

分数阶时滞的BAM神经网络的有限时间稳定性

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摘要

本文主要研究分数阶时滞双向联想记忆神经网络的有限时间稳定性。基于具有时滞的分数阶Gronwall不等式, 得到了一个保证分数阶时滞双向联想记忆神经网络有限时间稳定的充分条件, 降低了已有准则的保守性, 并通过两个数值例子验证了主要结果的有效性和可行性。

关键词

分数阶双向联想记忆神经网络, 有限时间稳定性, 时滞, Gronwall不等式

Finite-Time Stability of Fractional-Order BAM Neural Networks with Time Delays

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Abstract

The finite-time stability of time-delayed fractional order bidirectional associative memory neural networks is studied in this paper. Based on the fractional Gronwall inequality with time delay, a sufficient condition for the finite-time stability of fractional bidirectional associative memory neural networks with time delays is obtained, which reduces the conservatism of the existing criteria. At last, two examples are given to confirm the validity and feasibility of the main results.

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Keywords

Fractional Order Bidirectional Associative Memory Neural Networks, Finite-Time Stability, Time Delays, Gronwall Inequality

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1. 引言

近年来,分数阶微积分已被广泛应用于各种领域(见[1] [2] [3] [4])。众所周知,神经网络在人工智能、模式识别、非线性优化等方面起了重要作用,因此将神经网络和分数阶微积分结合是十分必要且具有重要意义的,而且由于分数阶神经网络相较于整数阶能更准确地描述联想和记忆的特点,因此引起了大量学者的关注。而稳定性是神经网络一个重要的问题,因此,分数阶神经网络的稳定性分析也成为了学者们研究的一个热门问题,这当中涉及到了指数稳定[5]、一致稳定[6]、Mittag-Leffler 稳定[7]、有限时间稳定[8]等。

在神经网络的许多类型中,分数阶双向联想记忆神经网络(FOBAMNNs)引起了广泛关注。时滞对于系统的稳定性将产生重大影响,在对神经网络的稳定性分析时应当把时滞因素考虑在内,因此,研究具有时滞的 BAMNN 的稳定性具有十分重要的现实意义。近年来,有许多关于 FOBAMNNs 的稳定性研究成果[9] [10] [11] [12]。在[13]中,首先引入了分数阶时滞系统的有限时间稳定(FTS)概念。此后,FOBAMNNs 的有限时间稳定出现了许多有意义的结果。在[9]中研究了 FOBAMNNs 的全局渐近稳定性。同时考虑了时滞和脉冲效应,基于 Lyapunov 稳定性定理、分数阶 Barbalat 引理和 Razumikhin 型稳定性定理,得到不依赖于特定时滞形式的稳定性条件。在[10]中研究了一类具有时滞的分数阶 Cohen-Grossberg BAM 神经网络的有限时间稳定性问题。利用一些不等式技术、微分均值定理和压缩映象原理,提出了保证该分数阶神经网络有限时间稳定性的充分条件。在[11]中利用 Lyapunov 函数和同态映射定理,推导了具有时变时滞的脉冲分数阶复值 FOBAMNNs 存在唯一平衡解和全局渐近稳定性的充分条件。在[12]中推导出了保证具有混合时变时滞的分数阶双向关联记忆神经网络的有限时间稳定性的充分条件。研究 FOBAMNN 的 FTS 所面临的一个困难是如何处理系统中的延迟项。分数阶 Gronwall 不等式中的分布式延迟为获得 FTS 标准带来了新的挑战。但,在[13]中所建立的 FTS 标准是依据没有时滞的 Gronwall 不等式。因此,对于分数阶 Gronwall 不等式的研究是非常有意义的。最近,在[14]中提出了一个新的分数阶 Gronwall 不等式,受此启发,基于上述讨论,本文利用新的 Gronwall 不等式,研究了具有时滞的 FOBAMNNs 的 FTS。

本文的主要结构如下。在第 2 节中介绍了 Caputo 导数的定义、性质和一些重要不等式。在第 3 节中,研究了一类 FOBAMNNs 的 FTS 问题。基于新的分数阶 Gronwall 不等式,获得了一个实现系统有限时间稳定的充分条件。在第四节中,通过两个数值例子来验证主要结果的有效性。最后给出全文主要结论。

2. 模型描述及预备知识

2.1. 预备知识

本节介绍了一些基本不等式和 Caputo 分数阶导数的定义。此外,给出了一类 FOBAMNNs 的数学描

述。

$\|B\| = \max_{1 \leq j \leq n} \sum_{1 \leq i \leq n} |b_{ij}|$ 和 $\|\mu(t)\| = \sum_{1 \leq i \leq n} |\mu_i(t)|$ 是矩阵范数和向量范数, 其中 b_{ij} 和 $\mu_i(t)$ 是矩阵 $A \in \mathbb{R}^{n \times n}$ 和向量 $\mu(t) \in \mathbb{R}^n$ 的元素。

定义 2.1 [15] 设 $h(t) \in C^m([0, +\infty], \mathbb{R}^n)$, $h(t)$ 的 ℓ 阶 Caputo 导数定义为:

$${}_0^C D_t^\ell f(t) = \frac{1}{\Gamma(l-\ell)} \int_0^t (t-s)^{l-\ell-1} f^{(l)}(s) ds,$$

其中 $\ell > 0$, l 是一个正整数, 使得 $l-1 < \ell < l$, $\Gamma(\cdot)$ 为 Gamma 函数, 即: $\Gamma(\rho) = \int_0^{+\infty} t^{\rho-1} e^{-t} dt$.

引理 1 [16] 如果 $\mu: [-\zeta, H] \rightarrow \mathbb{R}^m$ 是连续可微函数, 则 $\mu(t)$ 是系统的解, 当且仅当

$$\mu(t) = \begin{cases} \omega(0) + \frac{1}{\Gamma(\ell)} \int_0^t (t-\rho)^{\ell-1} [A_0 x(\rho) + A_1 x(\rho-\tau) + f(\rho, x(\rho), x(\rho-\tau))] d\rho, & 0 \leq t \leq T, \\ \omega(t), & -\zeta \leq t \leq 0. \end{cases} \quad (1)$$

引理 2 [14] 设 $\bar{\omega}(t), k(t), l(t) \in C([0, T], \mathbb{R}^+)$, $\omega(t) \in C([- \tau, 0], \mathbb{R}^+)$, $\bar{\omega}(t), \mathfrak{I}(t)$ 为非减函数, $\bar{\omega}(0) = \mathfrak{I}(0)$ 。若 $\mu(t) \in C([- \zeta, T], \mathbb{R}^+)$, 且

$$\begin{cases} v(t) \leq \bar{\omega}(t) + \int_0^t \frac{(t-s)^{\ell-1}}{\Gamma(\ell)} [k(s)v(s) + l(s)v(s-\tau)] ds, & t \in [0, T], \\ v(t) \leq \mathfrak{I}(t), & t \in [-\tau, 0], \end{cases}$$

则

$$\mu(t) \leq \left\{ F(t) \exp\left(\int_0^t [K(s) + L(s)] ds\right) \right\}^\lambda, \quad (2)$$

其中 $L(t) = \frac{4^{\frac{1}{\lambda}-1}}{\Gamma^\lambda(\ell)} \left(B\left(\frac{\ell-\lambda}{1-\lambda}, \frac{1-\ell}{1-\lambda}\right) \right)^{\frac{1-\lambda}{\lambda}} t^{\frac{\ell-\lambda}{\lambda}} l^{\frac{1}{\lambda}}(t)$, $K(t) = \frac{4^{\frac{1}{\lambda}-1}}{\Gamma^\lambda(\ell)} \left(B\left(\frac{\ell-\lambda}{1-\lambda}, \frac{1-\ell}{1-\lambda}\right) \right)^{\frac{1-\lambda}{\lambda}} t^{\frac{\ell-\lambda}{\lambda}} k^{\frac{1}{\lambda}}(t)$,

$F(t) = 2^{\frac{1}{\lambda}-1} \bar{\omega}^{\frac{1}{\lambda}}(t)$, $0 < \lambda < \ell < 1$, $B(\cdot, \cdot)$ 是 beta 函数, 定义为 $B(\mu, \nu) = \int_0^1 t^{\mu-1} (1-t)^{\nu-1} dt$ ($\text{Re}(\mu) > 0, \text{Re}(\nu) > 0$)。

引理 3 [17] 假设 $\mathbb{R}^+ = [0, +\infty)$, $T < +\infty$, $Q = [t_0, T] \in \mathbb{R}$, $\mu(t) \in C([t_0 - \zeta, T], \mathbb{R}^+)$, $\bar{\omega}(t), l(t), k(t) \in C(Q, \mathbb{R}^+)$, $\mathfrak{I}(t) \in C([t_0 - \zeta, t_0], \mathbb{R}^+)$, $\bar{\omega}(t_0) = \mathfrak{I}(t_0)$,

$$\begin{cases} \mu(t) \leq \bar{\omega}(t) + \int_{t_0}^t \frac{(t-\rho)^{\ell-1}}{\Gamma(\ell)} [k(\rho)v(\rho) + l(\rho)v(\rho-\zeta)] d\rho, & t \in [t_0, T], \\ \mu(t) \leq \mathfrak{I}(t), & t \in [t_0 - \tau, t_0], \end{cases}$$

如果 $\bar{\omega}(t)$ 和 $\mathfrak{I}(t)$ 是非减 C^1 函数, 则

1) 当 $\ell > \frac{1}{2}$ 时,

$$\mu(t) \leq \sqrt{M_3} \bar{\omega}(t) \exp\left\{ t - t_0 + \frac{3}{4^\ell} \Gamma(2\ell - 1) \int_{t_0}^t [l^2(s) + e^{-2\zeta} k^2(\rho)] d\rho \right\}, \quad (3)$$

其中 $t \in [t_0, T]$, $M_3 = \max\{3, e^{2\zeta}\}$;

2) 当 $0 < \ell \leq \frac{1}{2}$ 时,

$$\mu(t) \leq M_4^{\frac{1}{q}} \bar{\omega}(t) \exp \left\{ t - t_0 + \frac{3^{q-1} R_2^q}{q} \int_{t_0}^t [l^q(s) + e^{-q\varsigma} k^q(\rho)] d\rho \right\}, \quad (4)$$

其中 $t \in [t_0, T]$, $M_4 = \max \{3^{q-1}, e^{q\varsigma}\}$, 且 $q = 1 + \frac{1}{\ell}$, $R_2 = \left[\frac{\Gamma(1 - (1 - \ell)p)}{p^{1-p(1-\ell)}} \right]^{\frac{1}{p}}$.

2.2. 模型描述

考虑如下—类 FOBAMNNs:

$$\begin{cases} {}^C_0 D_t^\ell \mu_i(t) = -w_i \mu_i(t) + \sum_{j=1}^m a_{ij} h_{1j}(v_j(t)) + \sum_{j=1}^m b_{ij} g_{1j}(v_j(t) - \varsigma) + \aleph_i, \\ {}^C_0 D_t^\ell v_j(t) = -s_j v_j(t) + \sum_{i=1}^n c_{ji} h_{2i}(\mu_i(t)) + \sum_{i=1}^n d_{ji} g_{2i}(\mu_i(t) - \varsigma) + \mathfrak{R}_j, \end{cases} \quad (5)$$

其中 $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, $0 < \ell < 1$, $\mu_i(t)$ 和 $v_j(t)$ 分别表示时刻 t 第 i 个神经元和第 j 个神经元的状态, 常数 $w_i > 0$ 和 $s_j > 0$ 是神经元的自调节参数, 常数 $\varsigma > 0$ 是时滞, $A = (a_{ij})_{m \times n}$, $B = (b_{ij})_{m \times n}$, $C = (c_{ij})_{m \times n}$, $D = (d_{ij})_{m \times n}$ 是连接权重矩阵, \aleph_i 和 \mathfrak{R}_j 表示恒定的外部输入, $h_{1j}, g_{1j}, h_{2i}, g_{2i}$ 表示激活函数。

假设 1. 对于任意 $\mu, v \in \mathbb{R}$, 其中 $\xi_1, \eta_1, \xi_2, \eta_2$ 是一些正常数, 激活函数 $h_{1j}, g_{1j}, h_{2i}, g_{2i}$ 满足:

$$\begin{aligned} |h_{1j}(\mu) - h_{1j}(v)| &\leq \xi_1 |\mu - v|, & |g_{1j}(\mu) - g_{1j}(v)| &\leq \eta_1 |\mu - v|, \\ |h_{2i}(\mu) - h_{2i}(v)| &\leq \xi_2 |\mu - v|, & |g_{2i}(\mu) - g_{2i}(v)| &\leq \eta_2 |\mu - v|. \end{aligned}$$

设 $(\mu(t), v(t))$ 和 $(\bar{\mu}(t), \bar{v}(t))$ 是系统具有不同初始条件的任意两个解。令 $\tilde{h}(t) = \mu(t) - \bar{\mu}(t)$ 和 $\tilde{\varphi}(t) = v(t) - \bar{v}(t)$ 。假设 $\tilde{h}(t)$ 和 $\tilde{\varphi}(t)$ 满足以下初始条件: $\tilde{h}(t) = \varphi(t)$, $\tilde{\varphi}(t) = \psi(t)$, $t \in [-\varsigma, 0]$, 则有以下误差系统:

$$\begin{cases} {}^C D_t^\alpha \tilde{h}_i(t) = -w_i \tilde{h}_i(t) + \sum_{j=1}^m a_{ij} h_{1j}(v_j(t) - \bar{v}_j(t)) + \sum_{j=1}^m b_{ij} g_{1j}(v_j(t - \varsigma) - \bar{v}_j(t - \varsigma)), \\ {}^C D_t^\alpha \tilde{\varphi}_j(t) = -s_j \tilde{\varphi}_j(t) + \sum_{i=1}^n c_{ji} h_{2i}(\mu_i(t) - \bar{\mu}_i(t)) + \sum_{i=1}^n d_{ji} g_{2i}(\mu_i(t - \varsigma) - \bar{\mu}_i(t - \varsigma)). \end{cases} \quad (6)$$

定义 2.2 [18] 设 δ, ε 为任意正常数, 使得 $0 < \delta < \varepsilon$ 。如果 $\|\varphi\| + \|\psi\| < \delta$ 可推得 $\|\tilde{h}(t)\| + \|\tilde{\varphi}(t)\| < \varepsilon$, 则称系统(5)是关于 (δ, ε, T) 有限时间稳定的。其中 $\|\varphi\| = \sup_{s \in [-\varsigma, 0]} \|\varphi(s)\|$, $\|\psi\| = \sup_{s \in [-\varsigma, 0]} \|\psi(s)\|$, $t \in [0, T]$ 。

3. 主要结果

定理 1: 若

$$\begin{aligned} &2^{1-\lambda} \exp \left(\frac{\lambda^2}{\ell} \frac{4^{\frac{1}{\lambda}-1}}{\Gamma^{\frac{1}{\lambda}}(\ell)} \left(B \left(\frac{\alpha - \lambda}{1 - \lambda}, \frac{1 - \ell}{1 - \lambda} \right) \right)^{\frac{1-\lambda}{\lambda}} T^{\frac{1-\lambda}{\lambda}} \left((W + \xi_2 \|C\|)^{\frac{1}{\lambda}} + (\eta_2 \|D\|)^{\frac{1}{\lambda}} \right) \right) \\ &+ 2^{1-\lambda} \exp \left(\frac{\lambda^2}{\ell} \frac{4^{\frac{1}{\lambda}-1}}{\Gamma^{\frac{1}{\lambda}}(\ell)} \left(B \left(\frac{\alpha - \lambda}{1 - \lambda}, \frac{1 - \ell}{1 - \lambda} \right) \right)^{\frac{1-\lambda}{\lambda}} T^{\frac{1-\lambda}{\lambda}} \left((S + \xi_1 \|A\|)^{\frac{1}{\lambda}} + (\eta_1 \|B\|)^{\frac{1}{\lambda}} \right) \right) \leq \frac{\varepsilon}{\delta}, \end{aligned} \quad (7)$$

则系统(5)是有限时间稳定的。其中 $0 < \lambda < \ell < 1$, $|A| = (|a_{ij}|)_{m \times n}$, $|B| = (|b_{ij}|)_{m \times n}$, $|C| = (|c_{ji}|)_{m \times n}$, $|D| = (|d_{ji}|)_{m \times n}$, $W = \text{diag}(w_1, w_2, \dots, w_n)$, $S = \text{diag}(s_1, s_2, \dots, s_m)$ 。

证明: 考虑如下函数 $V(t) = \sum_{i=1}^n |\tilde{h}_i(t)| + \sum_{i=1}^n |\wp_i(t)|$ 。

由引理 1 有

$$\begin{aligned} \tilde{h}_i(t) &= \varphi_i(0) + \frac{1}{\Gamma(\ell)} \int_0^t (t-\rho)^{\ell-1} \left\{ -w_i \tilde{h}_i(\rho) + \sum_{j=1}^m a_{ij} h_{1j}(v_j(\rho) - \bar{v}_j(\rho)) \right. \\ &\quad \left. + \sum_{j=1}^m b_{ij} g_{1j}(v_j(\rho-\varsigma) - \bar{v}_j(\rho-\varsigma)) \right\} d\rho, \\ \wp_i(t) &= \psi_i(0) + \frac{1}{\Gamma(\ell)} \int_0^t (t-\rho)^{\ell-1} \left\{ -k_j \wp_j(\rho) + \sum_{i=1}^n c_{ji} h_{2i}(\mu_i(\rho) - \bar{\mu}_i(\rho)) \right. \\ &\quad \left. + \sum_{i=1}^n d_{ji} g_{2i}(\mu_i(\rho-\varsigma) - \bar{\mu}_i(\rho-\varsigma)) \right\} d\rho, \end{aligned}$$

由假设有

$$\begin{aligned} |\tilde{h}_i(t)| &\leq |\varphi_i(0)| + \frac{1}{\Gamma(\ell)} \int_0^t (t-\rho)^{\ell-1} \left\{ w_i |\tilde{h}_i(\rho)| + \sum_{j=1}^m \xi_1 |a_{ij}| |\wp_j(\rho)| + \sum_{j=1}^m \eta_1 |b_{ij}| |\wp_j(\rho-\varsigma)| \right\} d\rho, \\ |\wp_i(t)| &\leq |\psi_i(0)| + \frac{1}{\Gamma(\ell)} \int_0^t (t-\rho)^{\ell-1} \left\{ s_j |\wp_j(\rho)| + \sum_{i=1}^n \xi_2 |c_{ji}| |\tilde{h}_i(\rho)| + \sum_{i=1}^n \eta_2 |d_{ji}| |\tilde{h}_i(\rho-\varsigma)| \right\} d\rho, \end{aligned}$$

则

$$\begin{aligned} \|\tilde{h}(t)\| &\leq \|\varphi(0)\| + \frac{1}{\Gamma(\ell)} \int_0^t (t-\rho)^{\ell-1} \{W \|\tilde{h}(\rho)\| + \xi_1 \|A\| \|\wp(\rho)\| + \eta_1 \|B\| \|\wp(\rho-\varsigma)\|\} d\rho, \\ \|\wp(t)\| &\leq \|\psi(0)\| + \frac{1}{\Gamma(\ell)} \int_0^t (t-\rho)^{\ell-1} \{S \|\wp(\rho)\| + \xi_2 \|C\| \|\tilde{h}(\rho)\| + \eta_2 \|D\| \|\tilde{h}(\rho-\varsigma)\|\} d\rho. \end{aligned}$$

$$\begin{aligned} V(t) &\leq \|\tilde{h}(t)\| + \|\wp(t)\| \\ &\leq \|\varphi(0)\| + \|\psi(0)\| + \frac{1}{\Gamma(\ell)} \int_0^t (t-\rho)^{\ell-1} \{W \|\tilde{h}(\rho)\| + \xi_1 \|A\| \|\wp(\rho)\| + \eta_1 \|B\| \|\wp(\rho-\varsigma)\|\} \\ &\quad + S \|\wp(\rho)\| + \xi_2 \|C\| \|\tilde{h}(\rho)\| + \eta_2 \|D\| \|\tilde{h}(\rho-\varsigma)\|\} d\rho \\ &\leq \|\varphi(0)\| + \frac{1}{\Gamma(\ell)} \int_0^t (t-\rho)^{\ell-1} \{W \|\tilde{h}(\rho)\| + \xi_2 \|C\| \|\tilde{h}(\rho)\| + \eta_2 \|D\| \|\tilde{h}(\rho-\varsigma)\|\} d\rho \\ &\quad + \|\psi(0)\| + \frac{1}{\Gamma(\ell)} \int_0^t (t-\rho)^{\ell-1} \{S \|\wp(\rho)\| + \xi_1 \|A\| \|\wp(\rho)\| + \eta_1 \|B\| \|\wp(\rho-\varsigma)\|\} d\rho. \end{aligned}$$

对于 $t \in [-\varsigma, 0]$, $\|\tilde{h}(t)\| = \|\varphi(t)\| \leq \sup_{t \in [-\varsigma, 0]} \|\varphi(t)\| = \|\varphi\|$, $\|\wp(t)\| = \|\psi(t)\| \leq \sup_{t \in [-\varsigma, 0]} \|\psi(t)\| = \|\psi\|$,

$$\begin{aligned} V(t) &\leq \|\varphi\| + \frac{1}{\Gamma(\ell)} \int_0^t (t-\rho)^{\ell-1} \{(W + \xi_2 \|C\|) \|\tilde{h}(\rho)\| + \eta_2 \|D\| \|\tilde{h}(\rho-\varsigma)\|\} d\rho \\ &\quad + \|\psi\| + \frac{1}{\Gamma(\ell)} \int_0^t (t-\rho)^{\ell-1} \{(S + \xi_1 \|A\|) \|\wp(\rho)\| + \eta_1 \|B\| \|\wp(\rho-\varsigma)\|\} d\rho, \end{aligned} \tag{8}$$

由引理 2 有

$$\begin{aligned}
 V(t) &\leq \left\{ 2^{\frac{1}{\lambda}-1} \|\varphi\|^{\frac{1}{\lambda}} \exp \left[\int_0^t \frac{4^{\frac{1}{\lambda}-1}}{\Gamma^{\frac{1}{\lambda}}(\ell)} \left(B \left(\frac{\ell-\lambda}{1-\lambda}, \frac{1-\ell}{1-\lambda} \right) \right)^{\frac{1-\lambda}{\lambda}} t^{\frac{1-\lambda}{\lambda}} \left(K_1^{\frac{1}{\lambda}}(\rho) + L_1^{\frac{1}{\lambda}}(\rho) \right) \right] \right\}^{\lambda} \\
 &\quad + \left\{ 2^{\frac{1}{\lambda}-1} \|\psi\|^{\frac{1}{\lambda}} \exp \left[\int_0^t \frac{4^{\frac{1}{\lambda}-1}}{\Gamma^{\frac{1}{\lambda}}(\ell)} \left(B \left(\frac{\ell-\lambda}{1-\lambda}, \frac{1-\ell}{1-\lambda} \right) \right)^{\frac{1-\lambda}{\lambda}} t^{\frac{1-\lambda}{\lambda}} \left(K_2^{\frac{1}{\lambda}}(\rho) + L_2^{\frac{1}{\lambda}}(\rho) \right) \right] \right\}^{\lambda} \\
 &= \left\{ 2^{\frac{1}{\lambda}-1} \|\varphi\|^{\frac{1}{\lambda}} \exp \left[\int_0^t \frac{4^{\frac{1}{\lambda}-1}}{\Gamma^{\frac{1}{\lambda}}(\ell)} \left(B \left(\frac{\ell-\lambda}{1-\lambda}, \frac{1-\ell}{1-\lambda} \right) \right)^{\frac{1-\lambda}{\lambda}} t^{\frac{1-\lambda}{\lambda}} \left((W + \xi_2 \|C\|)^{\frac{1}{\lambda}} + (\eta_2 \|D\|)^{\frac{1}{\lambda}} \right) \right] \right\}^{\lambda} \\
 &\quad + \left\{ 2^{\frac{1}{\lambda}-1} \|\psi\|^{\frac{1}{\lambda}} \exp \left[\int_0^t \frac{4^{\frac{1}{\lambda}-1}}{\Gamma^{\frac{1}{\lambda}}(\ell)} \left(B \left(\frac{\ell-\lambda}{1-\lambda}, \frac{1-\ell}{1-\lambda} \right) \right)^{\frac{1-\lambda}{\lambda}} t^{\frac{1-\lambda}{\lambda}} \left((S + \xi_1 \|A\|)^{\frac{1}{\lambda}} + (\eta_1 \|B\|)^{\frac{1}{\lambda}} \right) \right] \right\}^{\lambda} \\
 &= 2^{1-\lambda} \|\varphi\| \exp \left[\frac{\lambda^2}{\ell} \frac{4^{\frac{1}{\lambda}-1}}{\Gamma^{\frac{1}{\lambda}}(\ell)} \left(B \left(\frac{\ell-\lambda}{1-\lambda}, \frac{1-\ell}{1-\lambda} \right) \right)^{\frac{1-\lambda}{\lambda}} t^{\frac{1-\lambda}{\lambda}} \left((W + \xi_2 \|C\|)^{\frac{1}{\lambda}} + (\eta_2 \|D\|)^{\frac{1}{\lambda}} \right) \right] \\
 &\quad + 2^{1-\lambda} \|\psi\| \exp \left[\frac{\lambda^2}{\ell} \frac{4^{\frac{1}{\lambda}-1}}{\Gamma^{\frac{1}{\lambda}}(\ell)} \left(B \left(\frac{\ell-\lambda}{1-\lambda}, \frac{1-\ell}{1-\lambda} \right) \right)^{\frac{1-\lambda}{\lambda}} t^{\frac{1-\lambda}{\lambda}} \left((S + \xi_1 \|A\|)^{\frac{1}{\lambda}} + (\eta_1 \|B\|)^{\frac{1}{\lambda}} \right) \right] \\
 &\leq 2^{1-\lambda} \|\varphi\| \exp \left[\frac{\lambda^2}{\ell} \frac{4^{\frac{1}{\lambda}-1}}{\Gamma^{\frac{1}{\lambda}}(\ell)} \left(B \left(\frac{\ell-\lambda}{1-\lambda}, \frac{1-\ell}{1-\lambda} \right) \right)^{\frac{1-\lambda}{\lambda}} T^{\frac{1-\lambda}{\lambda}} \left((W + \xi_2 \|C\|)^{\frac{1}{\lambda}} + (\eta_2 \|D\|)^{\frac{1}{\lambda}} \right) \right] \\
 &\quad + 2^{1-\lambda} \|\psi\| \exp \left[\frac{\lambda^2}{\ell} \frac{4^{\frac{1}{\lambda}-1}}{\Gamma^{\frac{1}{\lambda}}(\ell)} \left(B \left(\frac{\ell-\lambda}{1-\lambda}, \frac{1-\ell}{1-\lambda} \right) \right)^{\frac{1-\lambda}{\lambda}} T^{\frac{1-\lambda}{\lambda}} \left((S + \xi_1 \|A\|)^{\frac{1}{\lambda}} + (\eta_1 \|B\|)^{\frac{1}{\lambda}} \right) \right].
 \end{aligned} \tag{9}$$

由上式, 当 $\|\varphi\| + \|\psi\| < \delta$ 时, 可推得 $\|\hbar(t)\| + \|\wp(t)\| < \varepsilon$ 。

根据定义 2, 系统(5)是有限时间稳定的。

注 1: 对于不等式(8), 令 $\bar{\omega}_1(t) = \|\varphi\|$, $\iota_1(t) = W + \xi_2 \|C\|$, $\kappa_1(t) = \eta_2 \|D\|$, $\bar{\omega}_2(t) = \|\psi\|$, $\iota_2(t) = S + \xi_1 \|A\|$, $\kappa_2(t) = \eta_1 \|B\|$, 由引理 3 的(3), 对于 $\frac{1}{2} < \ell < 1$,

$$\begin{aligned}
 V(t) &\leq \sqrt{M_3} \|\varphi\| \exp \left\{ T + \frac{3}{4^\ell} \Gamma(2\ell-1) \left[(W + \xi_2 \|C\|)^2 + e^{-2\zeta} (\eta_2 \|D\|)^2 \right] T \right\} \\
 &\quad + \sqrt{M_3} \|\psi\| \exp \left\{ T + \frac{3}{4^\ell} \Gamma(2\ell-1) \left[(S + \xi_1 \|A\|)^2 + e^{-2\zeta} (\eta_1 \|B\|)^2 \right] T \right\},
 \end{aligned} \tag{10}$$

其中 $M_3 = \max\{3, e^{2\zeta}\}$;

由引理 3 的(4), 对于 $0 < \ell < \frac{1}{2}$,

$$\begin{aligned}
 V(t) &\leq M_4^{\frac{1}{q}} \|\varphi\| \exp \left\{ T + \frac{3^{q-1} R_2^q}{q} \left[(W + \xi_2 \|C\|)^q + e^{-q\zeta} (\eta_2 \|D\|)^q \right] T \right\} \\
 &\quad + C_4^{\frac{1}{q}} \|\psi\| \exp \left\{ T + \frac{3^{q-1} R_2^q}{q} \left[(S + \xi_1 \|A\|)^q + e^{-q\zeta} (\eta_1 \|B\|)^q \right] T \right\},
 \end{aligned} \tag{11}$$

其中 $M_4 = \max\{3^{q-1}, e^{q\zeta}\}$, 且 $q = 1 + \frac{1}{\ell}$, $R_2 = \left[\frac{\Gamma(1 - (1 - \ell)p)}{p^{1 - p(1 - \ell)}} \right]^{\frac{1}{p}}$.

因此, 对于 $\frac{1}{2} < \ell < 1$, 若

$$\begin{aligned} & \sqrt{M_3} \|\varphi\| \exp \left\{ T + \frac{3}{4^\ell} \Gamma(2\ell - 1) \left[(W + \xi_2 \|C\|)^2 + e^{-2\zeta} (\eta_2 \|D\|)^2 \right] T \right\} \\ & + \sqrt{M_3} \|\psi\| \exp \left\{ T + \frac{3}{4^\ell} \Gamma(2\ell - 1) \left[(S + \xi_1 \|A\|)^2 + e^{-2\zeta} (\eta_1 \|B\|)^2 \right] T \right\} \leq \frac{\varepsilon}{\delta}, \end{aligned} \tag{12}$$

则系统(5)是有限时间稳定的。

对于 $0 < \ell < \frac{1}{2}$, 若

$$\begin{aligned} & M_4^{\frac{1}{q}} \|\varphi\| \exp \left\{ T + \frac{3^{q-1} R_2^q}{q} \left[(W + \xi_2 \|C\|)^q + e^{-q\zeta} (\eta_2 \|D\|)^q \right] T \right\} \\ & + M_4^{\frac{1}{q}} \|\psi\| \exp \left\{ T + \frac{3^{q-1} R_2^q}{q} \left[(S + \xi_1 \|A\|)^q + e^{-q\zeta} (\eta_1 \|B\|)^q \right] T \right\} \leq \frac{\varepsilon}{\delta}, \end{aligned} \tag{13}$$

则系统(5)是有限时间稳定的。

注 2: 本文中的 FTS 准则(7)是通过一个新的分数阶延迟 Gronwall 不等式(见引理 2)获得的, 该不等式不需要不等式缩放技巧来处理延迟项。这可以帮助我们获得保守性更低和更简单的标准。

注 3: 对于 $0 < \ell < 1$, 近年来得到了各种具有时滞的分数阶神经网络的 FTS 准则。这些 FTS 准则是通过 Hölder 不等式和 Gronwall 不等式得到的。这些准则中的分数阶是分为 $0 < \ell < \frac{1}{2}$ 和 $\frac{1}{2} < \ell < 1$ 讨论得到的。本文在新的分数阶延迟 Gronwall 不等式(见引理 2)的基础上, 不需要分这两种情况讨论而得到一个新的统一的判据。

4. 数值例子

在本节中, 通过以下两个例子来说明第 3 节所得到结果的可行性。

例 1 考虑下面的 FOBAMNNs, 它可以写为

$$\begin{cases} {}^C D_t^\ell \mu_i(t) = -w_i \mu_i(t) + \sum_{j=1}^2 a_{ij} h_{1j}(v_j(t)) + \sum_{j=1}^2 b_{ij} g_{1j}(v_j(t) - \zeta) + \aleph_i, \\ {}^C D_t^\ell v_j(t) = -s_j v_j(t) + \sum_{i=1}^2 c_{ji} h_{2i}(\mu_i(t)) + \sum_{i=1}^2 d_{ji} g_{2i}(\mu_i(t) - \zeta) + \mathfrak{R}_j, \quad i, j = 1, 2, \end{cases} \tag{14}$$

其中 $\ell = 0.8$, $\mu = (\mu_1, \mu_2)^T$, $v = (v_1, v_2)^T$, $A = \begin{bmatrix} -0.15 & -0.24 \\ 0.24 & -0.25 \end{bmatrix}$, $B = \begin{bmatrix} -0.15 & 0.35 \\ 0.15 & 0.25 \end{bmatrix}$,

$C = \begin{bmatrix} 0.15 & -0.5 \\ 0.23 & -0.32 \end{bmatrix}$, $D = \begin{bmatrix} 0.25 & 0.05 \\ 0.25 & -0.22 \end{bmatrix}$, $\aleph = \begin{bmatrix} 0.35 \\ 0.25 \end{bmatrix}$, $\mathfrak{R} = \begin{bmatrix} -0.15 \\ 0.15 \end{bmatrix}$, $W = \text{diag}(0.22, 0.25)$,

$S = \text{diag}(0.18, 0.12)$, $\zeta = 0.2$,

$$h_1(\mu) = h_2(\mu) = g_1(\mu) = g_2(\mu) = \frac{1}{4} \cos \mu.$$

显然, $h_1(\bullet) = h_2(\bullet) = g_1(\bullet) = g_2(\bullet)$ 满足假设 1, 且 $\xi_1 = \xi_2 = \eta_1 = \eta_2 = \frac{1}{4}$ 。

假设系统(14)的两个解 $\mu(t)$ 、 $v(t)$ 具有如下的初始条件:

$$\psi(t) = (0.02, -0.03)^T, \theta(t) = (0.02, -0.01)^T;$$

$$\bar{\psi}(t) = (0.03, -0.04)^T, \bar{\theta}(t) = (-0.03, -0.02)^T.$$

当 $T = 0.5$, $\delta = 0.08$, $\lambda = 0.6$, 对于 $\|\psi - \theta\| = 0.08 \leq \delta$, 解的实际误差范数 $\|h(t)\| + \|\phi(t)\|$, 在有限时间间隔 $[0, 0.5]$ 内不超过估计的界限 $\varepsilon = 0.4154$ 。因此, 准则(7)是有效的。

对于 $\ell = 0.8 \in (\frac{1}{2}, 1)$, 评估函数 $H_1(T)$ 和 $H_2(T)$ 对于不同的 T , 在表 1 中列出了一些值。其中

$$H_1(T) = \delta 2^{1-\lambda} \exp \left\{ \frac{\lambda^2}{\ell} \frac{4^{\frac{1}{\ell}-1}}{\Gamma^{\frac{1}{\ell}}(\ell)} \left(B \left(\frac{\ell-\lambda}{1-\lambda}, \frac{1-\ell}{1-\lambda} \right) \right)^{\frac{1-\lambda}{\lambda}} T^{\frac{1-\lambda}{\lambda}} \left((W + \xi_2 \|C\|)^{\frac{1}{\lambda}} + (\eta_2 \|D\|)^{\frac{1}{\lambda}} \right) \right\}$$

$$+ \delta 2^{1-\lambda} \exp \left\{ \frac{\lambda^2}{\ell} \frac{4^{\frac{1}{\ell}-1}}{\Gamma^{\frac{1}{\ell}}(\ell)} \left(B \left(\frac{\ell-\lambda}{1-\lambda}, \frac{1-\ell}{1-\lambda} \right) \right)^{\frac{1-\lambda}{\lambda}} T^{\frac{1-\lambda}{\lambda}} \left((S + \xi_1 \|A\|)^{\frac{1}{\lambda}} + (\eta_1 \|B\|)^{\frac{1}{\lambda}} \right) \right\},$$

$$H_2(T) = \sqrt{\max(3, e^{2\zeta})} \delta \exp \left\{ T + \frac{3}{4^\ell} \Gamma(2\ell - 1) \left[(W + \xi_2 \|C\|)^2 + e^{-2\zeta} (\eta_2 \|D\|)^2 \right] T \right\}$$

$$+ \sqrt{\max(3, e^{2\zeta})} \delta \exp \left\{ T + \frac{3}{4^\ell} \Gamma(2\ell - 1) \left[(S + \xi_1 \|A\|)^2 + e^{-2\zeta} (\eta_1 \|B\|)^2 \right] T \right\}.$$

Table 1. The $\delta = 0.08$, $\ell = 0.8$ and different T in Example 1

表 1. 例 1 中 $\delta = 0.08$, $\ell = 0.8$ 及不同的 T

T	0	0.1	0.2	0.3	0.4	0.5
$H_1(T)$	0.1970	0.2213	0.2550	0.2975	0.3502	0.4154
$H_2(T)$	0.2771	0.3114	0.3510	0.3964	0.4481	0.5071

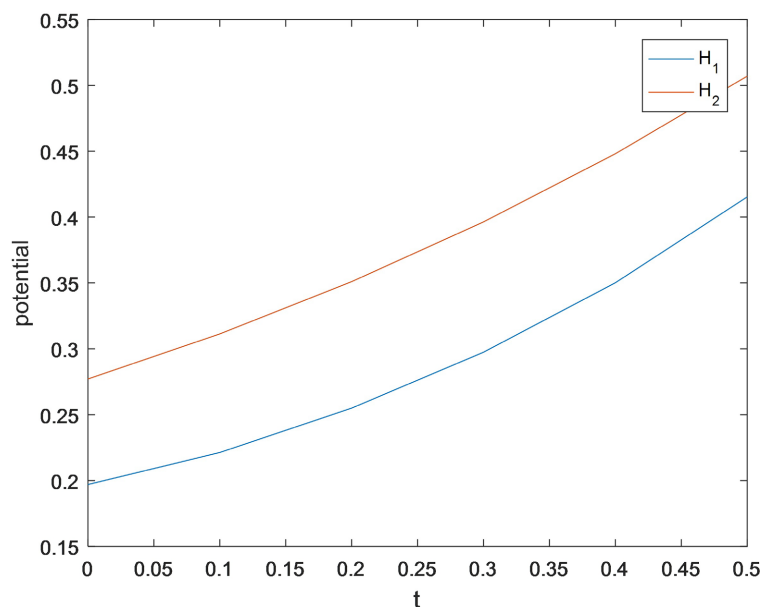


Figure 1. $H_1(T)$ and $H_2(T)$ in Example 1

图 1. 例 1 中的 $H_1(T)$ 和 $H_2(T)$

其中 $H_1(T)$ 和 $H_2(T)$ 分别由定理 1 中的(7)和注释中的(12)得到。由图 1 可知, 当固定 $0 < T < 0.5$ 时, $H_1(T) \leq H_2(T)$, 表明在有限时间区间 $[0, T], T \in (0, 0.5]$ 内, 准则(7)的保守性小于准则(12)。

例 2 考虑下面的 FOBAMNNs, 它可以写为

$$\begin{cases} {}^C_0 D_t^\ell \mu_i(t) = -w_i \mu_i(t) + \sum_{j=1}^3 a_{ij} h_j(v_j(t)) + \sum_{j=1}^3 b_{ij} g_{1j}(v_j(t) - \zeta) + \aleph_i, \\ {}^C_0 D_t^\ell v_j(t) = -s_j v_j(t) + \sum_{i=1}^3 c_{ji} h_{2i}(\mu_i(t)) + \sum_{i=1}^3 d_{ji} g_{2i}(\mu_i(t) - \zeta) + \mathfrak{R}_j, \quad i, j = 1, 2, 3, \end{cases} \quad (15)$$

其中 $\zeta = 0.4$, $\mu = (\mu_1, \mu_2, \mu_3)^T$, $v = (v_1, v_2, v_3)^T$, $A = \begin{bmatrix} -0.05 & 0.35 & -0.35 \\ 0.45 & -0.02 & 0.25 \\ 0.45 & -0.08 & 0.25 \end{bmatrix}$,

$$B = \begin{bmatrix} -0.15 & -0.25 & 0.45 \\ -0.15 & 0.25 & -0.5 \\ -0.15 & 0.25 & -0.25 \end{bmatrix}, \quad C = \begin{bmatrix} 0.35 & 0.15 & -0.5 \\ 0.3 & -0.2 & -0.35 \\ 0.23 & -0.24 & -0.35 \end{bmatrix}, \quad D = \begin{bmatrix} 0.25 & 0.05 & -0.3 \\ 0.25 & -0.15 & -0.5 \\ 0.35 & -0.22 & -0.35 \end{bmatrix}, \quad \aleph = \begin{bmatrix} 0.05 \\ 0.06 \\ 0.01 \end{bmatrix},$$

$$\mathfrak{R} = \begin{bmatrix} 0.05 \\ 0.06 \\ 0.01 \end{bmatrix}, \quad W = \text{diag}(0.22, 0.25, 0.15), \quad S = \text{diag}(0.18, 0.24, 0.12),$$

$$h_1(\mu) = h_2(\mu) = g_1(\mu) = g_2(\mu) = \frac{1}{4} \sin \mu, \quad \zeta = 0.2.$$

显然, $f_1(\cdot) = f_2(\cdot) = g_1(\cdot) = g_2(\cdot)$ 满足假设 1, 且 $\xi_1 = \xi_2 = \eta_1 = \eta_2 = \frac{1}{4}$ 。

假设系统(15)的两个解 $\mu(t)$ 、 $v(t)$ 具有如下的初始条件:

$$\begin{aligned} \psi(t) &= (0.02, -0.03, 0.04)^T, \theta(t) = (0.03, -0.01, -0.02)^T \\ \bar{\psi}(t) &= (0.03, 0.04, 0.05)^T, \bar{\theta}(t) = (0.03, 0.05, -0.04)^T. \end{aligned}$$

当 $T = 0.2$, $\delta = 0.1$, $\lambda = 0.2$, 对于 $\|\psi - \theta\| = 0.08 \leq \delta$, 解的实际误差范数 $\|\tilde{h}(t)\| + \|\tilde{\phi}(t)\|$, 在有限时间间隔 $[0, 0.2]$ 内不超过估计的界限 $\varepsilon = 0.5920$ 。因此, 准则(7)是有效的。

对于 $\ell = 0.4 \in \left(0, \frac{1}{2}\right)$, 评估函数 $G_1(T)$ 和 $G_2(T)$ 对于不同的 T , 在表 2 中列出了一些值。其中

$$\begin{aligned} G_1(T) &= \delta 2^{1-\lambda} \exp \left\{ \frac{\lambda^2 4^{\frac{1}{\lambda}-1}}{\alpha \Gamma^{\frac{1}{\lambda}}(\ell)} \left(B \left(\frac{\ell-\lambda}{1-\lambda}, \frac{1-\ell}{1-\lambda} \right) \right)^{\frac{1-\lambda}{\lambda}} T^{\frac{1-\lambda}{\lambda}} \left((W + \xi_2 \|C\|)^{\frac{1}{\lambda}} + (\eta_2 \|D\|)^{\frac{1}{\lambda}} \right) \right\} \\ &\quad + \delta 2^{1-\lambda} \exp \left\{ \frac{\lambda^2 4^{\frac{1}{\lambda}-1}}{\alpha \Gamma^{\frac{1}{\lambda}}(\ell)} \left(B \left(\frac{\ell-\lambda}{1-\lambda}, \frac{1-\ell}{1-\lambda} \right) \right)^{\frac{1-\lambda}{\lambda}} T^{\frac{1-\lambda}{\lambda}} \left((S + \xi_1 \|A\|)^{\frac{1}{\lambda}} + (\eta_1 \|B\|)^{\frac{1}{\lambda}} \right) \right\}, \\ G_2(T) &= \left(\max \{3^{q-1}, e^{q\zeta}\} \right)^{\frac{1}{q}} \delta \exp \left\{ T + \frac{3^{q-1} R_2^q}{q} \left[(W + \xi_2 \|C\|)^q + e^{-q\zeta} (\eta_2 \|D\|)^q \right] T \right\} \\ &\quad + \left(\max \{3^{q-1}, e^{q\zeta}\} \right)^{\frac{1}{q}} \delta \exp \left\{ T + \frac{3^{q-1} R_2^q}{q} \left[(S + \xi_1 \|A\|)^q + e^{-q\zeta} (\eta_1 \|B\|)^q \right] T \right\}. \end{aligned}$$

Table 2. The $\delta = 0.1$, $\ell = 0.4$ and different T in Example 1
表 2. 例 2 中 $\delta = 0.1$, $\ell = 0.4$ 及不同的 T

T	0	0.05	0.1	0.15	0.2
$G_1(T)$	0.3482	0.3599	0.3974	0.4690	0.5920
$G_2(T)$	0.7896	0.8321	0.8811	0.9375	1.0023

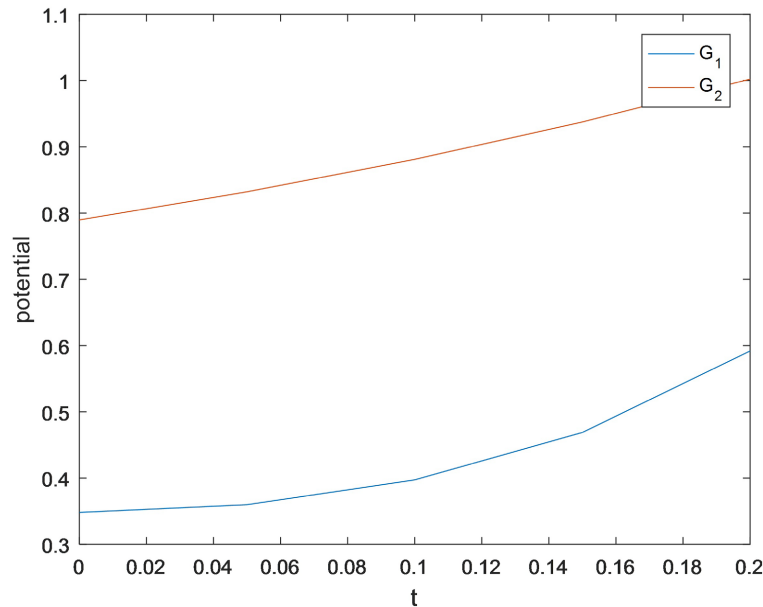


Figure 2. $G_1(T)$ and $G_2(T)$ in Example 2

图 2. 例 2 中的 $G_1(T)$ 和 $G_2(T)$

其中 $G_1(T)$ 和 $G_2(T)$ 分别由定理 1 中的(7)和注释中的(13)得到。由图 2 可知, 当固定 $0 < T < 0.2$ 时, $G_1(T) \leq G_2(T)$, 表明在有限时间区间 $[0, T], T \in (0, 0.2]$ 内, 准则(7)的保守性小于准则(13)。

5. 结论

本文讨论了具有时滞的 FOBAMNNs 的有限时间稳定性。利用新的分数阶 Gronwall 不等式, 证明得到 FOBAMNN 的有限时间稳定性的一个充分条件。数值例子表明, 与某些现有结果相比, 本文获得的有限时间稳定性条件保守性更低。

众所周知, 复值神经网络有着重要的应用, 因此将 BAMNNs 推广到复值情形也是十分有意义的, 这项工作可以作为今后的进一步研究内容。

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