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一类含有Soret项的Brinkman方程组的结构稳定性

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摘 要

考虑了具有Soret效应的Brinkman方程组的解对方程系数σ的连续依赖性。首先，运用微分不等式技术，得到温度和盐浓度的相关估计，尤其是获得了盐浓度的四阶范数估计；其次，利用先验估计，推导出能量函数所满足的微分不等式；最后，求解该不等式，建立了解对系数σ的连续依赖性结果，该结果表明系数σ的微小变化不会引起解的急剧变化，因此Brinkman方程组对Soret系数具有结构稳定性。

关键词

Brinkman方程组，连续依赖性，Brinkman系数，Soret系数

Structural Stability of a Class of Brinkman Equations with Soret Term

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Abstract

The continuous dependence of the solution of Brinkman equations with Soret effect on equation coefficient σ is considered. Firstly, the correlation estimates of temperature and salt concentration are obtained by using differential inequality technique. Especially, we can get the fourth-order norm estimates for the concentration of the salt. Secondly, the differential inequality satisfied by the energy function is derived by using a priori estimate. Finally, by solving the inequality, the continuous dependence of solution on coefficient σ is established, the results show that the

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small change of coefficient σ will not cause the sharp change of solution, so Brinkman equations have structural stability for Soret coefficients.

Keywords

Brinkman Equations, Continuous Dependence, Brinkman Coefficient, Soret Coefficient

1. 引言

Nield [1]和 Straughan [2]详细介质了多孔介质中的具有 Soret 效应的 Brinkman 方程，它具有双扩散效应，而且采用了 Boussinesq 逼近。Straughan 等[2]在有界区域内，建立了方程组的解对 Soret 系数的连续依赖性，其结果仅表明 Soret 系数微小的变化，不会引起解的急剧变化，此时方程组是稳定的，这类稳定性的研究称之为结构稳定性研究。结构稳定性研究主要是考察模型自身的微小变化对模型解的影响，而传统的稳定性主要是考虑初始数据的变化对解的影响，关于结构稳定性系统的介绍见文献[3] [4]。

近年来，人们越来越关注多孔介质中 Brinkman，Darcy 和 Forchheimer 流体方程组解的性态研究，它已经成为是数学与物理交叉学科领域的热点问题。Payne 等[5]研究了 Darcy 方程组的空间衰减性。文献[6]-[18]研究了这三类方程组的结构稳定性，取得了新的研究成果。盐浓度方程若有含 Soret 项，导致处理盐浓度的最大值难度大，从而使得研究具有 Soret 效应的方程组解的结构稳定性较少，本文考虑如下具有 Soret 效应非齐次边界条件下的 Brinkman 方程组：

$$\begin{cases}\dfrac{\partial u_i}{\partial t}=\nu\Delta u_i-u_i-p_i+g_iT+h_iC,\ (x,t)\in\Omega\times[0,\tau],\\ \dfrac{\partial u_i}{\partial x_i}=0,\ (x,t)\in\Omega\times[0,\tau],\\ \dfrac{\partial T}{\partial t}+u_i\dfrac{\partial T}{\partial x_i}=\Delta T,\ (x,t)\in\Omega\times[0,\tau],\\ \dfrac{\partial C}{\partial t}+u_i\dfrac{\partial C}{\partial x_i}=\Delta C+\sigma\Delta T,\ (x,t)\in\Omega\times[0,\tau],\end{cases}\tag{1}$$

其中 u_i，p，T，C 分别为速度，压强，温度和盐浓度。$g_i(x)$ 和 $h_i(x)$ 为重力函数且 $|g_i|,|h_i|\le 1$，Δ 为 Laplace 算子。σ，ν 分别为 Soret 和 Brinkman 系数。在方程组(1)中，ν，σ 都是大于零的常数。

方程组(1)在 $\Omega\times[0,\tau]$ 区域内成立，其中 Ω 是 $\boldsymbol{R}^3$ 中一个有界连通的凸区域，τ 是给定的常数且 $0\le\tau<\infty$。我们考虑绝缘材料有损坏，导致流体与外界有热交换，溶质通过边界的通量为零情形，相应的边界条件为

$$u_i=0,\ \frac{\partial T}{\partial n}+k_1T=f_1(x,t),\ \frac{\partial C}{\partial n}+k_2C=f_2(x,t),\ (x,t)\in\partial\Omega\times[0,\tau]\tag{2}$$

初始条件为

$$u_i(x,0)=u_{i0}(x),\ T(x,0)=T_0(x),\ C(x,0)=C_0(x),\ x\in\Omega\tag{3}$$

本文研究了方程组(1)的解对方程系数 σ 的连续依赖性。为了得到连续依赖的结果，通常的做法是需要利用温度与盐浓度的最大值估计，去处理交叉项。由于方程组(1)含有 Soret 项 $\sigma\Delta T$，导致盐浓度的最大值估计难度很大，为了克服这个难题，我们采用给出盐浓度的四阶范数估计。如何构造合适的函数进行盐浓度的四阶范数估计是本文的最大创新之处。

本文用逗号表示求偏导，i 表示对 x_i 求偏导，如：$u_{,i}$ 表示为 $\frac{\partial u}{\partial x_i}$，重复指标表示求和，$u_{i,i}=\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i}$，$u_{i,j}u_{i,j}=\sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j}\right)^2$。

2. 一些有用的估计

为了推导出本文的主要结果，首先给出一些引理。

引理 1 温度 T 满足以下最大值估计

$$\operatorname{Sup}_{[0,\tau]}\|T\|_\infty \le T_M, \tag{4}$$

其中 $T_M=\max\left\{\|T_0\|_\infty, \frac{1}{k_1}\operatorname*{Sup}_{[0,\tau]} f_{1_\infty}\right\}$。

证明 在方程组(1)第三个方程两边同时乘以 $2rT^{2r-1}(r\ge 1)$，并且在 $\Omega\times[0,t](t\in[0,\tau])$ 上积分，可得

$$2r\int_0^t\int_\Omega T^{2r-1}T_{,\eta}\mathrm{d}x\mathrm{d}\eta+2r\int_0^t\int_\Omega T^{2r-1}u_iT_{,i}\mathrm{d}x\mathrm{d}\eta=2r\int_0^t\int_\Omega T^{2r-1}\Delta T\mathrm{d}x\mathrm{d}\eta. \tag{5}$$

由散度定理，Hölder 不等式和 Young 不等式，可得

$$2r\int_0^t\int_\Omega T^{2r-1}u_iT_{,i}\mathrm{d}x\mathrm{d}\eta=\int_0^t\int_\Omega\left(T^{2r}\right)_{,i}u_i\mathrm{d}x\mathrm{d}\eta=0. \tag{6}$$

$$\begin{aligned}
&2r\int_0^t\int_\Omega T^{2r-1}\Delta T\mathrm{d}x\mathrm{d}\eta\\
&=2r\int_0^t\int_{\partial\Omega}T^{2r-1}\frac{\partial T}{\partial n}\mathrm{d}S\mathrm{d}\eta-\frac{2(2r-1)}{r}\int_0^t\int_\Omega\left(T^r\right)_{,i}\left(T^r\right)_{,i}\mathrm{d}x\mathrm{d}\eta\\
&=2r\int_0^t\int_{\partial\Omega}f_1T^{2r-1}\mathrm{d}S\mathrm{d}\eta-2rk_1\int_0^t\int_{\partial\Omega}T^{2r}\mathrm{d}S\mathrm{d}\eta-\frac{2(2r-1)}{r}\int_0^t\int_\Omega\left(T^r\right)_{,i}\left(T^r\right)_{,i}\mathrm{d}x\mathrm{d}\eta\\
&\le\left(\frac{2r-1}{2rk_1}\right)^{2r-1}\int_0^t\int_{\partial\Omega}f_1^{2r}\mathrm{d}S\mathrm{d}\eta.
\end{aligned} \tag{7}$$

联合式(5)~式(7)，可得

$$\int_\Omega T^{2r}\mathrm{d}x\le\int_\Omega T_0^{2r}\mathrm{d}x+\left(\frac{2r-1}{2rk_1}\right)^{2r-1}\int_0^t\int_{\partial\Omega}f_1^{2r}\mathrm{d}S\mathrm{d}\eta. \tag{8}$$

式(8)式从 0 到 t 积分，有

$$\left(\int_0^t\int_\Omega T^{2r}\mathrm{d}x\mathrm{d}\eta\right)^{\frac{1}{2r}}\le\left(\int_0^t\int_\Omega T_0^{2r}\mathrm{d}x\mathrm{d}\eta\right)^{\frac{1}{2r}}+\left(\frac{2r-1}{2rk_1}\right)^{\frac{2r-1}{2r}}\left(\int_0^t\int_0^\xi\int_{\partial\Omega}f_1^{2r}\mathrm{d}S\mathrm{d}\xi\mathrm{d}\eta\right)^{\frac{1}{2r}}. \tag{9}$$

在(9)式中，当 $r\to+\infty$，有

$$\operatorname{Sup}_{[0,\tau]}\|T\|_\infty\le\max\left\{\|T_0\|_\infty, \frac{1}{k_1}\operatorname*{Sup}_{[0,\tau]} f_{1_\infty}\right\}=T_M.$$

证毕！

引理 2 对于温度 T，有如下估计

$$\int_0^t \int_{\partial\Omega} T^4 \mathrm{d}s\mathrm{d}\eta \le m_1(t), \tag{10}$$

$$\int_0^t \int_{\partial\Omega} T^2 \mathrm{d}s\mathrm{d}\eta \le m_2(t), \tag{11}$$

其中 $m_1(t) = \frac{1}{2k_1^2}\int_0^t \int_{\partial\Omega} f_1^4 \mathrm{d}s\mathrm{d}\eta + \frac{1}{2k_1}\int_{\partial\Omega} T_0^4 \mathrm{d}s$， $m_2(t) = \frac{1}{2k_1^2}\int_0^t \int_{\partial\Omega} f_1^2 \mathrm{d}s\mathrm{d}\eta + \frac{1}{2k_1}\int_{\partial\Omega} T_0^2 \mathrm{d}s$。

证明 在方程组(1)第三个方程两边同时乘以 $4T^3$，并在 Ω 上积分得

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega T^4 \mathrm{d}x &\le 4\int_\Omega T^3\left(\Delta T - u_i T_{,i}\right)\mathrm{d}x\mathrm{d}\eta \\ &= -12\int_\Omega T^2 \left|\nabla T\right|^2 \mathrm{d}x\mathrm{d}\eta - 4k_1\int_{\partial\Omega} T^4 \mathrm{d}s + 4\int_{\partial\Omega} T^3 f_1 \mathrm{d}s.\end{aligned} \tag{12}$$

式(12)从 0 到t积分，并由 Young 不等式，可得

$$\int_0^t \int_{\partial\Omega} T^4 \mathrm{d}s\mathrm{d}\eta \le \frac{1}{2k_1^2}\int_0^t \int_{\partial\Omega} f_1^4 \mathrm{d}s\mathrm{d}\eta + \frac{1}{2k_1}\int_{\partial\Omega} T_0^4 \mathrm{d}s.$$

同理可知

$$\int_0^t \int_{\partial\Omega} T^2 \mathrm{d}s\mathrm{d}\eta \le \frac{1}{2k_1^2}\int_0^t \int_{\partial\Omega} f_1^2 \mathrm{d}s\mathrm{d}\eta + \frac{1}{2k_1}\int_{\partial\Omega} T_0^2 \mathrm{d}s.$$

证毕！

引理 3 对于温度 T 和盐浓度 C，有如下估计

$$\int_\Omega T^2 \mathrm{d}x + 2\int_0^t \int_\Omega T_{,i}T_{,i} \mathrm{d}x\mathrm{d}\eta \le m_3(t), \tag{13}$$

$$\int_\Omega C^2 \mathrm{d}x + \int_0^t \int_\Omega C_{,i}C_{,i} \mathrm{d}x\mathrm{d}\eta \le m_4(t), \tag{14}$$

其中

$$m_3(t) = \int_\Omega T_0^2 \mathrm{d}x + \frac{1}{2k_1}\int_0^t \int_{\partial\Omega} f_1^2 \mathrm{d}s\mathrm{d}\eta,$$

$$m_4(t) = \frac{\sigma^2}{2}m_3(t) + \int_\Omega C_0^2 \mathrm{d}x + \frac{1}{k_2}\int_0^t \int_{\partial\Omega} f_2^2 \mathrm{d}s\mathrm{d}\eta + \frac{2\sigma^2}{k_2}\int_0^t \int_{\partial\Omega} f_1^2 \mathrm{d}s\mathrm{d}\eta + \frac{2\sigma^2 k_1^2}{k_2}m_2(t).$$

证明 在方程组(1)第三个方程两边乘以 $2T$，并在 $\Omega\times[0,t]$ 上积分，由 Young 不等式，可得

$$\int_\Omega T^2 \mathrm{d}x + 2\int_0^t \int_\Omega T_{,i}T_{,i} \mathrm{d}x\mathrm{d}\eta \le \int_\Omega T_0^2 \mathrm{d}x + \frac{1}{2k_1}\int_0^t \int_{\partial\Omega} f_1^2 \mathrm{d}s\mathrm{d}\eta.$$

在方程组(1)第四个方程两边乘以 $2T$，并在 $\Omega\times[0,t]$ 上积分得

$$\begin{aligned}\int_\Omega C^2 \mathrm{d}x + 2\int_0^t \int_\Omega C_{,i}C_{,i} \mathrm{d}x\mathrm{d}\eta \le& \int_\Omega C_0^2 \mathrm{d}x - 2\sigma\int_0^t \int_\Omega C_{,i}T_{,i} \mathrm{d}x\mathrm{d}\eta \\ &+ 2\int_0^t \int_{\partial\Omega} C\left(f_2 - k_2 C\right)\mathrm{d}s\mathrm{d}\eta \\ &+ 2\int_0^t \int_{\partial\Omega} C\left(f_1 - k_1 T\right)\mathrm{d}s\mathrm{d}\eta.\end{aligned} \tag{15}$$

式(15)运用 Schwarz 不等式，可知

$$\int_\Omega C^2\mathrm{d}x+\int_0^t\int_\Omega C_{,i}C_{,i}\mathrm{d}x\mathrm{d}\eta\le\int_\Omega C_0^2\mathrm{d}x+\sigma^2\int_0^t\int_\Omega T_{,i}T_{,i}\mathrm{d}x\mathrm{d}\eta+\frac{1}{k_2}\int_0^t\int_{\partial\Omega}f_2^2\mathrm{d}s\mathrm{d}\eta \\ +\frac{2\sigma^2}{k_2}\int_0^t\int_{\partial\Omega}f_1^2\mathrm{d}s\mathrm{d}\eta+\frac{2\sigma^2k_1^2}{k_2}\int_0^t\int_{\partial\Omega}T^2\mathrm{d}s\mathrm{d}\eta. \tag{16}$$

将式(13)和式(4)代入式(16)，可得

$$\int_\Omega C^2\mathrm{d}x+\int_0^t\int_\Omega C_{,i}C_{,i}\mathrm{d}x\mathrm{d}\eta\le\frac{\sigma^2}{2}m_3(t)+\int_\Omega C_0^2\mathrm{d}x+\frac{1}{k_2}\int_0^t\int_{\partial\Omega}f_2^2\mathrm{d}s\mathrm{d}\eta \\ +\frac{2\sigma^2}{k_2}\int_0^t\int_{\partial\Omega}f_1^2\mathrm{d}s\mathrm{d}\eta+\frac{2\sigma^2k_1^2}{k_2}m_2(t).$$

证毕！

引理 4 对于盐浓度 C，有如下的 4 阶范数估计

$$\int_\Omega C^4\mathrm{d}x\le m_5(t), \tag{17}$$

其中 $m_5(t)=n_1\int_{\partial\Omega}f_1^4\mathrm{d}s+n_2\int_{\partial\Omega}f_2^4\mathrm{d}s+n_4m_4(t)+\frac{n_5}{2}m_3(t)+n_3m_1(t)$，$n_1,n_2,n_3,n_4,n_5$ 为大于零的常数。

证明 在方程组(1)第四个方程两边乘以 C^3，并在 Ω 上积分得

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega C^4\mathrm{d}x=4\int_\Omega C^3\left(\Delta C+\sigma\Delta T-u_iC_{,i}\right)\mathrm{d}x\\&=-12\int_\Omega C^2|\nabla C|^2\mathrm{d}x-12\sigma\int_\Omega C^2C_{,i}T_{,i}\mathrm{d}x+\int_{\partial\Omega}C^3\left(f_2-k_2C\right)\mathrm{d}s+\sigma\int_{\partial\Omega}C^3\left(f_1-k_1T\right)\mathrm{d}s\\&=-12\int_\Omega C^2|\nabla C|^2\mathrm{d}x-12\sigma\int_\Omega CC_{,i}\left(CT_{,i}+TC_{,i}\right)\mathrm{d}x+12\sigma\int_\Omega CT|\nabla C|^2\mathrm{d}x\\&\quad-4k_2\int_{\partial\Omega}C^4\mathrm{d}s+4\int_{\partial\Omega}C^3f_2\mathrm{d}s-4\sigma k_1\int_{\partial\Omega}C^3T\mathrm{d}s+4\sigma k_1\int_{\partial\Omega}C^3f_1\mathrm{d}s\\&\le-12\int_\Omega C^2|\nabla C|^2\mathrm{d}x+6\sigma\lambda_1\int_\Omega C^2|\nabla C|^2\mathrm{d}x+\frac{6\sigma}{\lambda_1}\int_\Omega\left(CT_{,i}+TC_{,i}\right)\left(CT_{,i}+TC_{,i}\right)\mathrm{d}x\\&\quad+6\sigma\lambda_2\int_\Omega C^2|\nabla C|^2\mathrm{d}x+\frac{6\sigma T_M^2}{\lambda_2}\int_\Omega|\nabla C|^2\mathrm{d}x-k_2\int_{\partial\Omega}C^4\mathrm{d}s\\&\quad+\frac{27}{k_2^3}\int_{\partial\Omega}f_2^4\mathrm{d}s+\frac{27\sigma^4k_1^4}{k_2^3}\int_{\partial\Omega}T^4\mathrm{d}s+\frac{27\sigma^4k_1^4}{k_2^3}\int_{\partial\Omega}f_1^4\mathrm{d}s,\end{aligned} \tag{18}$$

其中 λ_1，λ_2 是大于零的任意常数。

运用方程组(1)第三个和第四个方程以及 Hölder 不等式，可得

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega T^2C^2\mathrm{d}x=2\int_\Omega TC^2T_{,t}\mathrm{d}x+2\int_\Omega CT^2C_{,t}\mathrm{d}x\\&=2\int_\Omega TC^2\left(\Delta T-u_iT_{,i}\right)\mathrm{d}x+2\int_\Omega CT^2\left(\Delta C+\sigma\Delta T-u_iC_{,i}\right)\mathrm{d}x\\&=-2\int_\Omega\left(CT_{,i}+TC_{,i}\right)\left(CT_{,i}+TC_{,i}\right)\mathrm{d}x-4\int_\Omega TT_{,i}CC_{,i}\mathrm{d}x\\&\quad-4\sigma\int_\Omega TC|\nabla T|^2\mathrm{d}x-2\sigma\int_\Omega T^2T_{,i}C_{,i}\mathrm{d}x+2\int_{\partial\Omega}TC^2\left(f_1-k_1T\right)\mathrm{d}s\\&\quad+2\int_{\partial\Omega}CT^2\left(f_2-k_2C\right)\mathrm{d}s+2\sigma\int_{\partial\Omega}CT^2\left(f_1-k_1T\right)\mathrm{d}s\\&=-2\int_\Omega\left(CT_{,i}+TC_{,i}\right)\left(CT_{,i}+TC_{,i}\right)\mathrm{d}x-4\int_\Omega TT_{,i}CC_{,i}\mathrm{d}x\\&\quad+4\sigma\int_\Omega TT_{,i}\left(CT_{,i}+TC_{,i}\right)\mathrm{d}x+2\sigma\int_\Omega T^2T_{,i}C_{,i}\mathrm{d}x\\&\quad-2\left(k_1+k_2\right)\int_{\partial\Omega}T^2C^2\mathrm{d}s+2\int_{\partial\Omega}TC^2f_1\mathrm{d}s+2\int_{\partial\Omega}CT^2f_2\mathrm{d}s\\&\quad+2\sigma\int_{\partial\Omega}CT^2f_1\mathrm{d}s-2\sigma k_1\int_{\partial\Omega}CT^3\mathrm{d}s\end{aligned}$$

$$\begin{aligned}
&\le -2\int_\Omega \left(CT_{,i}+TC_{,i}\right)\left(CT_{,i}+TC_{,i}\right)\mathrm{d}x+2\lambda_3\int_\Omega C^2\left|\nabla C\right|^2\mathrm{d}x \\
&+\frac{2T_M^2}{\lambda_3}\int_\Omega T_{,i}T_{,i}\mathrm{d}x+2\sigma\lambda_4\int_\Omega\left(CT_{,i}+TC_{,i}\right)\left(CT_{,i}+TC_{,i}\right)\mathrm{d}x \\
&+\frac{T_M^2}{2\lambda_4}\int_\Omega T_{,i}T_{,i}\mathrm{d}x+\sigma T_M^2\int_\Omega T_{,i}T_{,i}\mathrm{d}x+\sigma T_M^2\int_\Omega C_{,i}C_{,i}\mathrm{d}x \\
&+\frac{k_2}{k}\int_{\partial\Omega}C^4\mathrm{d}s+\left(\frac{k}{4k_1^2}+\frac{\sigma^2}{2k_2}\right)\int_{\partial\Omega}f_1^4\mathrm{d}s+\left(\frac{1}{2}+\frac{\sigma^2}{2k_2}+\frac{\sigma^2k_1^2}{k_2^2}\right)\int_{\partial\Omega}T^4\mathrm{d}s+\frac{1}{2}\int_{\partial\Omega}f_2^4\mathrm{d}s,
\end{aligned} \tag{19}$$

其中 k， λ_3， λ_4 是大于零的任意常数。

联合式(18)和式(19)得

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega C^4\mathrm{d}x+k\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega T^2C^2\mathrm{d}x \\
&\le -\left(12-6\sigma\lambda_1-2k\lambda_3-6\sigma\lambda_2\right)\int_\Omega C^2\left|\nabla C\right|^2\mathrm{d}x \\
&-\left(2k-\frac{6\sigma}{\lambda_1}-2k\sigma\lambda_4\right)\int_\Omega\left(CT_{,i}+TC_{,i}\right)\left(CT_{,i}+TC_{,i}\right)\mathrm{d}x \\
&+\left(\frac{6\sigma T_M^2}{\lambda_2}+k\sigma T_M^2\right)\int_\Omega C_{,i}C_{,i}\mathrm{d}x+\left(\frac{2kT_M^2}{\lambda_3}+\frac{kT_M^2}{2\lambda_4}+k\sigma T_M^2\right)\int_\Omega T_{,i}T_{,i}\mathrm{d}x \\
&+n_1\int_{\partial\Omega}f_1^4\mathrm{d}s+n_2\int_{\partial\Omega}f_2^4\mathrm{d}s+n_3\int_{\partial\Omega}T^4\mathrm{d}s,
\end{aligned} \tag{20}$$

其中 $n_1=\frac{27\sigma^4k_1^4}{k_2^3}+\left(\frac{k}{4k_1^2}+\frac{\sigma^2}{2k_2}\right)k$， $n_2=\frac{k}{2}+\frac{27}{k_2^3}$， $n_3=\left(\frac{1}{2}+\frac{\sigma^2}{2k_2}+\frac{\sigma^2k_1^2}{k_2^2}\right)k+\frac{27\sigma^4k_1^4}{k_2^3}$。

式(20)中，取 $\lambda_1=\lambda_2=\frac{1}{3\sigma}$， $\lambda_3=\frac{1}{18\sigma^2}$， $\lambda_4=\frac{1}{2\sigma}$， $k=18\sigma^2$，可得

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega C^4\mathrm{d}x+k\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega T^2C^2\mathrm{d}x \\
&\le n_4\int_\Omega C_{,i}C_{,i}\mathrm{d}x+n_5\int_\Omega T_{,i}T_{,i}\mathrm{d}x+n_1\int_{\partial\Omega}f_1^4\mathrm{d}s+n_2\int_{\partial\Omega}f_2^4\mathrm{d}s+n_3\int_{\partial\Omega}T^4\mathrm{d}s,
\end{aligned} \tag{21}$$

其中 $n_4=18\sigma^2T_M^2\left(1+\sigma\right)$， $n_5=18\sigma^3T_M^2\left(1+36\sigma\right)+183\sigma T_M^2$。

积分式(21)并将式(10)，式(14)和(15)的结果代入，可得

$$\int_\Omega C^4\mathrm{d}x\le m_5\left(t\right).$$

证毕！

3. 解对 Soret 系数 σ 连续依赖性

在本节我们将建立方程组的解对 Soret 系数 σ 连续依赖性。

假设 $\left(u_i,T,C,p\right)$ 是如下 Brinkman 方程组的解

$$\begin{cases}
\frac{\partial u_i}{\partial t}=\nu\Delta u_i-u_i-p_{,i}+g_iT+h_iC,\ \left(x,t\right)\in\Omega\times\left[0,\tau\right], \\
\frac{\partial u_i}{\partial x_i}=0,\ \left(x,t\right)\in\Omega\times\left[0,\tau\right], \\
\frac{\partial T}{\partial t}+u_i\frac{\partial T}{\partial x_1}=\Delta T,\ \left(x,t\right)\in\Omega\times\left[0,\tau\right], \\
\frac{\partial C}{\partial t}+u_i\frac{\partial C}{\partial x_1}=\Delta C+\sigma_1\Delta T,\ \left(x,t\right)\in\Omega\times\left[0,\tau\right]
\end{cases} \tag{22}$$

边界条件为

$$u_i = 0,\ \frac{\partial T}{\partial n} + k_1 T = f_1(x,t),\ \frac{\partial C}{\partial n} + k_2 C = f_2(x,t),\ (x,t) \in \partial\Omega \times [0,\tau] \tag{23}$$

初始条件为

$$u_i(x,0) = u_{i0}(x),\ T(x,0) = T_0(x),\ C(x,0) = C_0(x),\ x \in \Omega. \tag{24}$$

此外，假设 (u_i^*, T^*, C^*, p^*) 也是如下 Brinkman 方程组的解

$$\begin{cases} \dfrac{\partial u_i^*}{\partial t} = \nu \Delta u_i^* - u_i^* - p_{,i}^* + g_i T^* + h_i C^*,\ (x,t) \in \Omega \times [0,\tau], \\ \dfrac{\partial u_i^*}{\partial x_i} = 0,\ (x,t) \in \Omega \times [0,\tau], \\ \dfrac{\partial T^*}{\partial t} + u_i^* \dfrac{\partial T^*}{\partial x_i} = \Delta T^*, (x,t) \in \Omega \times [0,\tau], \\ \dfrac{\partial C^*}{\partial t} + u_i^* \dfrac{\partial C^*}{\partial x_i} = \Delta C^* + \sigma_2 \Delta T^*, (x,t) \in \Omega \times [0,\tau]. \end{cases} \tag{25}$$

边界条件为

$$u_i^* = 0,\ \frac{\partial T^*}{\partial n} + k_1 T^* = f_1(x,t),\ \frac{\partial C^*}{\partial n} + k_2 C^* = f_2(x,t),\ (x,t) \in \partial\Omega \times [0,\tau]. \tag{26}$$

初始条件为

$$u_i^*(x,0) = u_{i0}(x),\ T^*(x,0) = T_0(x),\ C^*(x,0) = C_0(x),\ x \in \Omega. \tag{27}$$

定义解的差为：$\omega_i = u_i - u_i^*$， $\theta = T - T^*$， $S = C - C^*$， $\pi = p - p^*$， $\sigma = \sigma_1 - \sigma_2$ 则 $(\omega_i, \theta, S, \pi)$ 满足如下初边值问题

$$\begin{cases} \dfrac{\partial \omega_i}{\partial t} = \nu \Delta \omega_i - \omega_i - \pi_{,i} + g_i \theta + h_i S,\ (x,t) \in \Omega \times [0,\tau], \\ \dfrac{\partial \omega_i}{\partial x_i} = 0,\ (x,t) \in \Omega \times [0,\tau], \\ \dfrac{\partial \theta}{\partial t} + \omega_i T_{,i} + u_i^* \theta_{,i} = \Delta \theta,\ (x,t) \in \Omega \times [0,\tau], \\ \dfrac{\partial S}{\partial t} + \omega_i C_{,i} + u_i^* S_{,i} = \Delta S + \sigma \Delta T + \sigma_2 \Delta \theta,\ (x,t) \in \Omega \times [0,\tau]. \end{cases} \tag{28}$$

边界条件为

$$\omega_i = 0,\ \frac{\partial \theta}{\partial n} + k_1 \theta = 0,\ \frac{\partial S}{\partial n} + k_2 S = 0,\ (x,t) \in \partial\Omega \times [0,\tau]. \tag{29}$$

初始条件为

$$\omega_i(x,0) = 0,\ \theta(x,0) = 0,\ S(x,0) = 0,\ x \in \Omega. \tag{30}$$

定理 1 假设 (u_i, T, C, p) 是式(22)~式(24)初边值问题的古典解，(u_i^*, T^*, C^*, p^*) 是式(25)~式(27)初边值问题的古典解，$(\omega_i, \theta, S, \pi)$ 为这两个解的差，则当方程系数 σ 趋于 0 时，解 (u_i, T, C, p) 收敛于解 (u_i^*, T^*, C^*, p^*)，且有下列不等式成立

$$\varepsilon_1\|\omega\|^2+\varepsilon_2\|\theta\|^2+\|S\|^2\le\sigma^2\mathrm{e}^{\varepsilon_3 t}m_6(t),\tag{31}$$

其中 $\varepsilon_1,\varepsilon_2,\varepsilon_3$ 是大于零的常数，$m_6(t)=\frac{1}{k_2}\int_0^t\int_{\partial\Omega}f_1^2\mathrm{d}s\mathrm{d}\eta+m_2(t)+m_3(t)$。

证明 在方程组(28)第一个方程两边乘以 ω_i，并在 Ω 上积分得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|^2+\nu\int_\Omega\omega_{i,j}\omega_{i,j}\mathrm{d}x=-\int_\Omega\omega_i\omega_i\mathrm{d}x+\int_\Omega\omega_i g_i\theta\mathrm{d}x+\int_\Omega\omega_i h_i S\mathrm{d}x.$$

上式运用 Schwarz 不等式得

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega^2+2\nu\int_\Omega\omega_{i,j}\omega_{i,j}\mathrm{d}x\le\int_\Omega\theta^2\mathrm{d}x+\int_\Omega S^2\mathrm{d}x\tag{32}$$

在方程组(28)第三个方程两边同时乘以 θ，并在 Ω 上积分得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|^2+\int_\Omega\theta_{,i}\theta_{,i}\mathrm{d}x=-\int_\Omega\omega_i T_{,i}\theta\mathrm{d}x-k_1\int_{\partial\Omega}\theta^2\mathrm{d}s=\int_\Omega\omega_i T\theta_{,i}\mathrm{d}x-k_1\int_{\partial\Omega}\theta^2\mathrm{d}s.$$

上式运用 Schwarz 不等式得

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|^2+2k_1\int_{\partial\Omega}\theta^2\mathrm{d}x+\int_\Omega\theta_{,i}\theta_{,i}\mathrm{d}x\le T_M^2\|\omega\|^2.\tag{33}$$

在方程组(28)第四个方程两边同时乘以 S，并在 Ω 上积分得

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|S\|^2+\int_\Omega S_{,i}S_{,i}\mathrm{d}x+k_2\int_{\partial\Omega}S^2\mathrm{d}s\\&=\int_\Omega\omega_i CS_{,i}\mathrm{d}x+\sigma\int_{\partial\Omega}Sf_1\mathrm{d}s-\sigma k_1\int_{\partial\Omega}ST\mathrm{d}s-\sigma\int_\Omega S_{,i}T_{,i}\mathrm{d}x\\&\quad-\sigma_2k_1\int_{\partial\Omega}S\theta\mathrm{d}s-\sigma_2\int_\Omega S_{,i}\theta_{,i}\mathrm{d}x.\end{aligned}$$

上式运用 Young 和 Hölder 不等式得

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|S\|^2\le&\left(\int_\Omega(\omega_i\omega_i)^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega C^4\mathrm{d}x\right)^{\frac{1}{2}}+\frac{\sigma^2}{k_2}\int_{\partial\Omega}f_1^2\mathrm{d}s+\frac{2\sigma^2k_1^2}{k_2}\int_{\partial\Omega}T^2\mathrm{d}s\\&+2\sigma^2\int_\Omega T_{,i}T_{,i}\mathrm{d}x+\frac{2\sigma_2^2k_1^2}{k_2}\int_{\partial\Omega}\theta^2\mathrm{d}s+2\sigma_2^2\int_\Omega\theta_{,i}\theta_{,i}\mathrm{d}x.\end{aligned}\tag{34}$$

对于满足在边界上为零的函数 G，由[17]的结论，有如下 Sobolev 不等式成立

$$\int_\Omega|G|^4\mathrm{d}x\le c_1\left(\int_\Omega|G|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega G_{i,j}G_{i,j}\mathrm{d}x\right)^{\frac{3}{2}}\le c_2\left(\int_\Omega G_{i,j}G_{i,j}\mathrm{d}x\right)^2,\tag{35}$$

其中 c_1，c_2 是大于零的常数。

在式(35)中，取 $G=\omega_i$，可知

$$\int_\Omega(\omega_i\omega_i)^2\mathrm{d}x\le c_2\left(\int_\Omega\omega_{i,j}\omega_{i,j}\mathrm{d}x\right)^2.\tag{36}$$

联合式(34)和式(36)得

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}S^2\le&\sqrt{c_2m_5(\tau)}\int_\Omega\omega_{i,j}\omega_{i,j}\mathrm{d}x+\frac{\sigma^2}{k_2}\int_{\partial\Omega}f_1^2\mathrm{d}s+\frac{2\sigma^2k_1^2}{k_2}\int_{\partial\Omega}T^2\mathrm{d}s\\&+2\sigma^2\int_\Omega T_{,i}T_{,i}\mathrm{d}x+\frac{2\sigma_2^2k_1^2}{k_2}\int_{\partial\Omega}\theta^2\mathrm{d}s+2\sigma_2^2\int_\Omega\theta_{,i}\theta_{,i}\mathrm{d}x.\end{aligned}\tag{37}$$

定义

$$F(t)=\varepsilon_1\|\omega\|^2+\varepsilon_2\|\theta\|^2+\|S\|^2,$$

其中 $\varepsilon_1=\frac{\sqrt{c_2m_5(\tau)}}{\nu}$， $\varepsilon_2=\max\left\{\frac{\sigma_2^2k_1}{k_2},2\sigma_2^2\right\}$。

联合式(32)，(33)和(37)，可得

$$\frac{\mathrm{d}F(t)}{\mathrm{d}t}\le\varepsilon_3F(t)+\sigma^2\left(\frac{1}{k_2}\int_{\partial\Omega}f_1^2\mathrm{d}s+\frac{2k_1^2}{k_2}\int_{\partial\Omega}T^2\mathrm{d}s+2\int_{\Omega}T_{,i}T_{,i}\mathrm{d}x\right),\tag{38}$$

其中 $\varepsilon_3=\max\left\{\frac{T_M^2\varepsilon_2}{\varepsilon_1},\frac{\varepsilon_1}{\varepsilon_2},\varepsilon_1\right\}$。

对式(38)从 0 到 t 积分，并由 Gronwall 不等式，可得

$$F(t)\le\sigma^2\mathrm{e}^{\varepsilon_3t}\left(\frac{1}{k_2}\iint_0^t\int_0^t\int_{\partial\Omega}f_1^2\mathrm{d}s\mathrm{d}\eta+\frac{2k_1^2}{k_2}\int_0^t\int_{\partial\Omega}T^2\mathrm{d}s\mathrm{d}\eta+2\int_{\Omega}T_{,i}T_{,i}\mathrm{d}x\mathrm{d}\eta\right).\tag{39}$$

联合式(12)、式(14)和式(39)得

$$F(t)\le\sigma^2\mathrm{e}^{\varepsilon_3t}m_6(t).$$

证毕！

4. 结论

本文考虑了 Brinkman 方程组的解对 Soret 系数 σ 的连续依赖性。文中为了推导出盐浓度 C 四阶范数估计而提出的新解决办法，为以后先验估计提供新的路径。利用本文类似的方法，依然可以得到 Brinkman 方程组的解对其它系数的连续依赖性和收敛性。接下来，我们将研究在无界区域内 Brinkman 方程组的解对边界系数的结构稳定性。

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