模型可以用来模拟反应扩散方程。

## 2. 格子 Runge-Kutta-Boltzmann 模型

### 2.1. 不同时间尺度的系列偏微分方程

将一个 $D$ 维空间离散成网格，网格中心与相邻的 $b$ 个格点连线作为速度矢量，加上网格中心的静止速度，这样就得到一个 $b+1$ 速度模型。定义 $t$ 时刻，$\mathbf{x}$ 位置的分布函数为 $f_\alpha^\sigma(\mathbf{x},t)$，用来表示粒子的密度；定义该时刻该位置的速度为 $\mathbf{e}_\alpha$。则标准格子 Boltzmann 模型演化方程为

$$f_\alpha^\sigma(\mathbf{x}+\Delta\mathbf{x},t+\Delta t)-f_\alpha^\sigma(\mathbf{x},t)=-\frac{1}{\tau}\left[f_\alpha^\sigma(\mathbf{x},t)-f_\alpha^{\sigma,eq}(\mathbf{x},t)\right]+\omega_\alpha^\sigma(\mathbf{x},t), \tag{1}$$

在方程(1)中，$\sigma=1,2,\cdots,M$ 表示组分数；$\omega_\alpha^\sigma$ 为附加项。为得到可以应用于稳态的宏观量，假设存在平衡态分布函数 $f_\alpha^{\sigma,eq}(\mathbf{x},t)$，并且满足守恒条件 $\sum_\alpha f_\alpha^{\sigma,eq}(\mathbf{x},t)=\sum_\alpha f_\alpha^\sigma(\mathbf{x},t)$。

粒子状态的演化可以分为两步：流动，粒子沿速度方向运动到相邻格点；碰撞，不同速度的粒子发生碰撞并改变速度。所以，方程(1)也可以表示为

$$f_\alpha^\sigma(\mathbf{x},t+\Delta t)-f_\alpha^\sigma(\mathbf{x},t)=-\frac{1}{\tau}[f_\alpha^\sigma(\mathbf{x},t)-f_\alpha^{\sigma,eq}(\mathbf{x},t)]+\omega_\alpha^\sigma(\mathbf{x},t), \tag{2a}$$

$$f_\alpha^\sigma(\mathbf{x}+\Delta\mathbf{x},t+\Delta t)=f_\alpha^\sigma(\mathbf{x},t+\Delta t)。 \tag{2b}$$

引入 Knudsen 数 $\varepsilon$，选择 $\varepsilon=\Delta t$，$\Delta t$ 为时间步长[9]，将四阶 Runge-Kutta 公式[18]带入方程(2a)，得到格子 Runge-Kutta-Boltzmann 方程

$$f_\alpha^\sigma(\mathbf{x}+\mathbf{e}_\alpha\varepsilon,t+\varepsilon)-f_\alpha^\sigma(\mathbf{x},t)=\frac{\varepsilon}{6}(k_1+2k_2+2k_3+k_4), \tag{3a}$$

$$k_1=f\left(t,f_\alpha^\sigma(\mathbf{x},t)\right), \tag{3b}$$

$$k_2 = f\left(t+\frac{1}{2}\varepsilon, f_\alpha^\sigma\left(\boldsymbol{x},t\right)+\frac{1}{2}\varepsilon k_1\right), \tag{3c}$$

$$k_3 = f\left(t+\frac{1}{2}\varepsilon, f_\alpha^\sigma\left(\boldsymbol{x},t\right)+\frac{1}{2}\varepsilon k_2\right), \tag{3d}$$

$$k_4 = f\left(t+\varepsilon, f_\alpha^\sigma\left(\boldsymbol{x},t\right)+\varepsilon k_3\right), \tag{3e}$$

$$f_\alpha^\sigma\left(\boldsymbol{x}+\boldsymbol{e}_\alpha\varepsilon, t+\varepsilon\right) = f_\alpha^\sigma\left(\boldsymbol{x}, t+\varepsilon\right)。 \tag{3f}$$

其中，$f\left(t, f_\alpha^\sigma\left(\boldsymbol{x},t\right)\right) = -\frac{1}{\tau}\left[f_\alpha^\sigma\left(\boldsymbol{x},t\right) - f_\alpha^{\sigma,eq}\left(\boldsymbol{x},t\right)\right] + \omega_\alpha^\sigma\left(\boldsymbol{x},t\right)$。

对(3a)式进行 Taylor 展开，并保留到 $\mathrm{O}\left(\varepsilon^7\right)$，得

$$f_\alpha^\sigma\left(\boldsymbol{x}+\boldsymbol{e}_\alpha\varepsilon, t+\varepsilon\right) - f_\alpha^\sigma\left(\boldsymbol{x},t\right) = \sum_{n=1}^{6}\frac{\varepsilon^n}{n!}\Delta^n f_\alpha^\sigma + \mathrm{O}\left(\varepsilon^7\right), \tag{4}$$

其中，偏微分算子 $\Delta \equiv \frac{\partial}{\partial t_0} + \boldsymbol{e}_\alpha\frac{\partial}{\partial \boldsymbol{x}}$。

在小 Knudsen 数假设下，对 $f_\alpha^\sigma$ 进行 Chapman-Enskog 展开[19]

$$f_\alpha^\sigma = f_\alpha^{\sigma,(0)} + \sum_{n=1}^{6}\varepsilon^n f_\alpha^{\sigma,(n)} + \mathrm{O}\left(\varepsilon^7\right), \tag{5}$$

其中 $f_\alpha^{\sigma,(0)} = f_\alpha^{\sigma,eq}$。

引入不同的时间尺度 $t_0, t_1, \cdots, t_6$，其中 $t_i = \varepsilon^i t$，$i = 0,1,2,\cdots,6$ 并且满足 $\frac{\partial}{\partial t} = \sum_{i=1}^{6}\varepsilon^i\frac{\partial}{\partial t_i} + \mathrm{O}\left(\varepsilon^7\right)$。

假设反应和扩散的影响为 $\varepsilon^4$，即 $\omega_\alpha^\sigma\left(\boldsymbol{x},t\right) = \varepsilon^4\varphi^\sigma\left(\boldsymbol{x},t\right)$，求得不同时间尺度的格子 Runge-Kutta-Boltzmann 模型的系列方程为

$$\Delta f_\alpha^{\sigma,eq} = \gamma_1 f_\alpha^{\sigma,(1)}, \tag{6}$$

$$\frac{\partial}{\partial t_1} f_\alpha^{\sigma,eq} + \tau_1\Delta^2 f_\alpha^{\sigma,eq} = \gamma_1 f_\alpha^{\sigma,(2)}, \tag{7}$$

$$\frac{\partial}{\partial t_2} f_\alpha^{\sigma,eq} + 2\tau_1\Delta\frac{\partial}{\partial t_1} f_\alpha^{\sigma,eq} + \tau_2\Delta^3 f_\alpha^{\sigma,eq} = \gamma_1 f_\alpha^{\sigma,(3)}, \tag{8}$$

$$\begin{aligned}&\frac{\partial}{\partial t_3} f_\alpha^{\sigma,eq} + 2\tau_1\Delta\frac{\partial}{\partial t_2} f_\alpha^{\sigma,eq} + 3\tau_2\Delta^2\frac{\partial}{\partial t_1} f_\alpha^{\sigma,eq}\\ &+\tau_3\Delta^4 f_\alpha^{\sigma,eq} + \tau_1\frac{\partial^2}{\partial t_1^2} f_\alpha^{\sigma,eq} = \gamma_1 f_\alpha^{\sigma,(4)} + \gamma_2\varphi^\sigma\end{aligned}, \tag{9}$$

$$\begin{aligned}&\frac{\partial}{\partial t_4} f_\alpha^{\sigma,eq} + 2\tau_1\Delta\frac{\partial}{\partial t_3} f_\alpha^{\sigma,eq} + 3\tau_2\Delta^2\frac{\partial}{\partial t_2} f_\alpha^{\sigma,eq} + 4\tau_3\Delta^3\frac{\partial}{\partial t_1} f_\alpha^{\sigma,eq}\\ &+\tau_4\Delta^5 f_\alpha^{\sigma,eq} + 3\tau_3\Delta\frac{\partial^2}{\partial t_1^2} f_\alpha^{\sigma,eq} + 2\tau_2\frac{\partial^2}{\partial t_1\partial t_2} f_\alpha^{\sigma,eq} - \frac{\gamma_2}{\gamma_1}\Delta\varphi^\sigma = \gamma_1 f_\alpha^{\sigma,(5)}\end{aligned}, \tag{10}$$

$$\begin{aligned}&\tau_5\Delta^6 f_\alpha^{eq} + 5\tau_4\Delta^4\frac{\partial}{\partial t_1} f_\alpha^{eq} + 4\tau_3\Delta^3\frac{\partial}{\partial t_2} f_\alpha^{eq} + 3\tau_2\Delta^2\frac{\partial}{\partial t_3} f_\alpha^{eq}\\ &+2\tau_1\Delta\frac{\partial}{\partial t_4} f_\alpha^{eq} + \frac{\partial}{\partial t_5} f_\alpha^{eq} + 6\tau_3\Delta^2\frac{\partial^2}{\partial t_1^2} f_\alpha^{eq} + 6\tau_2\Delta\frac{\partial^2}{\partial t_1\partial t_2} f_\alpha^{eq}\\ &+2\tau_1\frac{\partial^2}{\partial t_1\partial t_3} f_\alpha^{eq} + \tau_2\frac{\partial^3}{\partial t_1^3} f_\alpha^{eq} + \tau_1\frac{\partial^2}{\partial t_2^2} f_\alpha^{eq} - \left(\frac{3\gamma_2}{2\gamma_1^2}\Delta^2 + \frac{\gamma_2}{\gamma_1}\frac{\partial}{\partial t_1}\right)\varphi^\sigma = \gamma_1 f_\alpha^{(6)}\end{aligned}。 \tag{11}$$

方程(6)~(11)中，$\tau_1,\tau_2,\cdots,\tau_5$ 是 Chapmann 多项式，用来表示修正的反应扩散方程中耗散项和色散项系数，

$$\tau_1=\frac{1}{2}+\frac{1}{\gamma_1}, \tag{12}$$

$$\tau_2=\frac{1}{\gamma_1^2}+\frac{1}{\gamma_1}+\frac{1}{6}, \tag{13}$$

$$\tau_3=\frac{1}{\gamma_1^3}+\frac{3}{2\gamma_1^2}+\frac{7}{12\gamma_1}+\frac{1}{24}, \tag{14}$$

$$\tau_4=\frac{1}{\gamma_1^4}+\frac{2}{\gamma_1^3}+\frac{5}{4\gamma_1^2}+\frac{1}{4\gamma_1}+\frac{1}{120}, \tag{15}$$

$$\tau_5=\frac{1}{\gamma_1^5}+\frac{5}{2\gamma_1^4}+\frac{13}{6\gamma_1^3}+\frac{3}{4\gamma_1^2}+\frac{31}{360\gamma_1}+\frac{1}{720}, \tag{16}$$

$$\gamma_1=-\frac{1}{\tau}+\frac{1}{2\tau^2}-\frac{1}{6\tau^3}+\frac{1}{24\tau^4}, \tag{17}$$

$$\gamma_2=1-\frac{1}{2\tau}+\frac{1}{6\tau^2}-\frac{1}{24\tau^3}。 \tag{18}$$

### 2.2. 反应扩散方程

含源项的一维反应扩散方程为

$$\frac{\partial u^\sigma(x,t)}{\partial t}=\xi^\sigma\frac{\partial^2 u^\sigma(x,t)}{\partial x^2}+\psi^\sigma(u^\sigma), \tag{19}$$

定义宏观量 $u^\sigma(x,t)=\sum_\alpha f_\alpha^\sigma(x,t)$。根据守恒条件有 $\sum_\alpha f_\alpha^{\sigma,(n)}(x,t)=0,n\ge 1$。对于反应扩散方程，选取平衡态分布函数的矩为

$$f_1^{\sigma,eq}=\frac{1}{6c^4}\left(4m^0c^3+4\pi^0c^2-Q^0-P^0c\right), \tag{20}$$

$$f_2^{\sigma,eq}=\frac{1}{6c^4}\left(-4m^0c^3+4\pi^0c^2-Q^0+P^0c\right), \tag{21}$$

$$f_3^{\sigma,eq}=\frac{1}{24c^4}\left(2P^0c-2m^0c^3+Q^0-\pi^0c^2\right), \tag{22}$$

$$f_4^{\sigma,eq}=\frac{1}{24c^4}\left(-2P^0c+2m^0c^3+Q^0-\pi^0c^2\right), \tag{23}$$

$$f_5^{\sigma,eq}=u^\sigma-f_1^{\sigma,eq}-f_2^{\sigma,eq}-f_3^{\sigma,eq}-f_4^{\sigma,eq}。 \tag{24}$$

其中 $\xi^\sigma$ 为扩散系数，且 $\eta=\frac{3\tau_1\tau_2-\tau_1^3}{\tau_3}$，$\lambda^\sigma=\frac{\xi^\sigma}{-\varepsilon\tau_1}$。

### 2.3. 误差分析

将 $(6)+\varepsilon\times(7)+\varepsilon^2\times(8)+\varepsilon^3\times(9)+\varepsilon^4\times(10)+\varepsilon^5\times(11)$ 式两端对 $\alpha$ 求和，得

$$\frac{\partial u^\sigma}{\partial t}+\lambda^\sigma\varepsilon\tau_1\frac{\partial^2 u^\sigma}{\partial x^2}=(b+1)\varepsilon^3\gamma_2\varphi_\alpha^\sigma+E_4+E_5+\mathrm{O}(\varepsilon^6)。 \tag{25}$$

如取方程(19)的源项为$\psi^{\sigma}\left(u^{\sigma}\right)=(b+1)\varepsilon^{3}\gamma_{2}\varphi^{\sigma}\left(u^{\sigma}\right)$，则修正后的一维反应扩散方程为

$$\frac{\partial u^{\sigma}}{\partial t}=\xi^{\sigma}\frac{\partial}{\partial x^{2}}u^{\sigma}+(b+1)\varepsilon^{3}\gamma_{2}\varphi^{\sigma}\left(u^{\sigma}\right)+E_{4}+E_{5}+\mathrm{O}\left(\varepsilon^{6}\right), \tag{26}$$

其中，

$$\begin{aligned}E_{4}=\varepsilon^{4}\Bigg[&2\tau_{1}\sum_{\alpha}\Delta\frac{\partial}{\partial t_{3}}f_{\alpha}^{\sigma,eq}+3\tau_{2}\sum_{\alpha}\Delta^{2}\frac{\partial}{\partial t_{2}}f_{\alpha}^{\sigma,eq}+4\tau_{3}\sum_{\alpha}\Delta^{3}\frac{\partial}{\partial t_{1}}f_{\alpha}^{\sigma,eq}\\&+\tau_{4}\sum_{\alpha}\Delta^{5}f_{\alpha}^{\sigma,eq}+3\tau_{3}\sum_{\alpha}\Delta\frac{\partial^{2}}{\partial t_{1}^{2}}f_{\alpha}^{\sigma,eq}+2\tau_{2}\sum_{\alpha}\frac{\partial^{2}}{\partial t_{1}\partial t_{2}}f_{\alpha}^{\sigma,eq}-\frac{\gamma_{2}}{\gamma_{1}}\sum_{\alpha}\Delta\varphi^{\sigma}\Bigg]=0\end{aligned}, \tag{27}$$

$$\begin{aligned}E_{5}=\varepsilon^{5}\Bigg[&\tau_{5}\sum_{\alpha}\Delta^{6}f_{\alpha}^{eq}+5\tau_{4}\sum_{\alpha}\Delta^{4}\frac{\partial}{\partial t_{1}}f_{\alpha}^{eq}+4\tau_{3}\sum_{\alpha}\Delta^{3}\frac{\partial}{\partial t_{2}}f_{\alpha}^{eq}+3\tau_{2}\sum_{\alpha}\Delta^{2}\frac{\partial}{\partial t_{3}}f_{\alpha}^{eq}\\&+2\tau_{1}\sum_{\alpha}\Delta\frac{\partial}{\partial t_{4}}f_{\alpha}^{eq}+6\tau_{3}\sum_{\alpha}\Delta^{2}\frac{\partial^{2}}{\partial t_{1}^{2}}f_{\alpha}^{eq}+6\tau_{2}\sum_{\alpha}\Delta\frac{\partial^{2}}{\partial t_{1}\partial t_{2}}f_{\alpha}^{eq}+2\tau_{1}\sum_{\alpha}\frac{\partial^{2}}{\partial t_{1}\partial t_{3}}f_{\alpha}^{eq}\\&+\tau_{2}\sum_{\alpha}\frac{\partial^{3}}{\partial t_{1}^{3}}f_{\alpha}^{eq}+\tau_{1}\sum_{\alpha}\frac{\partial^{2}}{\partial t_{2}^{2}}f_{\alpha}^{eq}-\sum_{\alpha}\left(\frac{3\gamma_{2}}{2\gamma_{1}^{2}}\Delta^{2}+\frac{\gamma_{2}}{\gamma_{1}}\frac{\partial}{\partial t_{1}}\right)\varphi^{\sigma}\Bigg]\\=\varepsilon^{5}&\left(U\frac{\partial^{6}u}{\partial x^{6}}+V\frac{\partial^{2}\varphi^{\sigma}}{\partial x^{2}}-\lambda^{\sigma}\tau_{1}\frac{\gamma_{2}}{\gamma_{1}}\frac{\mathrm{d}\varphi_{\alpha}^{\sigma}}{\mathrm{d}u^{\sigma}}\frac{\partial^{2}u^{\sigma}}{\partial x^{2}}\right)\end{aligned}, \tag{28}$$

$$U=\left(6\tau_{1}^{2}\tau_{3}-5\eta\tau_{1}\tau_{4}-\tau_{1}^{3}\tau_{2}\right)\left(\lambda^{\sigma}\right)^{3}+5\eta\tau_{5}\left(\lambda^{\sigma}\right)^{2}-4c^{2}\lambda^{\sigma}, \tag{29}$$

$$V=\left(3\tau_{2}-2\tau_{1}^{2}\right)\gamma_{2}(b+1)\lambda^{\sigma}+\frac{3bc^{2}\gamma_{2}}{2D\gamma_{1}^{2}}。 \tag{30}$$

方程(28)的主要项为$\varepsilon^{5}U\frac{\partial^{6}u}{\partial x^{6}}$。根据 Hirt 启发式稳定性理论，格子 Boltzmann 模型正耗散条件为$\varepsilon^{5}U>0$。

## 3. 数值算例

下面应用本文所提出的格子 Runge-Kutta-Boltzmann 模型求解反应扩散方程。

**例 1：**考虑方程

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t}=v(t),\quad \frac{\mathrm{d}v(t)}{\mathrm{d}t}=u(t),\quad (0\le t\le 1)。 \tag{31}$$

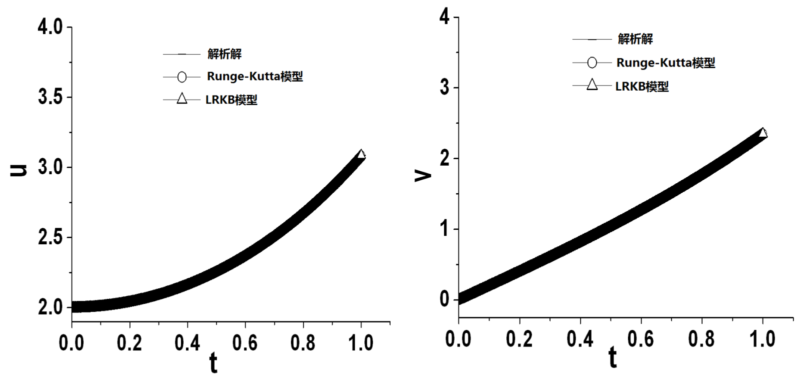
边界条件为

$$u(0)=2,\quad v(0)=0。 \tag{32}$$

该方程的解析解为

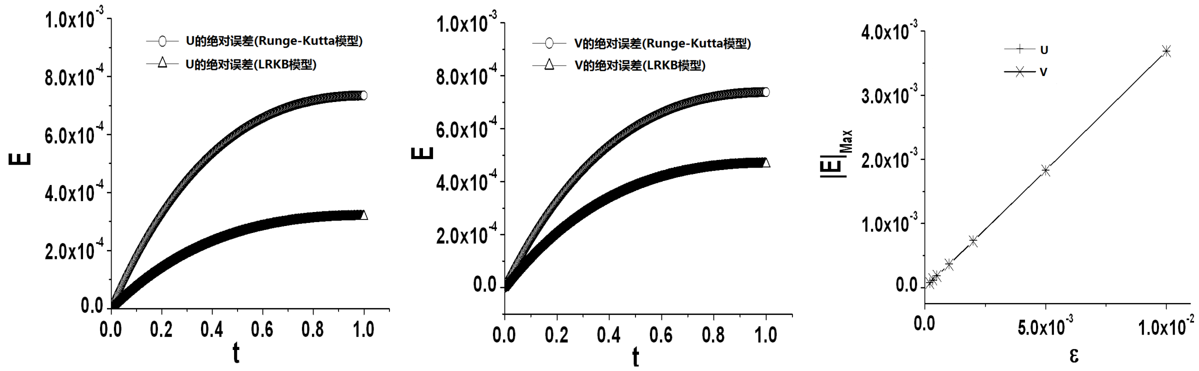
$$u(t)=\mathrm{e}^{t}+\mathrm{e}^{-t},\quad v(t)=\mathrm{e}^{t}-\mathrm{e}^{-t}。 \tag{33}$$

图 1 和图 2 给出了数值计算结果。其中图 1(a)和图 1(b)分别为 $u$-$t$ 和 $v$-$t$ 的 Runge-Kutta 算法计算结果、本文的格子 Runge-Kutta-Boltzmann 模型(在图中表示为 LRKB 模型)计算结果与解析解的比较；图 2(a)和图 2(b)分别为 $u$ 和 $v$ 的绝对误差曲线，图 2(c)为绝对误差 $E$ 对 Knudsen 数 $\varepsilon$ 的无穷范数曲线。其中 $E=\left|u^{N}-u^{E}\right|$，$u^{N}$ 为本文的格子 Runge-Kutta-Boltzmann 模型计算结果，$u^{E}$ 为解析解。其它参数为：$\tau=1.80$、$c=5.0$、$\xi^{1}=1.0\times10^{-3}$、$\xi^{2}=1.0\times10^{-3}$、格子数为 51、时间步长为 $\Delta t=0.001$。从图 1 和图 2




(a) (b)

**Figure 1.** Comparison between the Runge-Kutta solution, lattice Runge-Kutta-Boltzmann solution and the analytical result: (a) $u$ versus $t$; (b) $v$ versus $t$
**图 1.** Runge-Kutta 模拟结果、格子 Runge-Kutta-Boltzmann 模拟结果与解析解的比较：(a) $u$ 对 $t$ 图；(b) $v$ 对 $t$ 图




(a) (b) (c)

**Figure 2.** (a) The absolute error of $u$; (b) The absolute error of $v$; (c) Curves of the infinite norm of the absolute error $E$ versus the Knudsen number $\varepsilon$
**图 2.** (a) $u$ 的绝对误差曲线；(b) $v$ 的绝对误差曲线；(c) 绝对误差 $E$ 对 Knudsen 数$\varepsilon$的无穷范数曲线

中可以看出，格子 Runge-Kutta-Boltzmann 模型模拟结果与解析解有较好的一致性。

**例 2：**考虑反应扩散方程

$$\frac{\partial u}{\partial t}=\xi^1\nabla^2 u-(v+w)\text{，} \tag{34}$$

$$\frac{\partial v}{\partial t}=\xi^2\nabla^2 v+u+Av\text{，} \tag{35}$$

$$\frac{\partial w}{\partial t}=\xi^3\nabla^2 w+B+w(u-C)\text{。} \tag{36}$$

其中，$A$、$B$、$C$ 为参数。当 $\xi^\sigma\to 0$，$(\sigma=1,2,3)$，方程(34)~(36)简化为 Rössler 方程[20]

$$\frac{\partial u}{\partial t}=-(v+w)\text{，} \tag{37}$$

$$\frac{\partial v}{\partial t}=u+Av\text{，} \tag{38}$$

$$\frac{\partial w}{\partial t}=B+w(u-C)\text{。} \tag{39}$$

在数值模拟中，令方程(26)中附加项为

$$\varphi^1=\frac{-(v+w)}{\varepsilon^3(b+1)\gamma_2},\ \varphi^2=\frac{u+Av}{\varepsilon^3(b+1)\gamma_2},\ \varphi^3=\frac{B+w(u-C)}{\varepsilon^3(b+1)\gamma_2}。\tag{40}$$

$$\lambda^1=-\frac{\xi^1}{\varepsilon\tau_1},\ \lambda^2=-\frac{\xi^2}{\varepsilon\tau_1},\ \lambda^3=-\frac{\xi^3}{\varepsilon\tau_1}。\tag{41}$$

取计算域为$[0,1]$；初始条件为$u(0)^\sigma=u_0^\sigma$，$(\sigma=1,2,3)$；边界条件为$\frac{\partial u^\sigma(0,t)}{\partial x}=\frac{\partial u^\sigma(1,t)}{\partial x}=0$。

图 3 至图 5 给出了$A=B=0.20$，$C=5.00$时通过 Runge-Kutta 算法、文献[20]中的格子 Boltzmann 模型和本文的格子 Runge-Kutta-Boltzmann 模型绘制的相图。其中图 3(a)、图 4(a)和图 5(a)为使用 Runge-Kutta 算法得到的 *v*-*u*、*w*-*u*、*w*-*v* 相图；图 3(b)、图 4(b)和图 5(b)是通过文献[20]中的模型计算的 *v*-*u*、*w*-*u*、*w*-*v* 相图；图 3(c)、图 4(c)和图 5(c)给出了使用本文格子 Runge-Kutta-Boltzmann 模型绘制的 *v*-*u*、*w*-*u*、*w*-*v* 相图。其它参数为：$c=5.0$，$\tau=1.2$，$b=4$，$D=1$，$\xi^1=1.0\times10^{-3}$，$\xi^2=2.0\times10^{-3}$，$\xi^3=3.0\times10^{-3}$，格子数为$M=51$。从图中可以得出，本文模型所得结果与经典四阶 Runge-Kutta 算法吻合较好，并且比文献[20]的结果更精细。

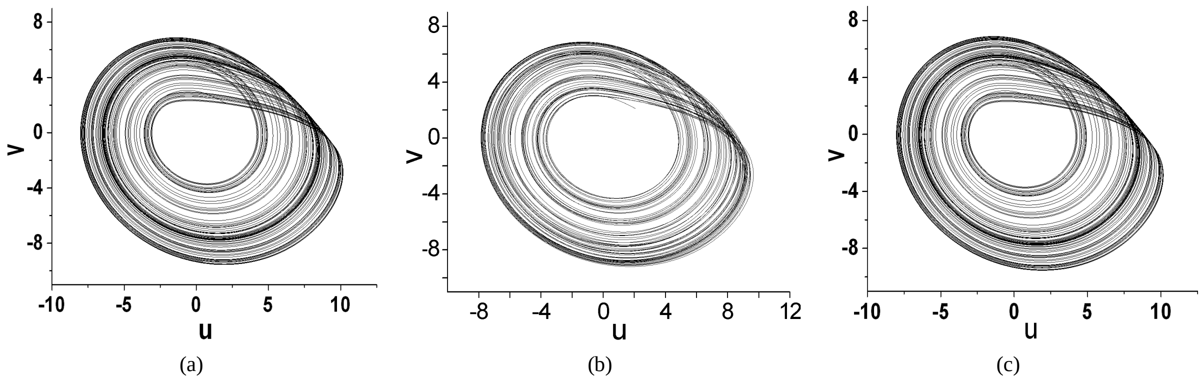



**Figure 3.** The phase figures of *v* versus *u*: (a) the Runge-Kutta formula; (b) the lattice Boltzmann result in Ref. [20]; (c) the result of the lattice Runge-Kutta-Boltzmann scheme

**图 3.** *v* 对 *u* 的相图：(a) Runge-Kutta 计算结果；(b) 文献[20]模拟结果；(c) 本文格子 Runge-Kutta-Boltzmann 模拟结果

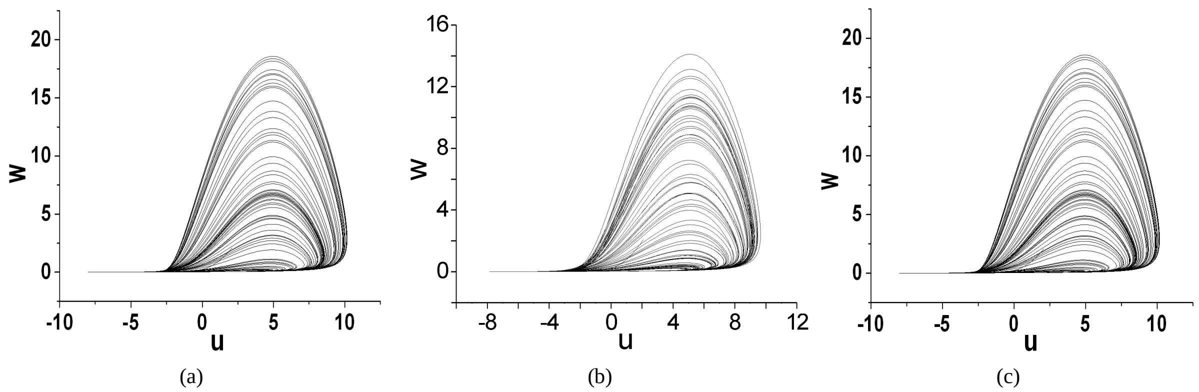
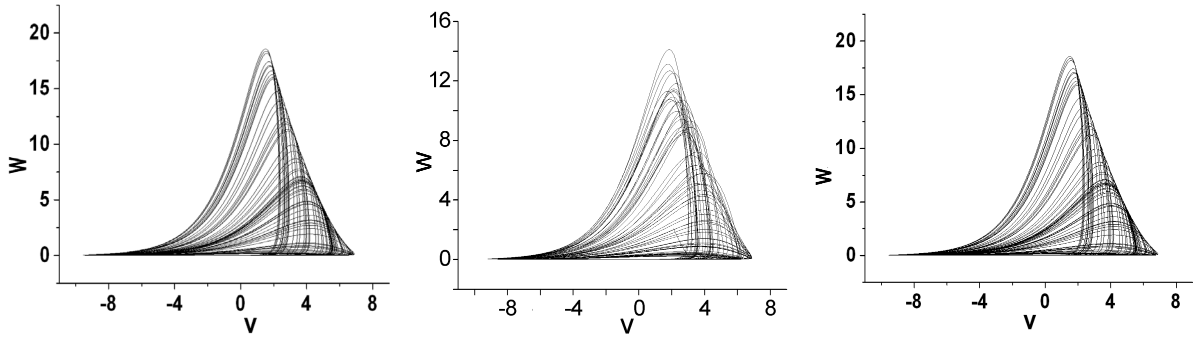



**Figure 4.** The phase figures of *w* versus *u*: (a) the Runge-Kutta formula; (b) the lattice Boltzmann result in Ref. [20]; (c) the result of the lattice Runge-Kutta-Boltzmann scheme

**图 4.** *w* 对 *u* 的相图：(a) Runge-Kutta 计算结果；(b) 文献[20]模拟结果；(c) 本文格子 Runge-Kutta-Boltzmann 模拟结果




(a) (b) (c)

**Figure 5.** The phase figures of *w* versus *v*: (a) the Runge-Kutta formula; (b) the lattice Boltzmann result in Ref. [20]; (c) the result of the lattice Runge-Kutta-Boltzmann scheme

**图 5.** *w* 对 *v* 的相图：(a) Runge-Kutta 计算结果；(b) 文献[20]模拟结果；(c) 本文格子 Runge-Kutta-Boltzmann 模拟结果

## 4. 结论

本文使用经典 Runge-Kutta 算法构建了用于反应扩散方程的格子 Runge-Kutta-Boltzmann 模型，获得了具有高阶精度的截断误差。数值结果表明，该模型的计算结果与经典 Runge-Kutta 模型吻合的较好。本文的主要思路，包括不同时间尺度的系列偏微分方程以及平衡态分布函数的形式，可以用来求解其它非线性偏微分方程。

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