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Dynamic Analysis of a Dissolved Oxygen-Plankton Model with Two Time Delays

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Abstract

Based on the internal dynamic characteristics of marine ecosystem and the dynamic mechanism of marine biological population, this paper constructed a dissolved oxygen-plankton dynamic model involving two time delays, and made theoretical analysis of corresponding dynamics. We obtained the locally asymptotic stability of positive equilibrium and the threshold conditions of occurring Hopf bifurcation were gained. Therefore, we analyzed the dynamics of Hopf bifurcation in detail. This result provides a great help for the interaction between dissolved oxygen and marine plankton in dynamics, and is helpful to deeply understand how the delay affects the dynamic trend of the marine ecosystem, and furthermore provide certain theoretical support for the study of dynamic growth mechanism of mutual restriction and mutual coordination in marine plankton.

Keywords

Dissolved Oxygen, Plankton, Two Time Delays, Hopf Bifurcation, Stability

一类具有双时滞效应的溶解氧-浮游生物模型的动力学分析

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摘　要

基于海洋生态系统内在动态变化特性和海洋生物种群之间的动态作用机制，本论文构建了一类具有双时滞效应的溶解氧-浮游生物动态模型，并对其相关动力学性质进行理论分析，解析出模型内平衡点具有局部渐近稳定性和发生Hopf分支的阈值条件，并详细探析了Hopf分支的相关动力学性质。本研究成果有利于从动力学的角度揭示溶解氧与海洋浮游生物之间的相互作用机制，有助于深入理解时滞效应如何影响海洋生态系统的动态运行趋势，为进一步研究海洋浮游生物之间相互制约、相互协调的生长动态机制提供一定的理论支撑。

关键词

溶解氧，浮游生物，时滞，Hopf分支，稳定性

1. 引言

海洋浮游生物是指悬浮在水层中常随水流移动的海洋生物，包括海洋浮游植物和海洋浮游动物，其特点是这类生物缺乏发达的运动器官，没有或仅有微弱的游动能力和多数分布于水体的上层或表层。海洋浮游植物是指在海水中营浮游生活的微小植物，主要包括单细胞藻类和细菌，其营养方式为自养方式，其中，藻类具有叶绿素或其它色素，能进行光合作用制造有机物，是水域生态系统中的生产者，细菌是生态系统的还原者(也可以是生产者)，它们一般分布于海洋的真光层。海洋浮游动物一类经常在海水中浮游，本身不能制造有机物的异养型无脊椎动物和脊索动物幼体的总称，它们必须依赖已有的有机物为营养来源，多为滤食性。同时，浮游动物是海洋生态系统中的消费者，可分布于真光层，也可分布于较深的水层。

众所周知，海洋浮游植物是海洋生态系统中重要的生产者，是食物链的基础环节，在海洋生态系统的物质循环和能量转化过程中起着重要作用。为海洋中的生命活动提供了能量，有助于勘探海底石油，有助于研究海洋古地质和古环境，有助于研究、防治海洋环境污染。其生长繁殖除主要受到自身生物学特性影响外，还受到周围环境因素如海流、扰动、温度、盐度、营养盐和其他生物因素的影响。海洋浮游动物是海洋生态系统中非常重要的一大生态类群，种类组成繁殖、数量大、分布广，有着极其重要的生态学意义。浮游动物通过捕食作用控制浮游植物的数量，同时作为鱼类等高层营养者的饵料，其数量变化可以直接影响鱼类等的资源量，在海洋生态系统的结构和功能中起着重要的调控作用。同时，海洋浮游动物既是初级生产的消费者，也是次级生产者，在浮游植物和游泳动物食物链之间扮演重要角色，在海洋生态系统动态变化中起着重要的调控作用，在海洋环境保护研究中有重要意义。因此研究海洋浮游生物生长动态机制与相互制约、相互协调生长动态机制具有至关重要的现实意义。就目前来说，应用数学动态模型与数值模拟方法来研究此类问题是具有一定的优越性。因而数学模型已成为一些学者研究浮游生物动态生长及其变化趋势的常用工具之一[1]-[7]。例如，论文[8]建立了一类 N-P-Z 数学动态模型，讨论了系统的边界性和 Hopf 分支存在性等问题，给出了系统具有此类特性的阈值条件及其生物学意义。

溶解在海水中的氧(溶解氧)是海洋生命活动不可缺少的物质，它的含量在海洋中的分布，既受化学过

程和生物过程的影响，还受物理过程的影响。最为重要的是海水中溶解氧是海洋水生动物、植物、微生物等进行新陈代谢的必要物质。随着研究的深入，一些学者发现溶解氧对藻类的爆发有着举足轻重的影响[9]-[17]。论文[14]建立了一类溶解氧-浮游生物动力学模型，研究了系统平衡点的存在性与稳定性，探讨了溶解氧如何影响浮游生物种群数量的动态变化趋势。论文[15]构建了一类数学动态模型去预测水体中溶解氧耗散动态变化情况，获得一些比较有意义的结果。论文[18]提出了一类数学动态模型来探析溶解氧如何影响浮游生物动力学性质，研究了模型平衡点的存在性与稳定性，探讨了系统分支的存在性及其相关动力学性质，其动态模型如下：

$$\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}t} &= \frac{rP}{(\xi + D_0 - D)} - \delta_1 P - \frac{\beta PZ}{(P+a)}, \\
\frac{\mathrm{d}D}{\mathrm{d}t} &= \eta(D_0 - D) - \alpha_1 P - \alpha_2 Z, \\
\frac{\mathrm{d}Z}{\mathrm{d}t} &= \frac{\beta_1 PZ}{(P+a)} - \delta_2 Z.
\end{aligned} \tag{1}$$

考虑到生物学意义，所有的参数都为正，P, Z 分别代表浮游植物和浮游动物的种群数量，D 表示在任意时刻 t 的溶解氧的含量。此外 D_0 表示溶解氧在水体中的吸收量。浮游植物的内禀增长率随着 D 的增加而增加，且假设在 $D = D_0$ 时有最大的增长率 r/ξ。$r/(\xi + D_0 - D)$ 视为浮游植物吸收溶解氧的增长方程。δ_1 和 δ_2 分别表示浮游植物和浮游动物的死亡率，其中包括水体的自然冲刷。β 表示浮游植物被浮游动物觅食的比率，β_1 是浮游植物在浮游动物体内的转换率。η 表示水体中溶解氧的补给率。α_1, α_2 分别表示每单位被浮游植物和浮游动物消耗掉的溶解氧。

然而在现实海洋生态系统中，浮游生物之间的相互作用并不是瞬时的，而是存在一些时滞效应，这些时滞可能是由于浮游动物的妊娠期或者浮游生物对食物的消化等其它因素引起的，因此将时滞引入到生态系统中可以更加合理地描述与揭示现实生态系统动态运行规律。大量的研究表明时滞系统比常规系统有更加丰富的动力学性质[19]-[25]。论文[26]研究了一个包含由妊娠期引起的时滞生态动力学模型，分别在六种情况下讨论了时滞参数如何影响系统的动力学性质和解析出相关的临界条件。论文[27]构建了一类具有双时滞效应的浮游生态动力学模型，其时滞效应是由浮游植物产生毒素和浮游动物消化食物所引起的。同时，作者将随机因素引入该系统中，探讨了随机稳定性和随机 Hopf 分支的存在性，并利用李雅普诺夫函数证明了系统含有双时滞情况下内平衡点的渐近稳定性。

基于论文[18]中的相关结果，本论文将浮游动物消化浮游植物以及浮游生物吸收溶解氧所引起的时滞效应考虑在内，建立了一类具有双时滞效应的海洋生态动力学新模型，其可以表示如下：

$$\begin{aligned}
\frac{\mathrm{d}D}{\mathrm{d}t} &= \eta(D_0 - D) - \alpha_1 P - \alpha_2 Z, \\
\frac{\mathrm{d}P}{\mathrm{d}t} &= \frac{rP}{(\xi + D_0 - D(t-\tau_1))} - \delta_1 P - \frac{\beta PZ}{(P+a)}, \\
\frac{\mathrm{d}Z}{\mathrm{d}t} &= \frac{\beta_1 P(t-\tau_2) Z}{(P(t-\tau_2)+a)} - \delta_2 Z.
\end{aligned} \tag{2}$$

其中 τ_1 是浮游植物吸收溶解氧引起的时滞项，τ_2 是由于浮游动物消化食物所引起的时滞项。本论文的主要工作是解析时滞参数如何影响该生态系统的动力学行为及其给出相关临界条件，为后续研究提供理论基础。

2. 局部稳定性和 Hopf 分支

现在主要讨论系统(2)内平衡点的稳定性和 Hopf 分支的存在性。从论文[18]中得知：当 $\beta_1 > \delta_2$ 时系统(2)存在一个内平衡点，记为 $E^* = (D^*, P^*, Z^*)$，其中：

$$\begin{aligned}
&D^* = D_0 - \frac{a\alpha_1\delta_2}{\eta(\beta_1-\delta_2)} - \frac{\alpha_2 Z^*}{\eta},\\
&P^* = \frac{a\delta_2}{\beta_1-\delta_2},\\
&Z^* = \frac{a\beta_1}{\beta(\beta_1-\delta_2)}\left(\frac{r}{\xi+D_0-D^*}-\delta_1\right).
\end{aligned} \tag{3}$$

首先对系统(2)进行换元变形：$u_1(t)=D(t)-D^*$，$u_2(t)=P(t)-P^*$，$u_3(t)=Z(t)-Z^*$。将内平衡点 E^* 转化到原点，系统(2)可以重新写为

$$\begin{aligned}
&\dot{u}_1 = a_{11}u_1 + a_{12}u_2 + a_{13}u_3 + F_1,\\
&\dot{u}_2 = a_{21}u_1(t-\tau_1) + a_{22}u_2 + a_{23}u_3 + F_2,\\
&\dot{u}_3 = a_{32}u_2(t-\tau_2) + F_3,
\end{aligned} \tag{4}$$

其中：

$$a_{11} = -\eta < 0;\ a_{12} = -\alpha_1 < 0;\ a_{13} = -\alpha_2 < 0;\ a_{21} = \frac{rP^*}{(\xi+D_0-D^*)^2} > 0;$$

$$a_{22} = \frac{r}{\xi+D_0-D^*} - \delta_1 - \frac{\beta a Z^*}{(a+P^*)^2} > 0;\ a_{23} = -\frac{\beta P^*}{a+P^*} < 0;\ a_{32} = \frac{\beta_1 Z^* a}{(a+P^*)^2}.$$

$$\begin{aligned}
&f_1 = 0,\\
&f_2 = \left(-\frac{a\beta}{(a+P^*)^2}\right)y(t)z(t) + \frac{a\beta Z^*}{(a+P^*)^3}y^2(t) - \frac{a\beta Z^*}{(a+P^*)^4}y^3(t) + \cdots,\\
&f_3 = \frac{a\beta_1}{(a+P^*)^2}y(t-\tau_2)z(t) - \frac{a\beta_1 Z^*}{(a+P^*)^3}y^2(t-\tau_2) + \frac{a\beta_1 Z^*}{(a+P^*)^4}y^3(t-\tau_2) + \cdots.
\end{aligned}$$

则系统(4)对应的线性部分为：

$$\begin{aligned}
&\dot{u}_1 = a_{11}u_1 + a_{12}u_2 + a_{13},\\
&\dot{u}_2 = a_{21}u_1(t-\tau_1) + a_{22}u_2 + a_{23}u_3,\\
&\dot{u}_3 = a_{32}u_2(t-\tau_2),
\end{aligned} \tag{5}$$

系统(5)对应的特征方程如下：

$$\lambda^3 + A\lambda^2 + B\lambda + C\lambda\cdot e^{-\lambda\tau_1} + (D\lambda+E)\cdot e^{-\lambda\tau_2} + F\cdot e^{-\lambda(\tau_1+\tau_2)} = 0. \tag{6}$$

其中：

$$\begin{aligned}
&A = -(a_{11}+a_{22}),\ B = a_{11}a_{22},\ C = -a_{12}a_{21},\\
&D = -a_{32}a_{23},\ E = a_{11}a_{23}a_{32},\ F = -a_{13}a_{21}a_{32}.
\end{aligned}$$

第一种情况： $\tau_1 = \tau_2 = 0$。

定理 2.1： $\tau_1=\tau_2=0$ **时，满足条件** (H_1) **时，内平衡点** E^* **是局部稳定的。**

证明：当 $\tau_1=\tau_2=0$ 时，方程(6)变为

$$\lambda^3 + A_{11}\lambda^2 + A_{12}\lambda + A_{13} = 0, \tag{7}$$

其中：

$$A_{11}=A,\ A_{12}=B+C+D,\ A_{13}=E+F.$$

根据 Routh-Hurwitz 判据，满足条件(H_1)：$A_{11}>0$，$A_{11}A_{12}>A_{13}$时，E^*是局部渐近稳定的。

第二种情况： $\tau_1>0,\tau_2=0$。

定理 2.2：$\tau_1>0,\tau_2=0$时，满足条件(H_{21},H_{22})，系统内平衡点E^*在$\tau_1\in[0,\tau_{10})$是局部渐近稳定的，并且当$\tau_1=\tau_{10}$时出现 Hopf 分支。

证明：当$\tau_1>0,\tau_2=0$时，方程(6)化为

$$\lambda^3+A\lambda^2+(B+D)\lambda+(C\lambda+F)\cdot \mathrm{e}^{-\lambda\tau_1}+E=0. \tag{8}$$

假设$\lambda=i\omega_1\ (\omega_1>0)$是方程(8)的纯虚根，将其代入方程(8)，得到如下方程组

$$\begin{aligned}&-\omega_1^3+(B+D)\omega_1-F\sin(\omega_1\tau_1)+C\omega_1\cos(\omega_1\tau_1)=0,\\&-A\omega_1^2+E+C\omega_1\sin(\omega_1\tau_1)+F\cos(\omega_1\tau_1)=0.\end{aligned} \tag{9}$$

将两个方程平方并且相加得到方程(10)：

$$\omega_1^6+\left[A^2-2(B+D)\right]\omega_1^4+\left[(B+D)^2-2AE-C^2\right]\omega_1^2+E^2-F^2=0, \tag{10}$$

假设$v_1=\omega_1^2$，然后方程(10)变为

$$v_1^3+\left[A^2-2(B+D)\right]v_1^2+\left[(B+D)^2-2AE-C^2\right]v_1+E^2-F^2=0, \tag{11}$$

接下来定义一个方程：

$$f_1(v_1)=v_1^3+\left[A^2-2(B+D)\right]v_1^2+\left[(B+D)^2-2AE-C^2\right]v_1+E^2-F^2 \tag{12}$$

假设(H_{21})：文献[28]中定理(2.1)情况(i)，(iii)成立，则$f_1(v_1)$至少有一个正根，不失一般性，假设方程(11)的根为v_{11},v_{12},v_{13}，则有$\omega_{1k}=\sqrt{v_{1k}},(k=1,2,3)$根据方程组(9)得到：

$$\tau_{1k}^j=\frac{1}{\omega_{1k}}\left(\arccos\frac{C\omega_{1k}^4+\left[AF-(B+D)C\right]\omega_{1k}^2-EF}{C^2\omega_{1k}^2+F^2}+2j\pi\right),\ (j=0,1,2\cdots,k=1,2,3)$$

假设$\tau_{10}=\min_{k=1,2,3}\tau_{1k}^{(0)}$，由 Hopf 分支定理[29]得知，还需要证明横衔性条件，对方程(8)两边关于τ_1微分求导得

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_1}\right)^{-1}=\frac{3\lambda^2+2A\lambda+B+D+C\mathrm{e}^{-\lambda\tau_1}}{\lambda(C\lambda+F)\mathrm{e}^{-\lambda\tau_1}}-\frac{\tau_1}{\lambda}, \tag{13}$$

将$\lambda=i\omega_{10}$代入(13)，通过简单计算得到

$$\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_1}\right)^{-1}_{\lambda=\mathrm{i}\omega_{10}}=\frac{\omega_{10}^2}{\Delta}f_1'(\omega_{10}^2), \tag{14}$$

其中：

$$\Delta=\left(F\omega_{10}\sin(\omega_{10}\tau_1)-C\omega_{10}^2\cos(\omega_{10}\tau_1)\right)^2+\left(F\omega_{10}\cos(\omega_{10}\tau_1)+C\omega_{10}^2\sin(\omega_{10}\tau_1)\right)^2,$$

得知当$(H_{22}):f_1'(\omega_{10}^2)\neq0$成立时，则$\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_1}\right)^{-1}_{\lambda=\mathrm{i}\omega_{10}}\neq0$。证毕。

第三种情况： $\tau_2>0,\tau_1=0$。

定理 2.3： $\tau_2>0,\tau_1=0$**时，满足**(H_{21},H_{32})**，系统内平衡点**E^***在**$\tau_2\in[0,\tau_{20})$**是局部渐近稳定的，并且当**$\tau_2=\tau_{20}$**时出现 Hopf 分支。**

证明：当$\tau_2>0,\tau_1=0$时，方程(6)化为

$$\lambda^3+A\lambda^2+(B+C)\lambda+D\lambda\cdot e^{-\lambda\tau_2}+(E+F)\cdot e^{-\lambda\tau_2}=0. \tag{15}$$

与第二种情况相似，假设$\lambda=i\omega_2\,(\omega_2>0)$是方程(15)的根，将其代入得到：

$$\begin{aligned}&-\omega_2^3+(B+C)\omega_2-(E+F)\sin(\omega_2\tau_2)+D\omega_2\cos(\omega_2\tau_2)=0,\\&-A\omega_2^2+(E+F)\omega_2\cos(\omega_2\tau_2)+D\sin(\omega_2\tau_2)=0.\end{aligned} \tag{16}$$

对方程组平方并且相加得到

$$\omega_2^6+\left(A^2-2(B+C)\right)\omega_2^4+\left[(B+C)^2-D^2\right]\omega_2^2-(E+F)^2=0, \tag{17}$$

假设$v_2=\omega_2^2$，则方程(17)化为

$$v_2^3+\left(A^2-2(B+C)\right)v_2^2+\left[(B+D)^2-D^2\right]v_2-(E+F)^2=0, \tag{18}$$

然后定义

$$f_2(v_2)=v_2^3+\left(A^2-2(B+C)\right)v_2^2+\left[(B+D)^2-D^2\right]v_2-(E+F)^2 \tag{19}$$

假设(H_{21})成立，则方程(18)至少有一个正根，不失一般性，将方程(18)的正根记为v_{21},v_{22},v_{23}，然后有$\omega_{2k}=\sqrt{v_{2k}},(k=1,2,3)$。根据方程(16)得到：

$$\tau_{2k}^j=\frac{1}{\omega_{2k}}\left(\arccos\frac{D\omega_{2k}^4+\left[A(E+F)-(B+C)D\right]\omega_{2k}^2}{D^2\omega_{2k}^2+(E+F)^2}+2j\pi\right),(j=0,1,2\cdots,k=1,2,3)$$

假设$\tau_{20}=\min_{k=1,2,3}\tau_{2k}^{(0)}$与第二种情况相似，还需要证明横衔性条件，对方程(15)两边关于τ_2微分求导得

$$\left(\frac{d\lambda}{d\tau_2}\right)^{-1}=\frac{3\lambda^2+2A\lambda+B+C+De^{-\lambda\tau_2}}{\lambda(D\lambda+E+F)e^{-\lambda\tau_2}}-\frac{\tau_2}{\lambda}, \tag{20}$$

将$\lambda=i\omega_{20}$代入(20)，通过简单计算得到

$$\operatorname{Re}\left(\frac{d\lambda}{d\tau_2}\right)^{-1}_{\lambda=i\omega_{20}}=\frac{\omega_{20}^2}{\Delta}f_2'(\omega_{20}^2) \tag{21}$$

其中：

$$\Delta=\left(-D\omega_{20}^2\cos(\omega_{20}\tau_2)+(E+F)\omega_{20}\sin(\omega_{20}\tau_2)\right)^2+\left((E+F)\omega_{20}\cos(\omega_{20}\tau_2)+D\omega_{20}^2(\sin\omega_{20}\tau_2)\right)^2,$$

得知当$(H_{32}):f_2'(\omega_{20}^2)\neq0$，则$\operatorname{Re}\left(\frac{d\lambda}{d\tau_2}\right)^{-1}_{\lambda=i\omega_{20}}\neq0$。证毕。

第四种情况： $\tau_1=\tau_2=\tau$。

定理 2.4： $\tau_1=\tau_2=\tau$**时，满足**(H_{41},H_{42})**，系统内平衡点**E^***在**$\tau\in[0,\tau_{30})$**是局部渐近稳定的，并且当**$\tau=\tau_{30}$**时出现 Hopf 分支。**

证明：当 $\tau_1=\tau_2=\tau$ 时，方程(6)化为

$$\lambda^3+A\lambda^2+B\lambda+\left[(C+D)\lambda+E\right]\cdot e^{-\lambda\tau}+F\cdot e^{-2\lambda\tau}=0. \tag{22}$$

方程两边同时乘以 $e^{\lambda\tau}$，得到

$$\left(\lambda^3+A\lambda^2+B\lambda\right)\cdot e^{\lambda\tau}+(C+D)\lambda+E+F\cdot e^{-\lambda\tau}=0. \tag{23}$$

假设 $\lambda=i\omega_3\left(\omega_3>0\right)$ 是方程(23)的根，将其代入得到

$$\begin{aligned}&\sin\left(\omega_3\tau\right)\left(B\omega_3-\omega_3^3\right)+\cos\left(\omega_3\tau\right)\left(A\omega_3^2-F\right)=E,\\&\sin\left(\omega_3\tau\right)\left(A\omega_3^2+F\right)+\cos\left(\omega_3\tau\right)\left(\omega_3^3-B\omega_3\right)=(C+D)\omega_3.\end{aligned} \tag{24}$$

根据方程(24)，得到如下方程组：

$$\begin{aligned}\sin\left(\omega_3\tau\right)&=\frac{E\left(\omega_3^3-B\omega_3\right)-(C+D)\left(A\omega_3^2-F\right)\omega_3}{-\left(\omega_3^3-B\omega_3\right)^2+\left(F-A^2\omega_3^4\right)},\\\cos\left(\omega_3\tau\right)&=\frac{(C+D)\left(B\omega_3-\omega_3^3\right)\omega_3-E\left(A\omega_3^2+F\right)}{-\left(\omega_3^3-B\omega_3\right)^2+\left(F-A^2\omega_3^4\right)}.\end{aligned} \tag{25}$$

利用 $\sin^2\left(\omega_3\tau\right)+\cos^2\left(\omega_3\tau\right)=1$，就有

$$\omega_3^{12}+E_{41}\omega_3^{10}+E_{42}\omega_3^8+E_{43}\omega_3^6+E_{44}\omega_3^4+E_{45}\omega_3^2+E_{46}=0, \tag{26}$$

其中：

$$\begin{aligned}&E_{41}=2A^2-4B;\ E_{42}=6B^2+A^4-4A^2B-(C+D)^2;\\&E_{43}=-4B^3-2F^2+2A^2B^2-(C+D)^2\left(A^2-2B\right)-E^2;\\&E_{44}=B^4-2AF^2+4BF^2-4EF(C+D)-(C+D)^2\left(B^2-2AF\right)+E^2\left(A^2+2B\right);\\&E_{45}=-2B^2F^2+4BEF(C+D)-(C+D)^2F^2-E^2\left(B^2+2AF\right);\\&E_{46}=F^4+E^2F^2;\end{aligned}$$

假设 $v_3=\omega_3^2$，则方程(26)变为

$$v_3^6+E_{41}v_3^5+E_{42}v_3^4+E_{43}v_3^3+E_{44}v_3^2+E_{45}v_3+E_{46}=0, \tag{27}$$

接下来定义

$$f_3\left(v_3\right)=v_3^6+E_{41}v_3^5+E_{42}v_3^4+E_{43}v_3^3+E_{44}v_3^2+E_{45}v_3+E_{46}, \tag{28}$$

假设 $\left(H_{41}\right)$：方程(28)至少有一个正根成立。不失一般性，记方程(27)的正根为 $v_{31},v_{32},\cdots,v_{36}$，则有 $\omega_{3k}=\sqrt{v_{3k}},(k=1,2,\cdots,6)$，由方程(24)可以得到

$$\tau_{3k}^j=\frac{1}{\omega_{3k}}\left(\arccos\frac{(C+D)\left(B\omega_3-\omega_3^3\right)\omega_3-E\left(A\omega_3^2+F\right)}{-\left(\omega_3^3-B\omega_3\right)^2+\left(F-A^2\omega_3^4\right)}+2j\pi\right),(j=0,1,2\cdots,k=1,2,\cdots,6)$$

假设 $\tau_{30}=\min_{k=1,2,\cdots,6}\tau_{3k}^{(0)}$，然后对方程(23)两边关于 τ 微分求导得

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}=\frac{\left(3\lambda^2+2A\lambda+B\right)e^{\lambda\tau}+C+D}{F\lambda e^{-\lambda\tau}-\left(\lambda^3+A\lambda^2+B\lambda\right)\lambda e^{\lambda\tau}}-\frac{\tau}{\lambda}, \tag{29}$$

将 $\lambda=i\omega_{30}$ 代入(29)，通过简单计算得到

$$\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}_{\lambda=\mathrm{i}\omega_{30}}=\frac{F_{41}F_{43}+F_{42}F_{44}}{F_{43}^2+F_{44}^2},$$

其中：

$$\begin{aligned}
F_{41}&=-3\omega_{30}^2\cos(\omega_{30}\tau)-2A\omega_{30}\sin(\omega_{30}\tau)+B\cos(\omega_{30}\tau)+C+D;\\
F_{42}&=-3\omega_{30}^2\sin(\omega_{30}\tau)+2A\omega_{30}\cos(\omega_{30}\tau)+B\sin(\omega_{30}\tau);\\
F_{43}&=F\omega_{30}\sin(\omega_{30}\tau)-\omega_{30}^4\cos(\omega_{30}\tau)+B\omega_{30}^2\cos(\omega_{30}\tau)-A\omega_{30}^3\sin(\omega_{30}\tau);\\
F_{44}&=F\omega_{30}\cos(\omega_{30}\tau)-\omega_{30}^4\sin(\omega_{30}\tau)+B\omega_{30}^2\sin(\omega_{30}\tau)+A\omega_{30}^3\cos(\omega_{30}\tau).
\end{aligned}$$

得知当$(H_{42}):F_{41}F_{43}+F_{42}F_{44}\neq 0$，则$\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}_{\lambda=\mathrm{i}\omega_{30}}\neq 0$。证毕。

第五种情况： $\tau_1\in(0,\tau_{10}),\tau_2>0$，

定理 2.5: $\tau_1\in(0,\tau_{10}),\tau_2>0$ **时，满足** (H_{51},H_{52})**，系统内平衡点** E^* **在** $\tau_2\in[0,\tau_{20}^*)$ **是局部渐近稳定的，并且当** $\tau=\tau_{20}^*$ **时出现 Hopf 分支。**

证明：当$\tau_1\in(0,\tau_{10}),\tau_2>0$时，将$\tau_2$作为参数，假设$\lambda=\mathrm{i}\omega_2^*(\omega_2^*>0)$是方程(6)的根，将其代入得到

$$\begin{aligned}
R_{51}\sin(\omega_2^*\tau_2)+R_{52}\cos(\omega_2^*\tau_2)&=R_{53},\\
R_{51}\cos(\omega_2^*\tau_2)-R_{52}\sin(\omega_2^*\tau_2)&=R_{54}.
\end{aligned}\tag{30}$$

其中：

$$\begin{aligned}
&R_{51}=D\omega_2^*-F\sin(\omega_2^*\tau_1);\ R_{52}=E+F\cos(\omega_2^*\tau_1);\\
&R_{53}=A\omega_2^{*2}-C\omega_2^*\sin(\omega_2^*\tau_1);\ R_{54}=\omega_2^{*3}-B\omega_2^*-C\omega_2^*\cos(\omega_2^*\tau_1);
\end{aligned}$$

对方程组(30)平方相加得

$$F_1(\omega_2^*)+F_2(\omega_2^*)\sin(\omega_2^*\tau_1)+F_3(\omega_2^*)\cos(\omega_2^*\tau_1)=0,\tag{31}$$

其中：

$$\begin{aligned}
F_1(\omega_2^*)&=\omega_2^{*6}+(A^2-2B)\omega_2^{*4}+(C^2+B^2-D^2)\omega_2^{*2}-(E^2+F^2);\\
F_2(\omega_2^*)&=2DF\omega_2^*-2AC\omega_2^{*3};\\
F_3(\omega_2^*)&=2BC\omega_2^{*2}-2C\omega_2^{*4}-2EF;
\end{aligned}$$

假设(H_{51})：(31)存在有限个正根成立，且记为$\omega_{2k}^*,(k=1,2,\cdots,l_1)$。由方程(30)有

$$\tau_{2k}^{*(j)}=\frac{1}{\omega_{2k}^*}\left(\arccos\frac{R_{52}R_{53}+R_{51}R_{54}}{R_{51}^2+R_{52}^2}+2j\pi\right),\ (j=0,1,2\cdots,k=1,2,\cdots,l_1)$$

假设$\tau_{20}^*=\min\tau_{2k}^{*(0)}(k=1,2,\cdots,l_1)$，然后对方程(6)两边关于$\tau_2$微分求导得

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_2}\right)^{-1}=\frac{f_{20}+f_{21}\mathrm{e}^{-\lambda\tau_1}+f_{22}\mathrm{e}^{-\lambda\tau_2}+f_{23}\mathrm{e}^{-\lambda(\tau_1+\tau_2)}}{f_{24}\mathrm{e}^{-\lambda\tau_2}+f_{25}\mathrm{e}^{-\lambda(\tau_1+\tau_2)}}-\frac{\tau_2}{\lambda},\tag{32}$$

其中：

$$\begin{aligned}
&f_{20}=3\lambda^2+2A\lambda+B;\ f_{21}=C-C\lambda\tau_1;\ f_{22}=D;\\
&f_{23}=-\tau_1F;\ f_{24}=(D\lambda+E)\lambda;\ f_{25}=F\lambda;
\end{aligned}$$

将$\lambda=i\omega_{20}^*,\tau_2=\tau_{20}^*$代入(32)得

$$\operatorname{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_2}\right)^{-1}_{\lambda=\mathrm{i}\omega_{20}^*}=\frac{F_{51}F_{53}+F_{52}F_{54}}{F_{53}^2+F_{54}^2}$$

其中：

$$\begin{aligned}
F_{51}&=-3\omega_{20}^{*2}+B+C\cos\left(\omega_{20}^*\tau_1\right)-C\omega_{20}^*\tau_1\sin\left(\omega_{20}^*\tau_1\right)+D\cos\left(\omega_{20}^*\tau_{20}^*\right)\\
&\quad-\tau_1F\cos\left(\omega_{20}^*\tau_1\right))\cos\left(\omega_{20}^*\tau_{20}^*\right)+\tau_1F\sin\left(\omega_{20}^*\tau_1\right)\sin\left(\omega_{20}^*\tau_{20}^*\right);\\
F_{52}&=2A\omega_{20}^*-C\sin\left(\omega_{20}^*\tau_1\right)-C\omega_{20}^*\tau_1\cos\left(\omega_{20}^*\tau_1\right)-D\sin\left(\omega_{20}^*\tau_{20}^*\right)\\
&\quad+\tau_1F\sin\left(\omega_{20}^*\tau_1\right)\cos\left(\omega_{20}^*\tau_{20}^*\right)+\tau_1F\cos\left(\omega_{20}^*\tau_1\right)\sin\left(\omega_{20}^*\tau_{20}^*\right);\\
F_{53}&=-D\omega_{20}^{*2}\cos\left(\omega_{20}^*\tau_{20}^*\right)+E\omega_{20}^*\sin\left(\omega_{20}^*\tau_{20}^*\right)+F\omega_{20}^*\sin\left(\omega_{20}^*\tau_1\right)\cos\left(\omega_{20}^*\tau_{20}^*\right)\\
&\quad+F\omega_{20}^*\cos\left(\omega_{20}^*\tau_1\right)\sin\left(\omega_{20}^*\tau_{20}^*\right);\\
F_{54}&=D\omega_{20}^{*2}\sin\left(\omega_{20}^*\tau_{20}^*\right)+E\omega_{20}^*\cos\left(\omega_{20}^*\tau_{20}^*\right)+F\omega_{20}^*\cos\left(\omega_{20}^*\tau_1\right)\cos\left(\omega_{20}^*\tau_{20}^*\right)\\
&\quad-F\omega_{20}^*\sin\left(\omega_{20}^*\tau_1\right)\sin\left(\omega_{20}^*\tau_{20}^*\right).
\end{aligned}$$

假设$\left(H_{52}\right):F_{51}F_{53}+F_{52}F_{54}\neq 0$成立，则有$\operatorname{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_2}\right)^{-1}_{\lambda=\mathrm{i}\omega_{20}^*}\neq 0$。证毕。

第六种情况： $\tau_2\in\left(0,\tau_{20}\right),\tau_1>0$，

定理 2.6：$\tau_2\in\left(0,\tau_{20}\right),\tau_1>0$ 时，满足$\left(H_{61},H_{62}\right)$，系统内平衡点 E^* 在 $\tau_1\in\left[0,\tau_{10}^*\right)$ 是局部渐近稳定的，并且当 $\tau=\tau_{10}^*$ 时出现 Hopf 分支。

证明：当$\tau_2\in\left(0,\tau_{20}\right),\tau_1>0$时，将$\tau_1$作为参数，假设$\lambda=\mathrm{i}\omega_1^*\left(\omega_1^*>0\right)$是方程(6)的根，代入可得

$$\begin{aligned}
R_{61}\sin\left(\omega_1^*\tau_1\right)+R_{62}\cos\left(\omega_1^*\tau_1\right)&=R_{63},\\
R_{61}\cos\left(\omega_1^*\tau_1\right)-R_{62}\sin\left(\omega_1^*\tau_1\right)&=R_{64}.
\end{aligned}\tag{33}$$

其中：

$$\begin{aligned}
&R_{61}=C\omega_1^*-F\sin\left(\omega_1^*\tau_2\right);\ R_{62}=F\cos\left(\omega_1^*\tau_2\right);\\
&R_{63}=A\omega_1^{*2}-E\cos\left(\omega_1^*\tau_2\right)-D\omega_1^*\sin\left(\omega_1^*\tau_2\right);\\
&R_{64}=\omega_1^{*3}-B\omega_1^*+E\sin\left(\omega_1^*\tau_2\right)-D\omega_1^*\cos\left(\omega_1^*\tau_2\right);
\end{aligned}$$

对方称组(33)平方相加得

$$G_1\left(\omega_1^*\right)+G_2\left(\omega_1^*\right)\sin\left(\omega_1^*\tau_2\right)+G_3\left(\omega_1^*\right)\cos\left(\omega_1^*\tau_2\right)=0.\tag{34}$$

其中：

$$\begin{aligned}
&G_1\left(\omega_1^*\right)=\omega_1^{*6}+\left(A^2-2B\right)\omega_1^{*4}+\left(B^2+D^2-C^2\right)\omega_1^{*2}+E^2-F^2;\\
&G_2\left(\omega_1^*\right)=\left(2E-2AD\right)\omega_1^{*3}+\left(2CF-2BE\right)\omega_1^*;\\
&G_3\left(\omega_1^*\right)=-2D\omega_1^{*4}+\left(2BD-2AE\right)\omega_1^{*2};
\end{aligned}$$

假设$\left(H_{61}\right)$：(33)存在有限个正根成立，且记为$\omega_{1k}^*,\left(k=1,2,\cdots,l_2\right)$。由方程(33)得到

$$\tau_{1k}^{*(j)}=\frac{1}{\omega_{1k}^*}\arccos\left(\frac{R_{62}R_{63}+R_{61}R_{64}}{R_{61}^2+R_{62}^2}+2j\tau\right),\ \left(j=0,1,2\cdots,k=1,2,\cdots,l_2\right)$$

假设$\tau_{10}^*=\min\tau_{1k}^{*(0)}\left(k=1,2,\cdots,l_2\right)$，对方称(6)关于$\tau_1$微分求导，并将$\lambda=\mathrm{i}\omega_{10}^*$，$\tau_1=\tau_{10}^*$代入得

$$\text{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_1}\right)^{-1}_{\lambda=\mathrm{i}\omega_{10}^*}=\frac{F_{61}F_{63}+F_{62}F_{64}}{F_{63}^2+F_{64}^2} \tag{35}$$

其中：

$$\begin{aligned}
F_{61}&=-3\omega_{10}^{*\,2}+B+C\cos\left(\omega_{10}^*\tau_{10}^*\right)+D\cos\left(\omega_{10}^*\tau_2\right)-\tau_2E\cos\left(\omega_{10}^*\tau_2\right)-\tau_2D\omega_{10}^*\sin\left(\omega_{10}^*\tau_2\right)\\
&\quad-\tau_2F\cos\left(\omega_{10}^*\tau_{10}^*\right)\cos\left(\omega_{10}^*\tau_2\right)+\tau_2F\sin\left(\omega_{10}^*\tau_{10}^*\right)\sin\left(\omega_{10}^*\tau_2\right);\\
F_{62}&=2A\omega_{10}^*-C\sin\left(\omega_{10}^*\tau_{10}^*\right)-D\sin\left(\omega_{10}^*\tau_2\right)-\tau_2D\omega_{10}^*\cos\left(\omega_{10}^*\tau_2\right)+\tau_2E\sin\left(\omega_{10}^*\tau_2\right)\\
&\quad+\tau_2F\sin\left(\omega_{10}^*\tau_{10}^*\right)\cos\left(\omega_{10}^*\tau_2\right)+\tau_2F\sin\left(\omega_{10}^*\tau_2\right)\cos\left(\omega_{10}^*\tau_{10}^*\right);\\
F_{63}&=F\omega_{10}^*\sin\left(\omega_{10}^*\tau_{10}^*\right)\cos\left(\omega_{10}^*\tau_2\right)+F\omega_{10}^*\cos\left(\omega_{10}^*\tau_{10}^*\right)\sin\left(\omega_{10}^*\tau_2\right)-C\omega_{10}^{*\,2}\cos\left(\omega_{10}^*\tau_{10}^*\right);\\
F_{64}&=F\omega_{10}^*\cos\left(\omega_{10}^*\tau_{10}^*\right)\cos\left(\omega_{10}^*\tau_2\right)-F\omega_{10}^*\sin\left(\omega_{10}^*\tau_{10}^*\right)\sin\left(\omega_{10}^*\tau_2\right)+C\omega_{10}^{*\,2}\sin\left(\omega_{10}^*\tau_{10}^*\right);
\end{aligned}$$

假设$\left(H_{62}\right):F_{61}F_{63}+F_{62}F_{64}\neq0$成立，则有 $\text{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_1}\right)^{-1}_{\lambda=\mathrm{i}\omega_{10}^*}\neq0$。证毕。

3. 分支性质

本部分利用范数定理和中心流行定理[29]，讨论关于τ_1的 Hopf 分支性质，其中$\tau_2\in\left[0,\tau_{20}\right)$。假设在$\tau_{2*}\in\left[0,\tau_{20}\right)$条件下有$\tau_{2*}<\tau_{10}^*$，由前面的章节得知当$\tau=\tau_{10}^*$时系统(2)在正平衡点$E^*=\left(D^*,P^*,Z^*\right)$处存在一个 Hopf 分支。

假设$\tau_1=\mu+\tau_{10}^*$，则$\mu=0$是系统(2)的 Hopf 分支值。令$u_1(t)=D(t)-D^*$，$u_2(t)=P(t)-P^*$，$u_3(t)=Z(t)-Z^*$，并重新设置时滞为$t\to\left(t/\tau_1\right)$，将系统(2)重新写为

$$\dot{u}(t)=L_\mu u_t+F\left(\mu,u_t\right), \tag{36}$$

其中：

$$L_\mu\phi=\left(\tau_{10}^*+\mu\right)\left(A\phi(0)+B\phi(-1)+C\phi\left(-\frac{\tau_{2*}}{\tau_{10}^*}\right)\right),$$

$$F(\mu,\phi)=\left(\tau_{10}^*+\mu\right)\left(F_1,F_2,F_3\right)^{\mathrm{T}}.$$

$$\phi(\theta)=\left(\phi_1(\theta),\phi_2(\theta),\phi_3(\theta)\right)^{\mathrm{T}}\in C\left(\left[-1,0\right],R^3\right),$$

$$A=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\0&a_{22}&a_{23}\\0&0&0\end{pmatrix},B=\begin{pmatrix}0&0&0\\a_{21}&0&0\\0&0&0\end{pmatrix},C=\begin{pmatrix}0&0&0\\0&0&0\\0&a_{32}&0\end{pmatrix},$$

$$\begin{aligned}
F_1&=0,\\
F_2&=\left(-\frac{a\beta}{\left(a+P^*\right)^2}\right)\phi_2(0)\phi_3(0)+\frac{a\beta Z^*}{\left(a+P^*\right)^3}\phi_2^2(0)-\frac{a\beta Z^*}{\left(a+P^*\right)^4}\phi_2^3(0)+\cdots,\\
F_3&=\frac{a\beta_1}{\left(a+P^*\right)^2}\phi_2\left(-\frac{\tau_{2*}}{\tau_{10}^*}\right)\phi_3(0)-\frac{a\beta_1Z^*}{\left(a+P^*\right)^3}\phi_2^2\left(-\frac{\tau_{2*}}{\tau_{10}^*}\right)+\frac{a\beta_1Z^*}{\left(a+P^*\right)^4}\phi_2^3\left(-\frac{\tau_{2*}}{\tau_{10}^*}\right)+\cdots.
\end{aligned}$$

根据 Riesz 表示定理，对一个有界变量存在一个3×3的矩阵方程$\eta(\theta,\mu):\left[-1,0\right]\to R^{32}$，其中$\theta\in\left[-1,0\right]$，则有

$$L_{\mu}\phi=\int_{-1}^{0}\mathrm{d}\eta(\theta,\mu)\phi(\theta),\ \phi\in C\left([-1,0],R^3\right).$$

选择

$$\nu(\theta)=\begin{cases}\left(\tau_{10}^*+\mu\right)(A+B+C), & \theta=0,\\ \left(\tau_{10}^*+\mu\right)(B+C), & \theta\in\left[-\dfrac{\tau_{2*}}{\tau_{10}^*},0\right),\\ \left(\tau_{10}^*+\mu\right)B, & \theta\in\left(-1,-\dfrac{\tau_{2*}}{\tau_{10}^*}\right),\\ 0, & \theta=-1.\end{cases}$$

对 $\phi\in C\left([-1,0],R^3\right)$，定义

$$\Gamma(\mu)\phi=\begin{cases}\dfrac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, & \theta\in[-1,0),\\ \int_{-1}^{0}\mathrm{d}\eta(\theta,\mu)\phi(\theta), & \theta=0,\end{cases}$$

$$R(\mu)\phi=\begin{cases}0, & \theta\in[-1,0),\\ F(\mu,\phi), & \theta=0.\end{cases}$$

因此，系统(36)可以重新写为

$$\dot{u}_t=\Gamma(\mu)u_t+R(\mu)u_t,\tag{37}$$

其中 $u_t=u(t+\theta)=\left(u_1(t+\theta),u_2(t+\theta),u_3(t+\theta)\right)$，$\theta\in[-1,0]$。

对 $\varphi\in C^1\left([0,1],\left(R^3\right)^*\right)$，定义 Γ 的耦合算子 Γ^* 如下：

$$\Gamma^*(\varphi)=\begin{cases}-\dfrac{\mathrm{d}\varphi(s)}{\mathrm{d}s}, & s\in(0,1],\\ \int_{-1}^{0}\mathrm{d}\eta^{\mathrm{T}}(s,0)\varphi(-s), & s=0,\end{cases}$$

一个线性内积定义如下：

$$\langle\varphi(s),\phi(\theta)\rangle=\bar{\varphi}(0)\phi(0)-\int_{\theta=-1}^{0}\int_{\xi=0}^{\theta}\bar{\varphi}(\xi-\theta)\mathrm{d}\eta(\theta)\phi(\xi)\mathrm{d}\xi\tag{38}$$

其中：$\eta(\theta)=\eta(\theta,0)$。

由前面的研究得知 $\pm\mathrm{i}\omega_1^*\tau_{10}^*$ 是 $\Gamma(0)$ 的特征值，则 $\pm\mathrm{i}\omega_1^*\tau_{10}^*$ 同样是 $\Gamma^*(0)$ 的特征值。

假设 $q(\theta)=(1,q_2,q_3)^{\mathrm{T}}\,\mathrm{e}^{\mathrm{i}\omega_1^*\tau_{10}^*\theta}$ 是 $\Gamma(0)$ 关于特征值 $\mathrm{i}\omega_1^*\tau_{10}^*$ 的特征向量，同样假设 $q^*(s)=D\left(1,q_2^*,q_3^*\right)\mathrm{e}^{\mathrm{i}\omega_1^*\tau_{10}^*s}$ 是 $\Gamma^*(0)$ 关于特征值 $-\mathrm{i}\omega_1^*\tau_{10}^*$ 的特征向量。通过计算可以得到：

$$q_2=\frac{-\omega_1^{*2}-a_{11}\mathrm{i}\omega_1^*}{a_{12}\mathrm{i}\omega_1^*+a_{13}a_{32}\mathrm{e}^{-\mathrm{i}\omega_1^*\tau_{2*}}},\quad q_3=\frac{\left(\mathrm{i}\omega_1^*-a_{11}\right)\left(-a_{32}\mathrm{e}^{-\mathrm{i}\omega_1^*\tau_{2*}}\right)}{a_{13}a_{32}\mathrm{e}^{-\mathrm{i}\omega_1^*\tau_{2*}}-a_{12}\mathrm{i}\omega_1^*};$$

$$q_2^*=\frac{-\left(\mathrm{i}\omega_1^*-a_{11}\right)}{a_{21}\mathrm{e}^{-\mathrm{i}\omega_1^*\tau_{10}^*}},\qquad q_3^*=\frac{\left(\mathrm{i}\omega_1^*+a_{11}\right)a_{23}-a_{13}a_{21}\mathrm{e}^{-\mathrm{i}\omega_1^*\tau_{10}^*}}{\mathrm{i}\omega_1^*a_{21}\mathrm{e}^{-\mathrm{i}\omega_1^*\tau_{10}^*}}.$$

从方程(38)，得到

$$D=\left[1+q_2^*\bar{q}_2+q_3^*\bar{q}_3+\tau_{2*}\mathrm{e}^{\mathrm{i}\omega_1^*\tau_{10}^*}\left(q_2^*\tau_{10}^*a_{21}+\bar{q}_2q_3^*\tau_{2*}a_{32}\right)\right]^{-1}$$

满足 $\langle q^*(s), q(\theta)\rangle = 1$，$\langle q^*(s), \overline{q}(\theta)\rangle = 0$。

引用[30]给出的算法可以得到用来决定分支方向和分支周期解稳定性的参数如下：

$$g_{20} = 2\overline{D}\tau_{10}^*\left(\frac{aq_2q_3}{(a+P^*)^2}\left(\beta_1 e^{-i\omega_1^*\tau_{2*}}\overline{q_3^*} - \beta\overline{q_2^*}\right) + \frac{aZ^*}{(a+P^*)^3}\left(\beta q_2^2\overline{q_2^*} - \beta_1 q_2^2 e^{-2i\omega_1^*\tau_{2*}}\overline{q_3^*}\right)\right),$$

$$g_{11} = \overline{D}\tau_{10}^*\left(\frac{aq_3\overline{q_2}}{(a+P^*)^2}\left(\beta_1 e^{i\omega_1^*\tau_{2*}}\overline{q_3^*} - \beta\overline{q_2^*}\right) + \frac{a\overline{q_3}q_2}{(a+P^*)^2}\left(\beta_1 e^{-i\omega_1^*\tau_{2*}}\overline{q_3^*} - \beta\overline{q_2^*}\right) + \frac{2aZ^*q_2\overline{q_2}}{(a+P^*)^3}\left(\beta\overline{q_2^*} - \beta_1\overline{q_3^*}\right)\right),$$

$$g_{02} = 2\overline{D}\tau_{10}^*\left(\frac{a\overline{q_3q_2}}{(a+P^*)^2}\left(\beta_1 e^{i\omega_1^*\tau_{2*}}\overline{q_3^*} - \beta\overline{q_2^*}\right) + \frac{aZ^*\overline{q_2}^2}{(a+P^*)^3}\left(\beta\overline{q_2^*} - \beta_1 e^{2i\omega_1^*\tau_{2*}}\overline{q_3^*}\right)\right),$$

$$\begin{aligned}
g_{21} = 2\overline{D}\tau_{10}^*\Bigg\{&\overline{q_2^*}\left[\left(-\frac{a\beta}{(a+P^*)^2}\right)\left(W_{20}^{(2)}(0)\frac{1}{2}\overline{q_3} + W_{11}^{(2)}(0)q_3 + \overline{q_2}\frac{1}{2}W_{20}^{(3)}(0) + W_{11}^{(3)}(0)q_2\right)\right] \\
&+\overline{q_2^*}\frac{aZ^*\beta}{(a+P^*)^3}\left(2W_{11}^{(2)}(0)q_2 + W_{20}^{(2)}(0)\overline{q_2}\right) + \overline{q_3^*}\frac{a\beta_1}{(a+P^*)^2}\Bigg(W_{11}^{(3)}(0)e^{-i\omega_1^*\tau_{2*}}q_2 \\
&+\frac{1}{2}e^{i\omega_1^*\tau_{2*}}\overline{q_2}W_{20}^{(3)}(0) + \frac{1}{2}W_{20}^{(2)}\left(-\frac{\tau_{2*}}{\tau_{10}^*}\right)\overline{q_3} + W_{11}^{(2)}\left(-\frac{\tau_{2*}}{\tau_{10}^*}\right)q_3\Bigg) \\
&-\overline{q_3^*}\frac{a\beta_1 Z^*}{(a+P^*)^3}\left(2W_{11}^{(2)}\left(-\frac{\tau_{2*}}{\tau_{10}^*}\right)e^{-i\omega_1^*\tau_{2*}}q_2 + W_{20}^{(2)}\left(-\frac{\tau_{2*}}{\tau_{10}^*}\right)e^{i\omega_1^*\tau_{2*}}\overline{q_2}\right)\Bigg\}.
\end{aligned}$$

并且可以得到：

$$W_{20}(\theta) = \frac{ig_{20}q(0)}{\omega_1^*\tau_{10}^*}e^{i\omega_1^*\tau_{10}^*\theta} + \frac{i\overline{g}_{02}\overline{q}(0)}{3\omega_1^*\tau_{10}^*}e^{-i\omega_1^*\tau_{10}^*\theta} + E_1 e^{2i\omega_1^*\tau_{10}^*\theta}.$$

$$W_{11}(\theta) = \frac{-ig_{11}q(0)}{\omega_1^*\tau_{10}^*}e^{i\omega_1^*\tau_{10}^*\theta} + \frac{i\overline{g}_{11}\overline{q}(0)}{\omega_1^*\tau_{10}^*}e^{-i\omega_1^*\tau_{10}^*\theta} + E_2.$$

其中 E_1 和 E_2 可以由如下方程分别决定

$$\begin{pmatrix} 2i\omega_1^* - a_{11} & -a_{12} & -a_{13} \\ -a_{21}e^{-2i\omega_1^*\tau_{10}^*} & 2i\omega_1^* - a_{22} & -a_{23} \\ 0 & -a_{32}e^{-2i\omega_1^*\tau_{2*}} & 2i\omega_1^* \end{pmatrix} \cdot E_1 = 2\begin{pmatrix} K_{11} \\ K_{21} \\ K_{31} \end{pmatrix},$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & 0 \end{pmatrix} \cdot E_2 = -\begin{pmatrix} K_{12} \\ K_{22} \\ K_{32} \end{pmatrix}.$$

其中：

$$K_{11} = 0,\ K_{21} = -\frac{a\beta}{(a+P^*)^2}q_2q_3 + \frac{a\beta Z^*}{(a+P^*)^3}q_2^2,$$

$$K_{31} = \frac{a\beta_1}{(a+P^*)^2}q_2q_3e^{-i\omega_1^*\tau_{2*}} - \frac{a\beta_1 Z^*}{(a+P^*)^3}q_2^2e^{-2i\omega_1^*\tau_{2*}};$$

$$K_{12}=0,\ K_{22}=-\frac{a\beta}{\left(a+P^*\right)^2}\left(q_2\overline{q_3}+\overline{q_2}q_3\right)+\frac{a\beta Z^*}{\left(a+P^*\right)^3}2q_2\overline{q_2},$$

$$K_{31}=\frac{a\beta_1}{\left(a+P^*\right)^2}\left(\overline{q_2}q_3\mathrm{e}^{\mathrm{i}\omega_1^*\tau_{2*}}+q_2\overline{q_3}\mathrm{e}^{-\mathrm{i}\omega_1^*\tau_{2*}}\right)-\frac{a\beta_1 Z^*}{\left(a+P^*\right)^3}2q_2\overline{q_2}.$$

最后基于 $g_{20}, g_{11}, g_{02}, g_{21}$ 的表达式可以计算下面的值

$$c_1(0)=\frac{i}{2\omega_1^*\tau_{10}^*}\left(g_{20}g_{11}-2|g_{11}|^2-\frac{|g_{02}|^2}{3}\right)+\frac{g_{21}}{2},$$

$$\mu_2=-\frac{\mathrm{Re}\{c_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_{10}^*)\}},$$

$$\beta_2=2\,\mathrm{Re}\{c_1(0)\},$$

$$T_2=-\frac{\mathrm{Im}\{c_1(0)\}+\mu_2\,\mathrm{Im}\{\lambda'(\tau_{10}^*)\}}{\omega_1^*\tau_{10}^*}.$$

根据之前的结论，可以得到关于 Hopf 分支性质的定理如下：

定理 3.1：系统(2)在 $\tau_1=\tau_{10}^*$ 处的 Hopf 分支性质可由 $c_1(0), \mu_2, \beta_2, T_2$ 决定，且结论如下：

(i) 如果 $\mu_2>0(<0)$，Hopf 分支是超临界的(亚临界的)，

(ii) 如果 $\beta_2<0(>0)$，分支周期解是稳定的(不稳定的)，

(iii) 如果 $T_2>0(<0)$，系统(2)的分支周期解是增加的(减少的)。

4. 结论

本文建立了一类具有时滞效应的溶解氧-浮游生物动力学新模型，以此为基础对模型的相关动力学性质进行了数学分析与理论推导，解析出模型内平衡点具有局部渐近稳定性和发生 Hopf 分支的阈值条件，并进一步讨论了 Hopf 分支的相关性质，给出一些理论结果。本论文的研究结果有利于促进海洋生态动力学模型的长远发展，有助于剖析海洋生态系统中各类生态环境因素的影响驱动性。

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