Pure Mathematics 理论数学, 2022, 12(11), 1989-1994
Published Online November 2022 in Hans. http://www.hanspub.org/journal/pm
https://doi.org/10.12677/pm.2022.1211215

二阶中立型时滞微分方程的振动准则

林靖杰，林全文，伍思敏*

广东石油化工学院理学院，广东 茂名

收稿日期：2022年10月18日；录用日期：2022年11月17日；发布日期：2022年11月25日

摘 要

考虑二阶中立型时滞微分方程：$\left(r(t)\left|\varphi'(t)\right|^{\alpha-1}\varphi'(t)\right)'+q(t)\left|x(\delta(t))\right|^{\beta-1}x(\delta(t))=0$ 的振动性，在现有文献基础上，利用广义Riccati变换、函数单调性和经典不等式，对方程做了进一步研究，建立新准则，改进了文献的某些结果。并且给出了例子说明主要结果的先进性。

关键词

时滞微分方程，振动准则，二阶，广义Riccati变换

Oscillation Criterion of Second Order Differential Equations with Neutral Delay

Jingjie Lin, Quanwen Lin, Simin Wu*

School of Science, Guangdong University of Petrochemical Technology, Maoming Guangdong

Received: Oct. 18th, 2022; accepted: Nov. 17th, 2022; published: Nov. 25th, 2022

Abstract

In this paper, we study the oscillation of second order differential equations with neutral delay $\left(r(t)\left|\varphi'(t)\right|^{\alpha-1}\varphi'(t)\right)'+q(t)\left|x(\delta(t))\right|^{\beta-1}x(\delta(t))=0$ using generalized Riccati transformation, the classical inequality and functional monotonicity, some new oscillation criterion are obtained, and improved the results of the references. At last, some examples are given to illustrate the advancement of our results.

*通讯作者。

文章引用: 林靖杰, 林全文, 伍思敏. 二阶中立型时滞微分方程的振动准则[J]. 理论数学, 2022, 12(11): 1989-1994.
DOI: 10.12677/pm.2022.1211215

Keywords

Delay Differential Equation, Oscillation Criterion, Second Order, Generalized Riccati Transformation

Copyright © 2022 by author(s) and Hans Publishers Inc.
This work is licensed under the Creative Commons Attribution International License (CC BY 4.0).
http://creativecommons.org/licenses/by/4.0/
Open Access

1. 引言

本文研究的是一类二阶半线性中立型时滞微分方程

$$\left(r(t)\left|\varphi'(t)\right|^{\alpha-1}\varphi'(t)\right)'+q(t)\left|x(\delta(t))\right|^{\beta-1}x(\delta(t))=0,\quad t>t_1>t_0 \tag{1}$$

的振动性，其中 $\varphi(t)=x(t)+p(t)x(\tau(t))$，并且满足以下条件：$\alpha>0$，$\beta>0$；时滞函数 $\delta\in C^1([t_0,\infty),R)$，$\tau\in C^1([t_0,\infty),R)$，任意 $t\geq t_0$，都有 $\tau(t)\leq t$，$\delta(t)\leq t$，$\delta'(t)>0$，$\lim\limits_{t\to\infty}\delta(t)=\lim\limits_{t\to\infty}\tau(t)=\infty$；$r\in C^1([t_0,\infty),(0,\infty))$，记 $\pi(t)=\int_t^\infty r^{-\frac{1}{\alpha}}(t)\mathrm{d}s$，并且满足 $\pi(t_0)<\infty$；$q\in C([t_0,\infty),(0,\infty))$，$p\in C([t_0,\infty),(0,\infty))$，$q(t)>0$，$0\leq p(t)<1$；$p(t)<\dfrac{\pi(t)}{\pi(\tau(t))}$。

若方程(1)的解既不最终为正，也不是最终为负，则它为振动的；否则称它为非振动的。若方程(1)所有的解都是振动的，则称方程(1)是振动的。对二阶半线性中立型微分方程振动性建立了一系列准则并给出证明方法[1]-[11]，但研究所得结果还不够完善，本文对文献[1]的振动准则进行研究，进一步改进其结果，得到一个新的振动准则。为了能更好地研究这方面的知识，需要引入以下定理。

引理 1.1 [1]设 $\alpha>0,\beta>0$ 是两个奇数商，如果

$$\int_{t_0}^\infty\left(\frac{1}{r(t)}\int_{t_0}^t Q(s)\pi^\beta(\sigma(s))\mathrm{d}s\right)^{\frac{1}{\alpha}}\mathrm{d}t=\infty \tag{2}$$

成立，则方程 $\left(\left(r(t)\left(x(t)+p(t)x(\tau(t))\right)'\right)^\alpha\right)'+q(t)x^\beta(\sigma(t))=0$ 是振动的。

引理 1.2 [11]设 $\pi(t_0)<\infty$，若存在函数 $\rho\in C([t_0,\infty),R)$ 满足

$$\limsup_{t\to\infty}\int_{t_0}^t\left(\rho(s)G_1-\frac{(\rho_+'(s))^{\alpha+1}r(\sigma(t))}{(\alpha+1)^{\alpha+1}(\rho(s)\sigma'(s))^\alpha}\right)\mathrm{d}s=\infty \tag{3}$$

且

$$\limsup_{t\to\infty}\int_{t_0}^t\left(\pi^\alpha(s)G_2(s)-\frac{\eta}{\pi(s)r^{\frac{1}{\alpha}}(s)}\right)\mathrm{d}s=\infty \tag{4}$$

其中

$$G_1(t)=g(t)\left(1-p(\sigma(t))\right)^{\alpha},\quad G_2(t)=g(t)\left(1-p(t)\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))}\right)^{\alpha},\tag{5}$$

$$\rho_+'(t)=\max\{0,\rho'(t)\},\quad \eta=\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1},\quad \pi(t)=\int_t^{\infty}\left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}}\mathrm{d}s\tag{6}$$

则方程$\left(r(t)\left|\varphi'(t)\right|^{\alpha-1}\varphi'(t)\right)'+g(t)\left|x(\sigma(t))\right|^{\alpha-1}x(\sigma(t))=0$振动。

受文献[1]和[11]启发，建立新的准则，主要对条件改进得到相对完善的结果，使得只要文献[1]中的$\alpha>0,\beta>0$且α,β可取偶数而不仅仅限于$\alpha>0,\beta>0$的两个奇数商。

2. 主要结果

定理 2.1：设(1)~(5)都成立，若存在函数$\rho\in C([t_0,\infty),R)$，满足

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(\rho(t)Q_1(t)-\frac{\alpha^{\alpha}}{\beta^{\alpha}(\alpha+1)^{\alpha+1}}\frac{\left(\rho_+'(t)\right)^{\alpha+1}r(\delta(t))}{\left(\rho(t)\delta'(t)\right)^{\alpha}}\right)\mathrm{d}t=\infty\tag{7}$$

且

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(\left(\frac{1}{r(s)}\right)\int_{t_1}^{t}\pi^{\beta}(u)Q(u)\mathrm{d}u\right)^{\frac{1}{\alpha}}\mathrm{d}s=\infty\tag{8}$$

其中

$$Q_1(t)=q(t)\left(1-p(\delta(t))\right)^{\beta},\quad Q(t)=q(t)\left(1-p(\delta(t))\frac{\pi(\tau(\delta(t)))}{\pi(\delta(t))}\right)^{\beta},\tag{9}$$

$$\rho_+'(t)=\max\{0,\rho'(t)\},\tag{10}$$

则方程(1)是振动的。

证明：设方程有一个非振动解$x(t)$，不妨设当$t\ge t_1$时，使得$x(t)>0$，$x(\sigma(t))>0$，$x(\tau(t))>0$成立。

由方程(1)可得

$$\left(r(t)\left|\varphi'(t)\right|^{\alpha-1}\varphi'(t)\right)'=-q(t)\left|x(\delta(t))\right|^{\beta-1}x(\delta(t))\le 0\tag{11}$$

在$[t_1,t]$上，$\left(r(t)\left|\varphi'(t)\right|^{\alpha-1}\varphi'(t)\right)$是非增函数，$\varphi'(t)$是定号的，那么分$\varphi'(t)>0$，$\varphi'(t)<0$两种情况展开讨论：

(I) 假设$\varphi'(t)>0$，$t\ge t_1\ge t_0$，

因为$\tau(t)\le t$，所以$\varphi(t)\ge\varphi(\tau(t))$，即有

$$x(t)=\varphi(t)-p(t)x(\tau(t))\ge\varphi(t)-p(t)\varphi(\tau(t))\ge(1-p(t))\varphi(t)\tag{12}$$

将(12)代入(1)得，

$$\left(r(t)\varphi'(t)^{\alpha}\right)'\le -Q_1(t)\left(\varphi(\delta(t))\right)^{\beta}\le 0\tag{13}$$

考虑广义 Riccati 变换，得到

$$\nu(t)=\frac{\rho(t)r(t)(\varphi'(t))^{\alpha}}{\varphi^{\beta}(\delta(t))}>0,\quad t\ge t_1 \tag{14}$$

对(14)式中分子，分母同时乘上 $\rho(t)$，再对 t 进行求导，结合(13)得

$$\begin{aligned}\nu'(t)&\le-\rho(t)q(t)\big(1-p(\delta(t))\big)^{\beta}+\frac{\rho'(t)}{\rho(t)}\nu(t)-\frac{\beta\rho(t)r(t)(-\varphi'(t))^{\alpha}\big(\varphi'(\delta(t))\big)\delta'(t)}{\varphi^{\beta+1}(\delta(t))}\\&\le-\rho(t)Q_1(t)+\frac{\rho'(t)}{\rho(t)}\nu(t)-\frac{\beta\delta'(t)}{\big(\rho(t)r(\delta(t))\big)^{\frac{1}{\alpha}}}\nu^{\frac{\alpha+1}{\alpha}}(t)\end{aligned} \tag{15}$$

由经典不等式 $Ay-By^{\frac{\alpha+1}{\alpha}}\le\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\cdot\frac{A^{\alpha+1}}{B^{\alpha}}$，上式变成

$$\begin{aligned}\nu'(t)&\le-\rho(t)Q_1(t)+\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\left(\frac{\rho'(t)}{\rho(t)}\right)^{\alpha+1}\frac{\rho(t)r(\delta(t))}{(\beta\delta'(t))^{\alpha}}\\&\le-\rho(t)Q_1(t)+\frac{\alpha^{\alpha}}{\beta^{\alpha}(\alpha+1)^{\alpha+1}}\frac{(\rho_+'(t))^{\alpha+1}r(\delta(t))}{(\rho(t)\delta'(t))^{\alpha}}\end{aligned} \tag{16}$$

对(16)式在 $[t_1,t]$ 上积分，即有

$$\int_{t_1}^{t}\nu'(t)\mathrm{d}t=\nu(t)-\nu(t_1)\le\int_{t_1}^{t}\left(-\rho(t)Q_1(t)+\frac{\alpha^{\alpha}}{\beta^{\alpha}(\alpha+1)^{\alpha+1}}\frac{(\rho_+'(t))^{\alpha+1}r(\delta(t))}{(\rho(t)\delta'(t))^{\alpha}}\right)\mathrm{d}t \tag{17}$$

即

$$\int_{t_1}^{t}\left(\rho(t)Q_1(t)-\frac{\alpha^{\alpha}}{\beta^{\alpha}(\alpha+1)^{\alpha+1}}\frac{(\rho_+'(t))^{\alpha+1}r(\delta(t))}{(\rho(t)\delta'(t))^{\alpha}}\right)\mathrm{d}t\le\nu(t_1)-\nu(t)\le\nu(t_1) \tag{18}$$

显然式(18)与条件(7)矛盾。

(II) $\varphi'(t)<0$，$t\ge t_1\ge t_0$，由于

$$\varphi(t)\ge-\int_t^{\infty}r^{-\frac{1}{\alpha}}(t)r^{\frac{1}{\alpha}}(t)\varphi'(t)\mathrm{d}t\ge-\pi(t)r^{\frac{1}{\alpha}}(t)\varphi'(t) \tag{19}$$

则

$$\left(\frac{\varphi(t)}{\pi(t)}\right)'\ge0$$

所以

$$\frac{\varphi(t)}{\pi(t)}\ge\frac{\varphi(\tau(t))}{\pi(\tau(t))} \tag{20}$$

即有

$$x(t)\ge\varphi(t)-p(t)\varphi(\tau(t))\ge\left(1-p(t)\frac{\pi(\tau(t))}{\pi(t)}\right)\varphi(t) \tag{21}$$

由方程(1)和不等式(21)得

$$-\left(r(t)\left(-\varphi'(t)\right)^{\alpha}\right)' \le -Q(t)\varphi^{\beta}(t) \tag{22}$$

考虑到 $r(t)\left(-\varphi'(t)\right)^{\alpha}$ 的单调性，则有

$$r(t)\left(-\varphi'(t)\right)^{\alpha} \ge r(t_1)\left(-\varphi'(t_1)\right)^{\alpha} = \mu > 0,\quad t \ge t_1 \tag{23}$$

由式(19)可得

$$\varphi(t) \ge -\pi(t)r^{\frac{1}{\alpha}}(t)\varphi'(t) \ge -\pi(t)r^{\frac{1}{\alpha}}(t_1)\varphi'(t_1) = \pi(t)\mu^{\frac{1}{\alpha}} \tag{24}$$

又结合式(22)和式(24)可得

$$-\left(r(t)\left(-\varphi'(t)\right)^{\alpha}\right)' \le -Q(t)\varphi^{\beta}(t) \le -\pi^{\beta}(t)\mu^{\frac{\beta}{\alpha}}Q(t) \tag{25}$$

对式(25)从 t_1 到 t 积分

$$\begin{aligned} -r(t)\left(-\varphi'(t)\right)^{\alpha} &\le -r(t_1)\left(-\varphi'(t_1)\right)^{\alpha} - \mu^{\frac{\beta}{\alpha}}\int_{t_1}^{t}\pi^{\beta}(s)Q(s)\mathrm{d}s \\ &\le -\mu^{\frac{\beta}{\alpha}}\int_{t_1}^{t}\pi^{\beta}(s)Q(s)\mathrm{d}s \end{aligned} \tag{26}$$

对上式从 t_1 到 t 积分得

$$\varphi(t) \le \varphi(t_1) - \mu^{\frac{\beta}{2\alpha}}\int_{t_1}^{t}\left(\frac{1}{r(t)}\int_{t_1}^{t}\pi^{\beta}(u)Q(u)\mathrm{d}u\right)^{\frac{1}{\alpha}}\mathrm{d}s \tag{27}$$

即

$$\int_{t_1}^{t}\left(\left(\frac{1}{r(t)}\right)\int_{t_1}^{t}\pi^{\beta}(u)Q(u)\mathrm{d}u\right)^{\frac{1}{\alpha}}\mathrm{d}s \le \mu^{-\frac{\beta}{2\alpha}}\varphi(t_1) - \mu^{-\frac{\beta}{2\alpha}}\varphi(t) < \mu^{-\frac{\beta}{2\alpha}}\varphi(t_1) \tag{28}$$

显然式(28)与条件(8)矛盾。所以假设不成立，即方程(1)是振动的。

3. 二阶中立型时滞微分方程的振动准则的应用

例：考虑方程：

$$\left(t^4\left(\left(x(t)+\frac{1}{9}\times x\left(\frac{t}{3}\right)\right)'\right)^2\right)' + ktx^4\left(\frac{t}{3}\right) = 0 \tag{29}$$

的解振动性

证明：取 $r(t)=t^4$， $\pi(t)=\int_t^{\infty}\left(\frac{1}{t^4}\right)^{\frac{1}{2}}\mathrm{d}t=\frac{1}{t}$， $p(t)=\frac{1}{9}$， $q(t)=kt^8$， $\delta(t)=\frac{t}{3}$， $\tau(t)=\frac{t}{3}$， $k>0$，

显然

$$Q_1(t)=kt^8\left(1-\frac{1}{9}\right)^4=\left(\frac{8}{9}\right)^4kt^8,\quad Q(t)=kt^8\left(1-\frac{1}{9}\times\frac{9}{t}\times\frac{t}{3}\right)^4=\frac{16}{81}kt^8, \tag{30}$$

则有

$$\int_{t_0}^{t}\left(\left(\frac{8}{9}\right)^4 kt^8\right)\mathrm{d}t = \frac{2048}{729}k\left(t^9 - t_0^9\right) \tag{31}$$

$$\begin{aligned}\int_{t_0}^{t}\left(\left(\frac{1}{t^4}\right)\int_{t_0}^{t} t^{-4}\frac{16}{81}kt^8\mathrm{d}t\right)^{\frac{1}{2}}\mathrm{d}t &= \int_{t_0}^{t}\left(\left(\frac{1}{t^4}\right)\left(\frac{16}{405}k\left(t^5 - t_0^5\right)\right)\right)^{\frac{1}{2}}\mathrm{d}t \\ &= \frac{8}{405}k\left(t^2 - t_0^2\right) + \frac{16}{1215}kt_0^5\left(t^{-5} - t_0^{-5}\right)\end{aligned} \tag{32}$$

显然当$t\to\infty$时，方程(29)条件(7)和(8)成立，故方程(29)是振动的。

注：本例子是建立在文献[1]中方程的基础上，主要是针对于α,β是偶数情况下，文献[1]的定理准则不能判断方程(29)的振动性。本例子检验了$\alpha=2,\beta=4$时，仍可判断方程(29)是振动的，体现了本准则的优越性。

基金项目

教育部产学合作协同育人项目(No. 201901104007)，茂名市科技计划(No. 2022056)。

参考文献

[1] 王璐. 两类二阶中立型时滞微分方程的振动性[D]: [硕士学位论文]. 曲阜: 曲阜师范大学, 2018.

[2] Agarwal, R.P., Zhang, C.H. and Li, T.X. (2016) Some Remarks on Oscillantion of Second Order Neutral Differential Equations. *Applied Mathematics and Computation*, **274**, 178-181. https://doi.org/10.1016/j.amc.2015.10.089

[3] Han, Z., Li, T., Sun, S. and Sun, Y. (2010) Remarks on the Paper [Appl. Math. Comput. 207 (2009) 388-396]. *Applied Mathematics and Computation*, **215**, 3998-4007. https://doi.org/10.1016/j.amc.2009.12.006

[4] Jiang, J. and Li, X. (2003) Qscillation of Second Order Nonlinear Neutral Differential Equations. *Applied Mathematics and Computation*, **135**, 531-540. https://doi.org/10.1016/S0096-3003(02)00066-8

[5] Li, T.X., Rogovchenko, Y.V. and Zhang, C.H. (2013) Oscillation of Second-Order Neutral Differential Equations. *Funkcialaj Ekvacioj*, **56**, 111-120. https://doi.org/10.1619/fesi.56.111

[6] Ye, L. and Xu, Z. (2009) Oscillation Criteria for Second Order Quasilinear Neutral Delay Differential Equations. *Applied Mathematics and Computation*, **207**, 388-396. https://doi.org/10.1016/j.amc.2008.10.051

[7] Li, T., Agarwal, R.P. and Bohner, M. (2012) Some Oscillation Results for Second-Order Differential Equations. *Journal of the Indian Mathematical Society*, **79**, 97-106.

[8] Baculíková, B. and Džurina, I. (2012) Oscillation Theorems for Higher Order Neutral Differential Equations. *Applied Mathematics and Computation*, **219**, 3769-3778. https://doi.org/10.1016/j.amc.2012.10.006

[9] Hasanbulli, M. and Rogovchenko, Y.V. (2010) Oscillation Criteria for Second-Order Nonlinear Neutral Differential Equations. *Applied Mathematics and Computation*, **215**, 4392-4399. https://doi.org/10.1016/j.amc.2010.01.001

[10] Zhang, S. and Wang, Q. (2010) Oscillation of Second-Order Nonlinear Neutral Dynamic Equations on Time Scales. *Applied Mathematics and Computation*, **216**, 2837-2848. https://doi.org/10.1016/j.amc.2010.03.134

[11] Lin, J.J., Simin, W. and Lin, Q.W. (2019) Oscillation Criteria for Higher Order Functional Equations. *Journal of Mathematics Research*, **11**, 135-141. https://doi.org/10.5539/jmr.v11n2p135