Advances in Applied Mathematics 应用数学进展, 2017, 6(3), 417-422
Published Online May 2017 in Hans. http://www.hanspub.org/journal/aam
https://doi.org/10.12677/aam.2017.63048

Oscillation of Second-Order Semilinear Differential Equations

Xinxiao Su, Lina Dai, Simin Wu*, Quanwen Lin

Science of School, Department of Mathematics, Guangdong University of Petrochemical Technology, Maoming Guangdong
Email: *520wsmsam@163.com

Received: May 11th, 2017; accepted: May 28th, 2017; published: May 31st, 2017

Abstract

By generalized Riccati transformation, this paper mainly studies oscillation of a class of second-order semilinear equation of the form $\left(r(t)\left|\xi'(t)\right|^{\alpha-1}\xi'(t)\right)'+q(t)\left|x(\sigma(t))\right|^{\alpha-1}x(\sigma(t))=0$. A new oscillation criterion is given, and the results of some references are improved. The results are illustrated and some examples are given.

Keywords

Semilinear, Second-Order Differential Equation, Generalized Riccati Transformation

二阶半线性中立型微分方程的振动性

苏新晓，戴丽娜，伍思敏*，林全文

广东石油化工学院理学院数学系，广东 茂名
Email: *520wsmsam@163.com

收稿日期：2017年5月11日；录用日期：2017年5月28日；发布日期：2017年5月31日

摘 要

利用广义Riccati变换技巧，本文对二阶半线性中立型微分方程 $\left(r(t)\left|\xi'(t)\right|^{\alpha-1}\xi'(t)\right)'+q(t)\left|x(\sigma(t))\right|^{\alpha-1}x(\sigma(t))=0$ 做进一步的研究，给出了新的振动准则，改进了部分文献中的结果，为说明主要结果的应用给出了例子。

*通讯作者。

文章引用：苏新晓，戴丽娜，伍思敏，林全文．二阶半线性中立型微分方程的振动性[J]．应用数学进展，2017，6(3): 417-422. https://doi.org/10.12677/aam.2017.63048

关键词

半线性，二阶微分方程，广义Riccati变换

1. 引言

考虑二阶半线性中立型微分方程

$$\left(r(t)\left|\xi'(t)\right|^{\alpha-1}\xi'(t)\right)'+q(t)\left|x(\sigma(t))\right|^{\alpha-1}x(\sigma(t))=0,\quad t\ge t_0>0 \tag{E}$$

其中 $\xi(t)=x(t)+p(t)x(\tau(t)),\alpha>0,r,\sigma\in C^1([t_0,\infty),(0,\infty))$， $p,q,\tau\in C([t_0,\infty),R)$，对每一 $t\ge t_0$，都有 $\tau(t)\le t,\sigma(t)\le t,\sigma'(t)>0$，$\lim\limits_{t\to\infty}\tau(t)=\lim\limits_{t\to\infty}\sigma(t)=\infty,0\le p(t)\le 1,q(t)>0$，按照习惯，(E)的解称为振动的，如果它有任意大的零点；否则称它为非振动的。

[1]-[8]对二阶半线性时滞性微分方程

$$\left(r(t)\left(\left(x(t)+p(t)x(\tau(t))\right)'\right)^{\alpha}\right)'+q(t)x^{\alpha}(\sigma(t))=0 \tag{E_0}$$

建立了一系列的振动法则，但是，所得结果不够完善，[9]利用 Riccati 变换经典不等式 $Bx-Ax^{\frac{\alpha+1}{\alpha}}\le\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}}$，对方程($E_0$)建立了新的振动准则，它改进了[2]和[7]中的结果。

定理 A [9] 设 $\alpha\ge1$ 是两个奇数的商，$\pi(t)=\int_t^{\infty}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s,\pi(t_0)<\infty$，存在函数 $\rho,\delta\in C^1([t_0,\infty),(0,\infty))$ 满足

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\rho(s)q(s)\left(1-p(\sigma(s))\right)^{\alpha}-\frac{\left(\rho_+'(s)\right)^{\alpha+1}r(\sigma(s))}{(\alpha+1)^{\alpha+1}\left(\rho(s)\sigma'(s)\right)^{\alpha}}\right]\mathrm{d}s=\infty \tag{1.1}$$

和

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\psi(s)-\frac{\delta(s)r(s)\left(\varphi_+(s)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\right]\mathrm{d}s=\infty \tag{1.2}$$

其中 $\psi(t)=\delta(t)\left[q(t)\left(1-p(\sigma(t))\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))}\right)^{\alpha}+\frac{1-\alpha}{r^{\frac{1}{\alpha}}(t)\pi^{\alpha+1}(t)}\right]$， $p(t)<\frac{\pi(t)}{\pi(\tau(t))}$，

$$\varphi(t)=\frac{\delta'(t)}{\delta(t)}+\frac{1+\alpha}{r^{\frac{1}{\alpha}}(t)\pi(t)};\quad \rho_+'(s)=\max\{0,\rho'(s)\},\quad \varphi_+(t)=\max\{0,\varphi(t)\}.$$

则方程(E_0)振动。

我们注意到[9]中限制 $\alpha \ge 1$ 是两个奇数的商，在[9]工作的启发下，我们在本文中利用 Riccati 变换技巧，建立了方程(E)的振动准则，它改进了文献中的结果，为说明主要结果的应用给出了例子。

2. 主要结果

使用记号：对 $\rho,\sigma \in C^1\left(\left[t_0,\infty\right),\left(0,\infty\right)\right),\alpha>0$，令

$$\pi\left(t\right)=\int_t^{\infty}\left(\frac{1}{r\left(s\right)}\right)^{\frac{1}{\alpha}}\mathrm{d}s,\quad \varepsilon=\left(\frac{\alpha}{1+\alpha}\right)^{\alpha+1},\quad \rho_+'\left(t\right)=\max\left\{0,\rho'\left(t\right)\right\},$$

$$\xi\left(t\right)=x\left(t\right)+p\left(t\right)x\left(\tau\left(t\right)\right),\quad A_1\left(t\right)=q\left(t\right)\left(1-p\left(\sigma\left(t\right)\right)\right)^{\alpha},\quad A_2\left(t\right)=q\left(t\right)\left(1-p\left(\sigma\left(t\right)\right)\frac{\pi\left(\tau\left(\sigma\left(t\right)\right)\right)}{\pi\left(\sigma\left(t\right)\right)}\right)^{\alpha}$$

下面我们来建立(E)当 $\alpha>0$ 时的振动准则

定理 2.1 设 $\pi\left(t_0\right)<\infty$，若存在函数 $\rho\in C^1\left(\left[t_0,\infty\right),\left(0,\infty\right)\right)$ 满足

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\rho\left(s\right)A_1\left(s\right)-\frac{\left(\rho_+'\left(s\right)\right)^{\alpha+1}r\left(\sigma\left(s\right)\right)}{\left(\alpha+1\right)^{\alpha+1}\left(\rho\left(s\right)\sigma'\left(s\right)\right)^{\alpha}}\right]\mathrm{d}s=\infty \tag{2.1}$$

且

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\pi^{\alpha}\left(s\right)A_2\left(s\right)-\frac{\varepsilon}{\pi\left(s\right)r^{\frac{1}{\alpha}}\left(s\right)}\right]\mathrm{d}s=\infty \tag{2.2}$$

则方程(E)振动。

证：设方程有一个非振动解 $x\left(t\right)$，不失一般性，设存在 $t_1\ge t_0$，使 $x\left(t\right)>0,\ x\left(\sigma\left(t\right)\right)>0,\ x\left(\tau\left(t\right)\right)>0$，对所有 $t\ge t_1$ 都成立，此时 $\xi\left(t\right)\ge x\left(t\right)>0$，由方程(E)可得

$$\left(r\left(t\right)\left|\xi'\left(t\right)\right|^{\alpha-1}\xi'\left(t\right)\right)'\le 0 \tag{2.3}$$

即 $r\left(t\right)\left|\xi'\left(t\right)\right|^{\alpha-1}\xi'\left(t\right)$ 在 $t\ge t_1$ 上是减函数，因此 $\xi'\left(t\right)$ 是不变号的，设存在 $t_2\ge t_1$，分两种情况来展开讨论：

(i) 假设 $\xi'\left(t\right)>0$，$t\ge t_2$。

证明过程与文[7]的定理 2.1 证明类似，得到条件(2.1)，在此就不多做讨论了。

(ii) 假设 $\xi'\left(t\right)<0$，$t\ge t_2$。此时，(2.3)式变成

$$\left(r\left(t\right)\left(-\xi'\left(t\right)\right)^{\alpha}\right)'\ge 0$$

于是对于 $s\ge t$ 有 $r^{\frac{1}{\alpha}}\left(s\right)\left(-\xi'\left(s\right)\right)\ge r^{\frac{1}{\alpha}}\left(t\right)\left(-\xi'\left(s\right)\right)$，即

$$\xi'\left(s\right)\le r^{\frac{1}{\alpha}}\left(t\right)\xi'\left(t\right)\left(\frac{1}{r\left(s\right)}\right)^{\frac{1}{\alpha}} \tag{2.4}$$

在 $\left[t,l\right]$ 上对(2.4)式中的 s 进行积分，得

$$0<\xi\left(l\right)\le\xi\left(t\right)+r^{\frac{1}{\alpha}}\left(t\right)\xi'\left(t\right)\int_t^{l}\left(\frac{1}{r\left(s\right)}\right)^{\frac{1}{\alpha}}\mathrm{d}s$$

令 $l \to \infty$，得

$$\xi(t) \ge -\pi(t) r^{\frac{1}{\alpha}}(t)\xi'(t) \tag{2.5}$$

于是得到 $\left(\frac{\xi(t)}{\pi(t)}\right)' \ge 0$，则 $\xi(\tau(t)) \le \frac{\pi(\tau(t))}{\pi(t)}\xi(t)$，即有

$$\begin{aligned} x(t) &= \xi(t) - p(t)x(\tau(t)) \ge \xi(t) - p(t)\xi(\tau(t)) \\ &\ge \xi(t) - p(t)\frac{\pi(\tau(t))}{\pi(t)}\xi(t) = \left(1 - p(t)\frac{\pi(\tau(t))}{\pi(t)}\right)\xi(t) \end{aligned}$$

即

$$x^{\alpha}(\sigma(t)) \ge \left(1 - p(\sigma(t))\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))}\right)^{\alpha} \xi^{\alpha}(\sigma(t)) \ge \frac{A_2(t)}{q(t)}\xi^{\alpha}(t) \tag{2.6}$$

由方程(E)和(2.6)式得

$$\left(r(t)(-\xi'(t))^{\alpha}\right)' - A_2(t)\xi^{\alpha}(t) \ge 0$$

即

$$\left(r(t)(-\xi'(t))^{\alpha}\right)' \ge A_2(t)\xi^{\alpha}(t) \tag{2.7}$$

考虑广义 Riccati 变换

$$v(t) = \frac{r(t)(-\xi'(t))^{\alpha}}{\xi^{\alpha}(t)} > 0, \quad t \ge t_2 \tag{2.8}$$

(2.8)式对 t 进行求导，并由(2.7)知

$$\begin{aligned} v'(t) &= \frac{\left(r(t)(-\xi'(t))^{\alpha}\right)'}{\xi^{\alpha}(t)} - \frac{\alpha\left(r(t)(-\xi'(t))^{\alpha}\right)\xi'(t)}{\xi^{\alpha+1}(t)} \\ &\ge A_2(t) + \frac{\alpha r(t)(-\xi'(t))^{\alpha+1}}{\xi^{\alpha+1}(t)} = A_2(t) + \frac{\alpha}{r^{\frac{1}{\alpha}}(t)} v^{\frac{\alpha+1}{\alpha}}(t) \end{aligned} \tag{2.9}$$

又由(2.5)得：$\xi(t) \ge \pi(t) r^{\frac{1}{\alpha}}(t)(-\xi'(t))$，所以

$$0 < \frac{\pi^{\alpha}(t) r(t)(-\xi'(t))^{\alpha}}{\xi^{\alpha}(t)} = \pi^{\alpha}(t)v(t) \le 1$$

以 $\pi^{\alpha}(t)$ 乘(2.9)式两边，从 t_2 到 t 关于 s 积分，有

$$\begin{aligned} &\int_{t_2}^{t} \pi^{\alpha}(s)A_2(s)\mathrm{d}s \\ &\le \int_{t_2}^{t} \pi^{\alpha}(s)v'(s)\mathrm{d}s - \int_{t_2}^{t} \alpha r^{-\frac{1}{\alpha}}(s)\pi^{\alpha}(s)v^{\frac{\alpha+1}{\alpha}}(s)\mathrm{d}s \\ &\le \pi^{\alpha}(t)v(t) - \pi^{\alpha}(t_2)v(t_2) + \left(\int_{t_2}^{t} \alpha\pi^{\alpha-1}(s) r^{-\frac{1}{\alpha}}(s)v(s)\mathrm{d}s - \int_{t_2}^{t} \alpha r^{-\frac{1}{\alpha}}(s)\pi^{\alpha}(s)v^{\frac{\alpha+1}{\alpha}}(s)\mathrm{d}s\right) \end{aligned}$$

利用 $By-Ay^{\frac{\alpha+1}{\alpha}}\le\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}}$，上不等式变为

$$\int_{t_2}^{t}\pi^{\alpha}(s)A_2(s)\mathrm{d}s\le 1+\int_{t_2}^{t}\alpha\pi^{\alpha-1}(s)r^{-\frac{1}{\alpha}}(s)\left(v(s)-\pi(s)v^{\frac{\alpha+1}{\alpha}}(s)\right)\mathrm{d}s$$

$$\le 1+\int_{t_2}^{t}\left(\frac{\alpha}{1+\alpha}\right)^{\alpha+1}\frac{1}{\pi(s)r^{\frac{1}{\alpha}}(s)}\mathrm{d}s$$

即

$$\int_{t_2}^{t}\left[\pi^{\alpha}(s)A_2(s)-\frac{\varepsilon}{\pi(s)r^{\frac{1}{\alpha}}(s)}\right]\mathrm{d}s\le 1 \tag{2.10}$$

显然(2.10)式与条件(2.2)矛盾，因此，方程(E)振动。

3. 应用

例：考虑二阶微分方程

$$\left(t^4\left[\left(x(t)+\frac{1}{9}x\left(\frac{t}{3}\right)\right)'\right]^2\right)'+ktx^2\left(\frac{t}{3}\right)=0,t\ge t_0>0,k>\frac{2}{3} \tag{3.1}$$

其中，我们取 $r(t)=t^4,\ p(t)=\frac{1}{9},q(t)=kt,\tau(t)=\frac{t}{3},\sigma(t)=\frac{t}{3},\rho(t)=1$，

显然，$A_1(t)=kt\times\left(1-\frac{1}{9}\right)^2=\frac{64k}{81}t$，$\pi(t)=\int_t^{\infty}\left(\frac{1}{t^4}\right)^{\frac{1}{2}}\mathrm{d}s=\frac{1}{t}$，

$$A_2(t)=kt\left(1-\frac{1}{9}\times\frac{9}{t}\times\frac{t}{3}\right)^2=\frac{4}{9}kt,$$

于是，

$$\int_{t_0}^{t}\frac{64}{81}ks\mathrm{d}s=\frac{64}{162}k\left(t^2-t_0^2\right),\quad\int_{t_0}^{t}\left[\frac{1}{s^2}\times\frac{4}{9}ks-\frac{8}{27}\frac{1}{s}\right]\mathrm{d}s=\frac{(12k-8)}{27}\left(\ln(t)-\ln(t_0)\right)$$

显然当 $t\to\infty$ 时，方程满足条件(2.1)和(2.2)，故由定理 2.1 知方程(3.1)振动。

基金项目

国家自然科学基金(11271380)、茂名市科技局软科学项目(20140340；2015038)。

参考文献 (References)

[1] Baulikova, B. and Dzurina, J. (2012) Oscillation Theorems for Higher Neutral Differential Equations. *Appl. Math. Comput.*, 3769-3778.

[2] Han, Z., Li, T., Sun, S. and Sun, Y. (2010) Remarks on the Paper [Appl.Math.Comput.207(2009)388-396]. *Appl. Math. Comput.*, **215**, 3998-4007.

[3] Hasanbulli, M. and Yu, V. (2010) Rogovchenko, Oscillation Criteria for Second Order Nonlinear Neutral Differential Equations. *Appl. Math. Comput.*, **215**, 4392-4399.

[4] Jiang, J. and Li, X. (2003) Oscillation of Second Order Nonlinear Neutral Differential Equations. *Appl. Math. Comput.*,

135, 531-540.

[5] Li, T., Agarwal, R.P. and Bohner, M. (2012) Some Oscillation Results for Second-Order Neutral Differential Equations. *J. Indian Math. Soc.*, **79**, 97-106.

[6] Li, T., Yu, V., Rogovechko, and Zhang, C. (2013) Oscillation of Second-Order Neutral Differential Equations. *Funkcialaj Ekvacioj*, **56**, 111-120. https://doi.org/10.1619/fesi.56.111

[7] Ye, L. and Xu, Z. (2009) Oscillation Criteria for Second Order Quasilinear Neutral Delay Differential Equations. *Appl. Math. Comput.*, **207**, 388-396.

[8] Zhang, S. and Wang, Q. (2010) Oscillation of Second-Order Nonlinear Neutral Dynamic Equations on Time Scales. *Appl. Math. Comput.*, **216**, 2837-2848.

[9] Agarwal, R.P., Zhang, C. and Li, T. (2016) Some Remarks on Oscillation of Second Order Neutral Differential Equations. *Appl. Math. Comput.*, **274**, 178-181.