Pure Mathematics 理论数学, 2022, 12(10), 1826-1836
Published Online October 2022 in Hans. http://www.hanspub.org/journal/pm
https://doi.org/10.12677/pm.2022.1210195

Hans汉斯

二维广义压力梯度方程的简单波

申　慧

长安大学理学院，陕西　西安

收稿日期：2022年9月20日；录用日期：2022年10月18日；发布日期：2022年10月26日

摘　要

利用直接的方法讨论在自相似平面下气体动力学中二维广义压力梯度方程的特征分解理论，可以得到压强p与特征值$\Lambda_{\pm}$的特征分解。进一步，若流动来自常状态，还可得到速度(u,v)的特征分解。由此可以得到与常状态流动相邻的流动是简单波，且说明简单波的流动区域是被一族直线所覆盖，沿着每条直线p,u,v均为常数。该结论推广Courant和Friedrichs的《超音速流和冲击波》一书中关于可约方程的著名结果。

关键词

压力梯度系统，特征分解，简单波

Simple Waves of Two-Dimensional Generalized Pressure Gradient Equations

Hui Shen

School of Science, Chang'an University, Xi'an Shaanxi

Received: Sep. 20th, 2022; accepted: Oct. 18th, 2022; published: Oct. 26th, 2022

Abstract

In this paper, the characteristic decompositions of the two-dimensional generalized pressure gradient equations in the self-similar plane are discussed by direct approach. The decompositions can allow a proof for simple wave. The decompositions of the pressure p and characteristics $\Lambda_{\pm}$ are obtained. Furthermore, the velocity (u,v) can be also obtained if the flow comes from a con-

文章引用: 申慧. 二维广义压力梯度方程的简单波[J]. 理论数学, 2022, 12(10): 1826-1836.
DOI: 10.12677/pm.2022.1210195

stant state which is not previously discussed. This way, by the characteristic decomposition, we find that any wave adjacent to a constant state is a simple wave whose flow region is covered by an one-parametric family of independent lines, along each of which the pressure p and the velocity (u,v) are constant. This conclusion is devoted to extending the well-known result on reducible equations in Courant and Friedrichs' book "Supersonic Flow and Shock Waves".

Keywords

Pressure Gradient System, Characteristic Decomposition, Simple Wave

1. 引言

近年来，由于航空工业和喷气推进技术的迅速发展，对于空气动力学的研究的重要性不言而喻。空气动力学研究分为理论和实验两个方面，其中理论研究作为实验研究的基础有着不可替代的作用，同时它也是数值计算的理论基础，因此对于动力学系统的理论研究就显得尤为重要。本文的研究内容属于空气动力学中理论研究的一个部分，对于研究复杂流动等起着重要作用。

黎曼不变量的存在性在双曲守恒律方程组解的构造中起着重要作用。例如，构造波动方程的达朗贝尔公式，以及证明奇点的发展[1]。可以回顾双曲守恒律系统

$$u_t + f(u)_x = 0,\ \ x \in R,\ \ t > 0,$$

其中 $u = (u_1, u_2, \cdots, u_n)$， $f(u) = (f_1(u), f_2(u), \cdots, f_n(u))$。 $n = 2$ 时，上式可以写成

$$R_t + \lambda_1 R_x = 0,\quad S_t + \lambda_2 S_x = 0,$$

其中 (R,S) 是黎曼不变量且 $\lambda_i(u), i = 1,2$ 是特征值。通过 Riemann 不变量的存在，我们可以知道任何靠近常态的解都是简单波[2]。然而，由于 $n > 2$ 时的黎曼不变量通常不存在，这种方法具有局限性[3]。

另一个可以用来研究简单波的重要技术是特征分解。Dai 和 Zhang [4]首先为压力梯度系统揭示了特征分解作为构建全局光滑解补丁的强大工具，然后广泛用于研究其他系统，例如 Euler 方程。随后，Hu 和 Sheng 建立了一般 2 × 2 拟线性严格双曲型方程组特征分解存在的一个很好的充分条件[5]。之后，Xiao 和 Li 考虑一类压力定律的二维可压缩 Euler 系统，并使用特征分解来确定与恒定状态相邻的任何波都必须是简单波[6]。利用特征分解技术，不仅可以在某些双曲型方程组中找到黎曼不变量，还可以找到所谓的黎曼变量[7]。特征分解方法对于处理拟线性双曲型方程组的一些问题非常有效[7]-[16]。

使用特征分解的思想可以追溯到经典的一维波动方程

$$u_{tt} - c^2 u_{xx} = 0,$$

其中 c 是常数，且上面的方程可以分解为下面的形式

$$(\partial_t \pm c\partial_x)(\partial_t \mp c\partial_x)u = 0.$$

记

$$R = \partial_- u := \partial_t u - c\partial_x u,\quad S = \partial_+ u := \partial_t u + c\partial_x u.$$

那么有

$$\partial_+ R = 0, \quad \partial_- S = 0,$$

暗示了 R 和 S 分别沿正负特征族是常数。

简单波定义为一个区域内的流，其解仅取决于单个参数。简单波在气体动力学中起着重要作用，在描述和建立流动问题的解方面起着基础性作用。本文的主要目的是将著名的 Courant 和 Friedrichs 定理[1] (即任何与常态相邻的双曲状态都必须是简单波)推广到压力梯度方程的结果。从而类似地定义简单波，即一个非线性特征族是直的，压力沿其恒定。我们注意到，我们不要求速度为常数。这样，就会使问题变得复杂，研究也更为困难。然而通过特征分解，我们可以发现与恒定状态相邻的波是简单波。

本文首先引入一个二维的广义压力梯度系统，对其作自相似变换后将其化为二阶拟线性双曲型方程。然后通过计算得到关于压强 p 与特征值 $\Lambda_\pm$ 的特征分解形式(即性质 1 和性质 2)，由此得到定理 1 的结论。最后，进一步假设流动无旋，得到关于 u,v 的特征分解形式，可推得定理 2。

2. 广义压力梯度系统

我们考虑二维广义压力梯度系统：

$$\begin{cases} u_t + p_x = 0 \\ v_t + p_y = 0 \\ p_t + f(p)(u_x + v_y) = 0 \end{cases} \tag{1}$$

其中 (u,v) 是速度，p 是压力且 $f(x) \in C^2(R)$。

对方程(1)作自相似变换，即令 $\xi = x/t$，$\eta = y/t$，得

$$\begin{cases} \xi u_\xi + \eta u_\eta - p_\xi = 0 \\ \xi v_\xi + \eta v_\eta - p_\eta = 0 \\ \xi p_\xi + \eta p_\eta - f(p)(u_\xi + v_\eta) = 0 \end{cases} \tag{2}$$

将(2)写成下面的矩阵形式

$$\begin{pmatrix} \xi & 0 & -1 \\ 0 & \xi & 0 \\ -f(p) & 0 & \xi \end{pmatrix} \begin{pmatrix} u \\ v \\ p \end{pmatrix}_\xi + \begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta & -1 \\ 0 & -f(p) & \eta \end{pmatrix} \begin{pmatrix} u \\ v \\ p \end{pmatrix}_\eta = 0 \tag{3}$$

为了方便，记 $W = (u,v,p)^{\mathrm{T}}$，(3)式可以转化为下面的矩阵形式

$$W_\xi + AW_\eta = 0 \tag{4}$$

其中

$$A = \begin{pmatrix} \dfrac{\xi\eta}{\xi^2 - f(p)} & -\dfrac{f(p)}{\xi^2 - f(p)} & \dfrac{\eta}{\xi^2 - f(p)} \\ 0 & \dfrac{\eta}{\xi} & -\dfrac{1}{\xi} \\ \dfrac{\eta f(p)}{\xi^2 - f(p)} & -\dfrac{\xi f(p)}{\xi^2 - f(p)} & \dfrac{\xi\eta}{\xi^2 - f(p)} \end{pmatrix}$$

由(4)式可得其特征值为

$$\Lambda_0=\frac{\eta}{\xi},\quad \Lambda_\pm=\frac{\xi\eta\pm\sqrt{f(p)\left(\xi^2+\eta^2-f(p)\right)}}{\xi^2-f(p)},\tag{5}$$

和正负特征值 $\Lambda_\pm$ 所对应的左特征向量分别为

$$\begin{aligned}\boldsymbol{l}_+&=\left(\frac{\eta f(p)}{\sqrt{f(p)\left(\xi^2+\eta^2-f(p)\right)}},-\frac{\xi f(p)}{\sqrt{f(p)\left(\xi^2+\eta^2-f(p)\right)}},1\right),\\ \boldsymbol{l}_-&=\left(-\frac{\eta f(p)}{\sqrt{f(p)\left(\xi^2+\eta^2-f(p)\right)}},\frac{\xi f(p)}{\sqrt{f(p)\left(\xi^2+\eta^2-f(p)\right)}},1\right).\end{aligned}\tag{6}$$

(4)式左乘 $\boldsymbol{l}_\pm$，可得其特征形式为

$$\boldsymbol{l}_\pm\left(W_\xi+\Lambda_\pm W_\eta\right)=0.\tag{7}$$

注：(i) (2)式也可以写成下面的二阶拟线性双曲型微分方程

$$\left(f(p)-\xi^2\right)p_{\xi\xi}-2\xi\eta p_{\xi\eta}+\left(f(p)-\eta^2\right)p_{\eta\eta}+\frac{f'(p)}{f(p)}\left(\xi p_\xi+\eta p_\eta\right)^2-2\left(\xi p_\xi+\eta p_\eta\right)=0.\tag{8}$$

(ii) 为了方便，我们记

$$\partial^i=\partial_\xi+\Lambda_i\partial_\eta,\quad i=0,\pm\tag{9}$$

3. 关于 p 和 $\Lambda_\pm$ 的特征分解

性质 1 在 (ξ,η) 坐标下，对压力变量 p 有以下特征分解

$$\begin{cases}\partial^+\partial^-p=\left(a_2\partial^-p+a_3\partial^+p+a_4\right)\partial^-p,\\ \partial^-\partial^+p=\left(b_1\partial^+p+b_3\partial^-p+b_4\right)\partial^+p.\end{cases}\tag{10}$$

其中

$$a_2=\frac{f'\left(\Lambda_+^2+1\right)}{\left(\xi^2-f\right)\left(\Lambda_+-\Lambda_-\right)^2},\quad b_1=\frac{f'\left(\Lambda_-^2+1\right)}{\left(\xi^2-f\right)\left(\Lambda_+-\Lambda_-\right)^2},$$

$$a_3=\frac{f'}{f}+\frac{f'\left(\Lambda_-^2-2\Lambda_+\Lambda_--1\right)}{\left(\xi^2-f\right)\left(\Lambda_+-\Lambda_-\right)^2},\quad b_3=\frac{f'}{f}+\frac{f'\left(\Lambda_+^2-2\Lambda_+\Lambda_--1\right)}{\left(\xi^2-f\right)\left(\Lambda_+-\Lambda_-\right)^2},$$

$$a_4=-\frac{2\xi}{\xi^2-f},\quad b_4=-\frac{2\xi}{\xi^2-f}.$$

证明：假设对压力变量 p 的特征分解形式为

$$\begin{cases}\partial^+\partial^-p=a_1\left(\partial^+p\right)^2+a_2\left(\partial^-p\right)^2+a_3\partial^+p\partial^-p+a_4\partial^-p,\\ \partial^-\partial^+p=b_1\left(\partial^+p\right)^2+b_2\left(\partial^-p\right)^2+b_3\partial^+p\partial^-p+b_4\partial^+p.\end{cases}\tag{11}$$

首先，由(5)式可知

$$\Lambda_+ + \Lambda_- = \frac{2\xi\eta}{\xi^2 - f(p)}, \qquad \Lambda_+\Lambda_- = \frac{\eta^2 - f(p)}{\xi^2 - f(p)},$$
$$\Lambda_+ - \Lambda_- = \frac{2\sqrt{f(p)\left(\xi^2 + \eta^2 - f(p)\right)}}{\xi^2 - f(p)}, \tag{12}$$

由(8)式可知，其特征方程为

$$\left(f(p) - \xi^2\right)\Lambda_+^2 + 2\xi\eta\Lambda_+ + \left(f(p) - \eta^2\right) = 0, \tag{13}$$

$$\left(f(p) - \xi^2\right)\Lambda_-^2 + 2\xi\eta\Lambda_- + \left(f(p) - \eta^2\right) = 0. \tag{14}$$

让(13)式和(14)式分别对 ξ 沿负、正方向求导得

$$\left(f'\partial^- p - 2\xi\right)\Lambda_+^2 + 2\left(f - \xi^2\right)\Lambda_+\partial^-\Lambda_+ + 2\eta\Lambda_+ + 2\xi\Lambda_-\Lambda_+ + 2\xi\eta\partial^-\Lambda_+ + f'\partial^- p - 2\eta\Lambda_- = 0, \tag{15}$$

$$\left(f'\partial^+ p - 2\xi\right)\Lambda_-^2 + 2\left(f - \xi^2\right)\Lambda_-\partial^+\Lambda_- + 2\eta\Lambda_- + 2\xi\Lambda_-\Lambda_+ + 2\xi\eta\partial^+\Lambda_- + f'\partial^+ p - 2\eta\Lambda_+ = 0. \tag{16}$$

则由(15)式可得

$$\begin{aligned}\partial^-\Lambda_+ &= \frac{-\left(f'\partial^- p - 2\xi\right)\Lambda_+^2 - 2\eta\Lambda_+ - 2\xi\Lambda_+\Lambda_- - f'\partial^- p + 2\eta\Lambda_-}{2\left[\left(f - \xi^2\right)\Lambda_+ + \xi\eta\right]} \\ &= \frac{-f'\left(\Lambda_+^2 + 1\right)}{2\left[\left(f - \xi^2\right)\Lambda_+ + \xi\eta\right]}\partial^- p + \frac{\left(\Lambda_+ - \Lambda_-\right)\left(\xi\Lambda_+ - \eta\right)}{\left(f - \xi^2\right)\Lambda_+ + \xi\eta} \\ &= \frac{f'\left(\Lambda_+^2 + 1\right)}{\left(\Lambda_+ - \Lambda_-\right)\left(\xi^2 - f\right)}\partial^- p + \frac{-2\left(f\eta + \xi\sqrt{f\left(\xi^2 + \eta^2 - f\right)}\right)}{\left(\xi^2 - f\right)^2},\end{aligned} \tag{17}$$

同理，由(16)式有

$$\partial^+\Lambda_- = \frac{-f'\left(\Lambda_-^2 + 1\right)}{\left(\Lambda_+ - \Lambda_-\right)\left(\xi^2 - f\right)}\partial^+ p + \frac{2\left[-f\eta + \xi\sqrt{f\left(\xi^2 + \eta^2 - f\right)}\right]}{\left(\xi^2 - f\right)^2} \tag{18}$$

成立。

接下来，由(9)式和(12)式可以得到

$$\partial^+\partial^- p = p_{\xi\xi} - \frac{2\xi\eta}{f - \xi^2}p_{\xi\eta} + \frac{f - \eta^2}{f - \xi^2}p_{\eta\eta} + \partial^+\Lambda_- p_\eta. \tag{19}$$

又由(8)式得

$$\partial^+\partial^- p = \frac{f'}{f}\frac{\left(\xi p_\xi + \eta p_\eta\right)^2}{\xi^2 - f} - \frac{2\left(\xi p_\xi + \eta p_\eta\right)}{\xi^2 - f} + \partial^+\Lambda_- p_\eta. \tag{20}$$

简单计算可得

$$\frac{f'\xi^2}{f\left(\xi^2 - f\right)} = \frac{f'}{f} + \frac{f'}{\xi^2 - f}, \qquad \frac{f'\eta^2}{f\left(\xi^2 - f\right)} = \frac{f'}{f}\Lambda_+\Lambda_- + \frac{f'}{\xi^2 - f}. \tag{21}$$

将(18)和(21)式代入(20)式中，能够得到

$$\partial^+\partial^- p=\frac{f'}{f}p_\xi\partial^+ p+\frac{f'}{\xi^2-f}p_\xi^2+\frac{f'}{f}\Lambda_- p_\eta\partial^+ p+\frac{f'}{\xi^2-f}p_\eta^2-\frac{2\xi}{\xi^2-f}p_\xi$$
$$+\frac{-f'\left(\Lambda_-^2+1\right)p_\eta}{\left(\Lambda_+-\Lambda_-\right)\left(\xi^2-f\right)}\partial^+ p+\frac{-2\eta\left(\xi^2-f\right)+2\xi\sqrt{f\left(\xi^2+\eta^2-f\right)}-2f\eta}{\left(\xi^2-f\right)^2}p_\eta$$
$$=\frac{f'}{f}\partial^- p\partial^+ p-\frac{2\xi}{\xi^2-f}\partial^- p+\frac{f'}{\xi^2-f}\left[\left(\frac{\Lambda_-\partial^+ p-\Lambda_+\partial^- p}{\Lambda_+-\Lambda_-}\right)^2+\left(\frac{\partial^+ p-\partial^- p}{\Lambda_+-\Lambda_-}\right)^2\right.$$
$$\left.-\frac{\Lambda_-^2+1}{\Lambda_+-\Lambda_-}\left(\frac{\partial^+ p-\partial^- p}{\Lambda_+-\Lambda_-}\right)\partial^+ p\right],$$

同理可得

$$\partial^-\partial^+ p=\frac{f'}{f}\partial^+ p\partial^- p-\frac{2\xi}{\xi^2-f}\partial^+ p+\frac{f'}{\xi^2-f}\left[\left(\frac{\Lambda_-\partial^+ p-\Lambda_+\partial^- p}{\Lambda_+-\Lambda_-}\right)^2+\left(\frac{\partial^+ p-\partial^- p}{\Lambda_+-\Lambda_-}\right)^2\right.$$
$$\left.+\frac{\Lambda_+^2+1}{\Lambda_+-\Lambda_-}\left(\frac{\partial^+ p-\partial^- p}{\Lambda_+-\Lambda_-}\right)\partial^- p\right].$$

那么就有

$$a_1=\frac{f'}{\xi^2-f}\left[\frac{\Lambda_-^2}{\left(\Lambda_+-\Lambda_-\right)^2}+\frac{1}{\left(\Lambda_+-\Lambda_-\right)^2}-\frac{\Lambda_-^2+1}{\left(\Lambda_+-\Lambda_-\right)^2}\right]=0,$$

$$a_2=\frac{f'}{\xi^2-f}\left[\frac{\Lambda_+^2}{\left(\Lambda_+-\Lambda_-\right)^2}+\frac{1}{\left(\Lambda_+-\Lambda_-\right)^2}\right]=\frac{f'\left(\Lambda_+^2+1\right)}{\left(\xi^2-f\right)\left(\Lambda_+-\Lambda_-\right)^2},$$

$$a_3=\frac{f'}{f}+\frac{f'}{\xi^2-f}\left[\frac{-2\Lambda_+\Lambda_-}{\left(\Lambda_+-\Lambda_-\right)^2}+\frac{-2}{\left(\Lambda_+-\Lambda_-\right)^2}+\frac{\Lambda_-^2+1}{\left(\Lambda_+-\Lambda_-\right)^2}\right]$$
$$=\frac{f'}{f}+\frac{f'\left(\Lambda_-^2-2\Lambda_+\Lambda_--1\right)}{\left(\xi^2-f\right)\left(\Lambda_+-\Lambda_-\right)^2},$$

$$a_4=\frac{-2\xi}{\xi^2-f}.$$

同理，通过计算可得 $b_i\left(i=1,2,3,4\right)$ 的值。

性质 2. 在 (ξ,η) 坐标下，对特征值 $\Lambda_\pm$ 有以下特征分解

$$\begin{cases}\partial^+\partial^-\Lambda_-=\left(c_1\partial^+\Lambda_++c_2\partial^+\Lambda_-+c_3\partial^-\Lambda_-+c_4\right)\partial^-\Lambda_-,\\ \partial^-\partial^+\Lambda_+=\left(d_1\partial^-\Lambda_-+d_2\partial^-\Lambda_++d_3\partial^+\Lambda_++d_4\right)\partial^+\Lambda_+.\end{cases}\tag{22}$$

其中

$$c_1=\frac{-\Lambda_+^2+\Lambda_-^2-2\Lambda_+\Lambda_--2}{\left(\Lambda_+-\Lambda_-\right)\left(\Lambda_+^2+1\right)}+\frac{\Lambda_+-\Lambda_-}{\Lambda_+^2+1}\left(\frac{\xi^2-f}{f}+\frac{f''\xi^2-ff''+f'^2}{f'^2}\right),$$

$$d_1=\frac{-\Lambda_+^2+\Lambda_-^2+2\Lambda_+\Lambda_-+2}{\left(\Lambda_+-\Lambda_-\right)\left(\Lambda_-^2+1\right)}-\frac{\Lambda_+-\Lambda_-}{\Lambda_-^2+1}\left(\frac{\xi^2-f}{f}+\frac{f''\xi^2-ff''+f'^2}{f'^2}\right),$$

$$c_2 = \frac{-\Lambda_-^2 + 2\Lambda_+\Lambda_- + 1}{(\Lambda_+ - \Lambda_-)(\Lambda_-^2 + 1)}, \quad d_2 = \frac{\Lambda_+^2 - 2\Lambda_+\Lambda_- - 1}{(\Lambda_+ - \Lambda_-)(\Lambda_+^2 + 1)},$$

$$c_3 = -\frac{\Lambda_+^2 + 1}{(\Lambda_+ - \Lambda_-)(\Lambda_-^2 + 1)}, \quad d_3 = \frac{\Lambda_-^2 + 1}{(\Lambda_+ - \Lambda_-)(\Lambda_+^2 + 1)},$$

$$c_4 = -\frac{4\xi}{\xi^2 - f}, \quad d_4 = -\frac{4\xi}{\xi^2 - f}.$$

证明：首先，(13)式和(14)式对 ξ 分别沿正、负方向求导得

$$\left(f'\partial^+ p - 2\xi\right)\Lambda_+^2 + 2\left(f - \xi^2\right)\Lambda_+\partial^+\Lambda_+ + 2\eta\Lambda_+ + 2\xi\Lambda_+^2 + 2\xi\eta\partial^+\Lambda_+ + f'\partial^+ p - 2\eta\Lambda_+ = 0, \tag{23}$$

$$\left(f'\partial^- p - 2\xi\right)\Lambda_-^2 + 2\left(f - \xi^2\right)\Lambda_-\partial^-\Lambda_- + 2\eta\Lambda_- + 2\xi\Lambda_-^2 + 2\xi\eta\partial^-\Lambda_- + f'\partial^- p - 2\eta\Lambda_- = 0. \tag{24}$$

由上面(23)式和(12)式可知

$$\partial^+\Lambda_+ = \frac{\left(\Lambda_+^2 + 1\right) f'\partial^+ p}{(\Lambda_+ - \Lambda_-)\left(\xi^2 - f\right)}, \tag{25}$$

同理有

$$\partial^-\Lambda_- = \frac{-\left(\Lambda_-^2 + 1\right) f'\partial^- p}{(\Lambda_+ - \Lambda_-)\left(\xi^2 - f\right)}. \tag{26}$$

因此有

$$\partial^+ p = \frac{(\Lambda_+ - \Lambda_-)\left(\xi^2 - f\right)\partial^+\Lambda_+}{f'\left(\Lambda_+^2 + 1\right)}, \quad \partial^- p = \frac{-(\Lambda_+ - \Lambda_-)\left(\xi^2 - f\right)\partial^-\Lambda_-}{\left(\Lambda_-^2 + 1\right) f'}. \tag{27}$$

接下来，(25)式对 ξ 沿负方向求导得

$$\begin{aligned}\partial^-\partial^+\Lambda_+ = {} & \frac{\left(-2\Lambda_-\Lambda_+ + \Lambda_+^2 - 1\right) f'}{(\Lambda_+ - \Lambda_-)^2\left(\xi^2 - f\right)}\partial^+ p\partial^-\Lambda_+ + \frac{\left(\Lambda_+^2 + 1\right) f'}{(\Lambda_+ - \Lambda_-)^2\left(\xi^2 - f\right)}\partial^-\Lambda_-\partial^+ p \\ & + \frac{\left(\Lambda_+^2 + 1\right) f'}{(\Lambda_+ - \Lambda_-)\left(\xi^2 - f\right)}\partial^-\partial^+ p + \frac{\left(\Lambda_+^2 + 1\right)\left(\xi^2 f'' - ff'' + f'^2\right)}{(\Lambda_+ - \Lambda_-)\left(\xi^2 - f\right)^2}\partial^+ p\partial^- p \\ & - \frac{2\xi\left(\Lambda_+^2 + 1\right) f'}{(\Lambda_+ - \Lambda_-)\left(\xi^2 - f\right)^2}\partial^+ p.\end{aligned} \tag{28}$$

同理，(26)式对 ξ 沿正方向求导得

$$\begin{aligned}\partial^+\partial^-\Lambda_- = {} & \frac{\left(-2\Lambda_-\Lambda_+ + \Lambda_-^2 - 1\right) f'}{(\Lambda_+ - \Lambda_-)^2\left(\xi^2 - f\right)}\partial^- P\partial^+\Lambda_- + \frac{\left(\Lambda_-^2 + 1\right) f'}{(\Lambda_+ - \Lambda_-)^2\left(\xi^2 - f\right)}\partial^- P\partial^+\Lambda_+ \\ & - \frac{\left(\Lambda_-^2 + 1\right) f'}{(\Lambda_+ - \Lambda_-)\left(\xi^2 - f\right)}\partial^+\partial^- P + \frac{(\Lambda_-^2 + 1)(-\xi^2 f'' + ff'' - f'^2)}{(\Lambda_+ - \Lambda_-)\left(\xi^2 - f\right)^2}\partial^+ P\partial^- P \\ & + \frac{2\xi\left(\Lambda_-^2 + 1\right) f'}{(\Lambda_+ - \Lambda_-)\left(\xi^2 - f\right)^2}\partial^- P.\end{aligned} \tag{29}$$

将(10)式中第一式代入(29)式，可得

$$\begin{aligned}\partial^+\partial^-\Lambda_- &= \left[\frac{\left(-2\Lambda_-\Lambda_+ + \Lambda_-^2 - 1\right)f'}{\left(\Lambda_+ - \Lambda_-\right)^2\left(\xi^2 - f\right)}\partial^+\Lambda_- + \frac{\left(\Lambda_-^2 + 1\right)f'}{\left(\Lambda_+ - \Lambda_-\right)^2\left(\xi^2 - f\right)}\partial^+\Lambda_+\right]\partial^- p \\ &\quad - \frac{\left(\Lambda_-^2 + 1\right)f'}{\left(\Lambda_+ - \Lambda_-\right)\left(\xi^2 - f\right)}\frac{\left(\Lambda_+^2 + 1\right)f'}{\left(\Lambda_+ - \Lambda_-\right)^2\left(\xi^2 - f\right)}\left(\partial^- p\right)^2 + \left[\frac{\left(\Lambda_-^2 + 1\right)\left(-\xi^2 f'' + ff'' - f'^2\right)}{\left(\Lambda_+ - \Lambda_-\right)\left(\xi^2 - f\right)^2}\right. \\ &\quad \left. - \frac{\left(\Lambda_-^2 + 1\right)f'}{\left(\Lambda_+ - \Lambda_-\right)\left(\xi^2 - f\right)}\left(\frac{f'}{f} + \frac{\left(-2\Lambda_-\Lambda_+ + \Lambda_-^2 - 1\right)f'}{\left(\Lambda_+ - \Lambda_-\right)^2\left(\xi^2 - f\right)}\right)\right]\partial^+ p\partial^- p \\ &\quad + \left[\frac{\left(\Lambda_-^2 + 1\right)f'}{\left(\Lambda_+ - \Lambda_-\right)\left(\xi^2 - f\right)}\frac{2\xi}{\xi^2 - f} + \frac{2\xi\left(\Lambda_-^2 + 1\right)f'}{\left(\Lambda_+ - \Lambda_-\right)\left(\xi^2 - f\right)^2}\right]\partial^- p\end{aligned} \tag{30}$$

同理，有

$$\begin{aligned}\partial^-\partial^+\Lambda_+ &= \left[\frac{\left(-2\Lambda_-\Lambda_+ + \Lambda_+^2 - 1\right)f'}{\left(\Lambda_+ - \Lambda_-\right)^2\left(\xi^2 - f\right)}\partial^-\Lambda_+ + \frac{\left(\Lambda_+^2 + 1\right)f'}{\left(\Lambda_+ - \Lambda_-\right)^2\left(\xi^2 - f\right)}\partial^-\Lambda_-\right]\partial^+ p \\ &\quad + \frac{\left(\Lambda_+^2 + 1\right)f'}{\left(\Lambda_+ - \Lambda_-\right)\left(\xi^2 - f\right)}\frac{\left(\Lambda_-^2 + 1\right)f'}{\left(\Lambda_+ - \Lambda_-\right)^2\left(\xi^2 - f\right)}\left(\partial^+ p\right)^2 + \left[\frac{\left(\Lambda_+^2 + 1\right)\left(\xi^2 f'' - ff'' + f'^2\right)}{\left(\Lambda_+ - \Lambda_-\right)\left(\xi^2 - f\right)^2}\right. \\ &\quad \left. + \frac{\left(\Lambda_+^2 + 1\right)f'}{\left(\Lambda_+ - \Lambda_-\right)\left(\xi^2 - f\right)}\left(\frac{f'}{f} + \frac{\left(-2\Lambda_-\Lambda_+ + \Lambda_+^2 - 1\right)f'}{\left(\Lambda_+ - \Lambda_-\right)^2\left(\xi^2 - f\right)}\right)\right]\partial^+ p\partial^- p \\ &\quad - \left[\frac{\left(\Lambda_+^2 + 1\right)f'}{\left(\Lambda_+ - \Lambda_-\right)\left(\xi^2 - f\right)}\frac{2\xi}{\xi^2 - f} + \frac{2\xi\left(\Lambda_+^2 + 1\right)f'}{\left(\Lambda_+ - \Lambda_-\right)\left(\xi^2 - f\right)^2}\right]\partial^+ p.\end{aligned} \tag{31}$$

将(27)式代入(30)式，可得

$$\begin{aligned}c_1 &= \left\{\frac{\left(\Lambda_-^2 + 1\right)f'}{\left(\Lambda_+ - \Lambda_-\right)^2\left(\xi^2 - f\right)} + \left[\frac{\left(\Lambda_-^2 + 1\right)\left(-\xi^2 f'' + ff'' - f'^2\right)}{\left(\Lambda_+ - \Lambda_-\right)\left(\xi^2 - f\right)^2} - \frac{\left(\Lambda_-^2 + 1\right)f'}{\left(\Lambda_+ - \Lambda_-\right)\left(\xi^2 - f\right)}\right.\right. \\ &\quad \left.\left.\left(\frac{f'}{f} + \frac{\left(-2\Lambda_+\Lambda_- + \Lambda_-^2 - 1\right)f'}{\left(\Lambda_+ - \Lambda_-\right)^2\left(\xi^2 - f\right)}\right)\right]\frac{\left(\Lambda_+ - \Lambda_-\right)\left(\xi^2 - f\right)}{\left(\Lambda_+^2 + 1\right)f'}\right\}\left(-\frac{\left(\Lambda_+ - \Lambda_-\right)\left(\xi^2 - f\right)}{\left(\Lambda_-^2 + 1\right)f'}\right) \\ &= \frac{-\Lambda_+^2 + \Lambda_-^2 - 2\Lambda_+\Lambda_- - 2}{\left(\Lambda_+ - \Lambda_-\right)\left(\Lambda_+^2 + 1\right)} + \frac{\Lambda_+ - \Lambda_-}{\Lambda_+^2 + 1}\left(\frac{\xi^2 - f}{f} + \frac{f''\xi^2 - ff'' + f'^2}{f'^2}\right),\end{aligned}$$

$$c_2 = \frac{\left(\Lambda_-^2 - 2\Lambda_+\Lambda_- - 1\right)f'}{\left(\Lambda_+ - \Lambda_-\right)^2\left(\xi^2 - f\right)}\left(-\frac{\left(\Lambda_+ - \Lambda_-\right)\left(\xi^2 - f\right)}{\left(\Lambda_-^2 + 1\right)f'}\right) = \frac{-\Lambda_-^2 + 2\Lambda_+\Lambda_- + 1}{\left(\Lambda_+ - \Lambda_-\right)\left(\Lambda_-^2 + 1\right)},$$

$$\begin{aligned}c_3 &= -\frac{\left(\Lambda_-^2 + 1\right)f'}{\left(\Lambda_+ - \Lambda_-\right)\left(\xi^2 - f\right)}\frac{\left(\Lambda_+^2 + 1\right)f'}{\left(\Lambda_+ - \Lambda_-\right)^2\left(\xi^2 - f\right)}\left(\frac{\left(\Lambda_+ - \Lambda_-\right)\left(\xi^2 - f\right)}{\left(\Lambda_-^2 + 1\right)f'}\right)^2 \\ &= -\frac{\Lambda_+^2 + 1}{\left(\Lambda_+ - \Lambda_-\right)\left(\Lambda_-^2 + 1\right)},\end{aligned}$$

$$
\begin{aligned}
c_4 &= \frac{\left(\Lambda_-^2+1\right)f'}{\left(\Lambda_+-\Lambda_-\right)\left(\xi^2-f\right)}\frac{2\xi}{\xi^2-f}\left(-\frac{\left(\Lambda_+-\Lambda_-\right)\left(\xi^2-f\right)}{\left(\Lambda_-^2+1\right)f'}\right) \\
&+\frac{2\xi\left(\Lambda_-^2+1\right)f'}{\left(\Lambda_+-\Lambda_-\right)\left(\xi^2-f\right)^2}\left(-\frac{\left(\Lambda_+-\Lambda_-\right)\left(\xi^2-f\right)}{\left(\Lambda_-^2+1\right)f'}\right) \\
&=-\frac{4\xi}{\xi^2-f}.
\end{aligned}
$$

同理，将(27)式代入(31)式可得 $d_i\left(i=1,2,3,4\right)$ 的值。

注：对于上面的所有式子都有 $f=f\left(p\right)$ 成立。

4. 结论

对于气体动力学中的二维广义压力梯度方程，通过上面性质 1 和性质 2 中分别对 p 和 $\Lambda_\pm$ 的特征分解形式，可以得到下面的结论：

定理 1. 压力梯度系统中与恒定状态相邻的状态必须是简单波，其中 p 沿正(或负)特征族是恒定的。

进一步假设流动无旋，(2)式可以被写成

$$
\begin{aligned}
&\left(\xi^2-f\left(p\right)\right)u_\xi+\xi\eta\left(u_\eta+v_\xi\right)+\left(\eta^2-f\left(p\right)\right)v_\eta=0, \\
&u_\eta=v_\xi.
\end{aligned} \tag{32}
$$

由上式可得

$$
\begin{cases}
\partial^+u+\Lambda_-\partial^+v=0, \\
\partial^-u+\Lambda_+\partial^-v=0.
\end{cases} \tag{33}
$$

性质 3. 在 $\left(\xi,\eta\right)$ 坐标下，对 u,v 有以下特征分解形式

$$
\begin{cases}
\partial^-\partial^+v=\tilde{M}_1\partial^+v, \\
\partial^+\partial^-v=\tilde{M}_2\partial^-v.
\end{cases}
\quad
\begin{cases}
\partial^-\partial^+u=\tilde{M}_3\partial^+u, \\
\partial^+\partial^-u=\tilde{M}_4\partial^-u.
\end{cases} \tag{34}
$$

其中 $\tilde{M}_i\left(i=1,2,3,4\right)$ 是一个好的因子。

证明：由(7)式可得

$$
\begin{aligned}
&\eta f\left(p\right)\partial^+u-\xi f\left(p\right)\partial^+v+\sqrt{f\left(p\right)\left(\xi^2+\eta^2-f\left(p\right)\right)}\partial^+p=0, \\
&-\eta f\left(p\right)\partial^-u+\xi f\left(p\right)\partial^-v+\sqrt{f\left(p\right)\left(\xi^2+\eta^2-f\left(p\right)\right)}\partial^-p=0.
\end{aligned} \tag{35}
$$

又由(33)式，可得

$$
\begin{aligned}
&\partial^+u=-\frac{\Lambda_-\sqrt{f\left(p\right)\left(\xi^2+\eta^2-f\left(p\right)\right)}}{\eta f\left(p\right)\Lambda_-+\xi f\left(p\right)}\partial^+p,\quad \partial^+v=\frac{\sqrt{f\left(p\right)\left(\xi^2+\eta^2-f\left(p\right)\right)}}{\eta f\left(p\right)\Lambda_-+\xi f\left(p\right)}\partial^+p, \\
&\partial^-u=\frac{\Lambda_+\sqrt{f\left(p\right)\left(\xi^2+\eta^2-f\left(p\right)\right)}}{\eta f\left(p\right)\Lambda_++\xi f\left(p\right)}\partial^-p,\quad \partial^-v=-\frac{\sqrt{f\left(p\right)\left(\xi^2+\eta^2-f\left(p\right)\right)}}{\eta f\left(p\right)\Lambda_++\xi f\left(p\right)}\partial^-p.
\end{aligned} \tag{36}
$$

以(36)式中第二式为例，对 ξ 沿负方向求偏导，可得

$$\partial^{-}\partial^{+}v=\partial^{-}\left(\frac{\sqrt{f(p)\left(\xi^{2}+\eta^{2}-f(p)\right)}}{\eta f(p)\Lambda_{-}+\xi f(p)}\right)\partial^{+}p+\frac{\sqrt{f(p)\left(\xi^{2}+\eta^{2}-f(p)\right)}}{\eta f(p)\Lambda_{-}+\xi f(p)}\partial^{-}\partial^{+}p, \tag{37}$$

代入(10)式中第二式，可得

$$\partial^{-}\partial^{+}v=M_{1}\partial^{+}p. \tag{38}$$

又由(36)式中的第二式，可得

$$\partial^{-}\partial^{+}v=\tilde{M}_{1}\partial^{+}v. \tag{39}$$

同理可得(34)式中其余三式。因此就有下面的定理成立：

定理 2. 与方程(1)中的恒定状态相邻的是一个简单波(如图 1)，其中物理变量(u,v)沿一系列特征线是恒定的，这些特征线是直线。

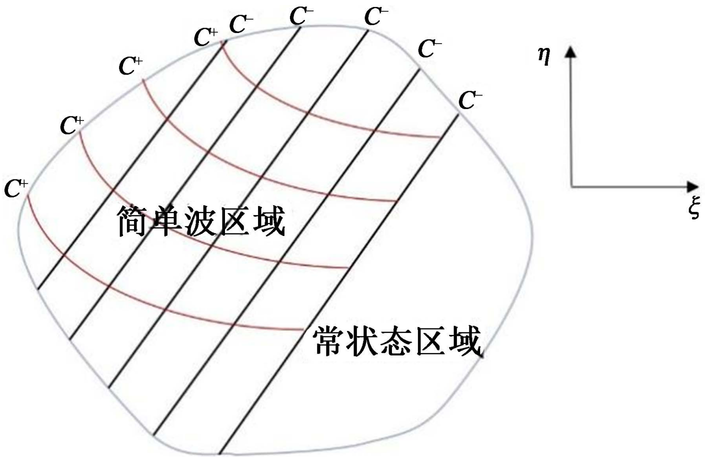

Figure 1. The flow is simple wave in constant state
图 1. 常状态下的流动是简单波

参考文献

[1] Lax, P.D. (1964) Development of Singularities of Solutions of Nonlinear Hyperbolic Partial Differential Equations. *Journal of Mathematical Physics*, **5**, 611-613. https://doi.org/10.1063/1.1704154

[2] Courant, R. and Friedrichs, K.O. (1948) Supersonic Flow and Shock Waves. Interscience Publishers Inc., New York, 38-45.

[3] Evans, L.C. (2010) Partial Differential Equations. American Mathematical Society, Providence. https://doi.org/10.1090/gsm/019

[4] Dai, Z. and Zhang, T. (2000) Existence of a Global Smooth Solution for a Degenerate Goursat Problem of Gas Dynamics. *Archive for Rational Mechanics and Analysis*, **155**, 277-298. https://doi.org/10.1007/s002050000113

[5] Hu, Y. and Sheng, W. (2012) Characteristic Decomposition of the 2 × 2 Quasilinear Strictly Hyperbolic Systems. *Applied Mathematics Letters*, **25**, 262-267. https://doi.org/10.1016/j.aml.2011.08.021

[6] Xiao, W. and Li, W. (2021) Simple Waves of the Two Dimensional Compressible Euler System for a Class of Pressure Laws. *Indian Journal of Pure and Applied Mathematics*, **52**, 1113-1122. https://doi.org/10.1007/s13226-021-00023-w

[7] Li, J. and Zheng, Y. (2009) Interaction of Rarefaction Waves of the Two-Dimensional Self-Similar Euler Equations. *Archive for Rational Mechanics and Analysis*, **193**, 623-657. https://doi.org/10.1007/s00205-008-0140-6

[8] Chen, X. and Zheng, Y. (2010) The Interaction of Rarefaction Waves of the Two-Dimensional Euler Equations. *Indiana University Mathematics Journal*, **59**, 231-256. https://doi.org/10.1512/iumj.2010.59.3752

[9] Li, M. and Zheng, Y. (2011) Semi-Hyperbolic Patches of Solutions to the Two-Dimensional Euler Equations. *Archive for Rational Mechanics and Analysis*, **201**, 1069-1096. https://doi.org/10.1007/s00205-011-0410-6

[10] Li, J. and Zheng, Y. (2010) Interaction of Four Rarefaction Waves in the Bi-Symmetric Class of the Two-Dimensional Euler Equations. *Communications in Mathematical Physics*, **296**, 303-321. https://doi.org/10.1007/s00220-010-1019-6

[11] Li, J., Zhang, T. and Zheng, Y. (2006) Simple Waves and a Characteristic Decomposition of the Two Dimensional Compressible Euler Equations. *Communications in Mathematical Physics*, **267**, 1-12. https://doi.org/10.1007/s00220-006-0033-1

[12] Zafar, M. and Sharma, V.D. (2014) Characteristic Decomposition of Compressible Euler Equations for a Non-Ideal Gas in Two-Dimensions. *Journal of Mathematical Physics*, **55**, Article ID: 093103. https://doi.org/10.1063/1.4896080

[13] Zafar, M. (2016) A Note on Characteristic Decomposition for Two-Dimensional Euler System in van der Waals Fluids. *International Journal of Non-Linear Mechanics*, **86**, 33-36. https://doi.org/10.1016/j.ijnonlinmec.2016.07.011

[14] Yao, A. and Sheng, W. (2022) Two-Dimensional Pseudo-Steady Supersonic Flow around a Sharp Corner. *ZAMM—Journal of Applied Mathematics and Mechanics*, **102**, e201800270. https://doi.org/10.1002/zamm.201800270

[15] Hu, Y. and Li, J. (2020) On a Global Supersonic-Sonic Patch Characterized by 2-D Steady Full Euler Equations. *Advances in Differential Equations*, **25**, 213-254.

[16] Hu, Y. and Chen, J. (2021) Sonic-Supersonic Solutions to a Mixed-Type Boundary Value Problem for the Two-Dimensional Full Euler Equations. *SIAM Journal on Mathematical Analysis*, **53**, 1579-1629. https://doi.org/10.1137/20M134589X