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变系数BBM方程的三角函数周期解

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摘 要

本文将利用一个简单技巧化简高次变系数BBM方程，并采取辅助方程法求该方程的三角函数周期解。

关键词

三角函数周期解，BBM方程，变系数

Trigonometric Period Solutions for the BBM Equations with Variable Coefficients

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Abstract

The order of the BBM equation with variable coefficients is reduced by an simple

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approach and trigonometric periodic solutions for the equation are investigated by using auxiliary equations.

Keywords

Trigonometric Periodic Solutions, BBM Equation, Variable Coefficients

1. 引言

BBM方程是由Benjamin，Bona 和Mahony [1]提出来的，有如下的形式：

$$u_t + au_x - bu_{xxt} + k(u^{\pm 2})_x = 0, \tag{1.1}$$

该方程是用来描述浅水波在水渠中的运动状态。

不同形式的BBM方程吸引了很多学者的注意 [2–8]。很多不同的方法都被提出来解决非线性偏微分方程。例如Bäcklund 变换方法 [9]，反散射方法 [10]，达布变换方法 [11]，Hirota's 双线性方法 [12]，同伦分析法 [13]，双曲正切函数法 [14] [15]，正弦-余弦方法 [16]，动力系统的分支方法 [17] [18]，辅助方程方法 [19]等。

近年来，由于BBM方程的广泛应用 [20–24]，许多研究者都对变系数的BBM方程特别感兴趣。通过变换

$$u(x,t) = v^{\frac{1}{n-1}}(x,t), \tag{1.2}$$

和辅助方程方法, Lv 等人[25] 获得了下面不同物理结构的广义BBM方程许多显式的行波解

$$u_t + au_x + b(u^n)_x + k(u^n)_{xxt} = 0, \quad n \neq 0, 1. \tag{1.3}$$

这些解中包含三角函数周期波解和双曲函数解。

这篇文章中,我们准备拓展文章 [25]中的结果，并寻找新的一般的变换来研究BBM 方程

$$u_t + a(t)u_x + b(t)(u^m)_x + k(t)(u^n)_{xxt} = 0, \quad m, n \neq 0, 1, \tag{1.4}$$

其中$a(t)b(t)k(t)\neq 0$。在这篇文章中，我们只研究BBM 方程的三角函数周期波解。

2. 辅助方程法

为了获得三角函数周期波解，我们将

$$u(x,t)=c_1+c_2z(\xi),\quad \xi=p(t)x+q(t), \tag{2.1}$$

代入到方程(1.4)中去，其中$z'=\epsilon\sqrt{a_1+a_2z+a_3z^2}$, $\epsilon=\pm 1$, a_1, a_2, a_3, c_1 与c_2 是实常数，$p(t)$ 与$q(t)$ 是未知函数。让$\Delta=a_2^2-4a_1a_3$, 就会有下面的事实 [12]。当$a_3<0$ 与$\Delta>0$，我们有$z(\xi)=\frac{\epsilon\sqrt{\Delta}}{2a_3}\sin(\sqrt{-a_3}\xi)-\frac{a_2}{2a_3}$ 与$z(\xi)=\frac{\epsilon\sqrt{\Delta}}{2a_3}\cos(\sqrt{-a_3}\xi)-\frac{a_2}{2a_3}$。

事实上，将(2.1) 代入到方程(1.4)中去，并让$x^sz^i(\xi)$ 的系数$\sqrt{a_1+a_2z+a_3z^2}(s,i=0,1,2,\cdots)$ 等于零，我们就会获得关于c_1, c_2，a_1, a_2, a_3, $p(t)$ 和$q(t)$ 的代数方程。通过求解这些方程，我们就会获得参数的显式表达式。然后我们就能获得BBM方程的三角函数周期波解。

3. 化简高次BBM方程

在这一节中，我们将介绍一个简单的手段来获得变换公式，借用这些公式化简高次BBM方程。

性质1. 当$m=n$，我们可以利用变换

$$u=v^p, \tag{3.1}$$

这里面$p=\frac{\alpha}{n-1}$, $\alpha\in Z$ 和$\alpha\neq 0$，方程(1.4) 可以化为如下的两种情形。

(i) 若$\alpha\geq 2$，方程(1.4) 能够变换为

$$\begin{aligned}&v_t+a(t)v_x+b(t)nv^{\alpha}v_x+k(t)n(np-1)(np-2)v^{\alpha-2}v_tv_x^2\\&+2k(t)n(np-1)v^{\alpha-1}v_xv_{xt}+k(t)n(np-1)v^{\alpha-1}v_tv_{xx}+k(t)nv^{\alpha}v_{xxt}=0.\end{aligned} \tag{3.2}$$

(ii) 若$\alpha<2$，方程(1.4) 能够变换为

$$\begin{aligned}&v^{2-\alpha}v_t+a(t)v^{2-\alpha}v_x+b(t)nv^2v_x+k(t)n(np-1)(np-2)v_tv_x^2\\&+2k(t)n(np-1)vv_xv_{xt}+k(t)n(np-1)vv_tv_{xx}+k(t)nv^2v_{xxt}=0.\end{aligned} \tag{3.3}$$

性质2. 当$m\neq n$ 且$\frac{n-1}{m-1}=\frac{\beta+2}{\alpha}\neq 1$，利用变换

$$u=v^p, \tag{3.4}$$

其中α, $\beta\in Z$, $\alpha\neq 0$, $\beta\neq -2$ 和$p=\frac{\alpha}{m-1}=\frac{\beta+2}{n-1}$，方程(1.4)能够化简为

$$v_t+a(t)v_x+b(t)mv^{\alpha}v_x+k(t)n(np-1)(np-2)v^{\beta}v_tv_x^2$$
$$+2k(t)n(np-1)v^{\beta+1}v_xv_{xt}+k(t)n(np-1)v^{\beta+1}v_tv_{xx}+k(t)nv^{\beta+2}v_{xxt}=0. \tag{3.5}$$

证明: 利用(3.1)这个式子，我们比较容易获得

$$\begin{cases} u_t=pv^{p-1}v_t, \\ u_x=pv^{p-1}v_x, \\ (u^m)_x=mpv^{mp-1}v_x, \\ (u^n)_{xxt}=np(np-1)(np-2)v^{np-3}v_tv_x^2 \\ +2np(np-1)v^{np-2}v_xv_{xt}+np(np-1)v^{np-2}v_tv_{xx}+npv^{np-1}v_{xxt}. \end{cases} \tag{3.6}$$

通过将(3.6) 代入到方程(1.4)中，得到

$$pv^{p-1}v_t+a(t)pv^{p-1}v_x+b(t)mpv^{mp-1}v_x+k(t)np(np-1)(np-2)v^{np-3}v_tv_x^2$$
$$+2k(t)np(np-1)v^{np-2}v_xv_{xt}+k(t)np(np-1)v^{np-2}v_tv_{xx}+k(t)npv^{np-1}v_{xxt}=0. \tag{3.7}$$

倘若(3.7)的两边同时乘以v^s，这样就有

$$v^s[pv^{p-1}v_t+a(t)pv^{p-1}v_x+b(t)mpv^{mp-1}v_x+k(t)np(np-1)(np-2)v^{np-3}v_tv_x^2$$
$$+2k(t)np(np-1)v^{np-2}v_xv_{xt}+k(t)np(np-1)v^{np-2}v_tv_{xx}+k(t)npv^{np-1}v_{xxt}=0, \tag{3.8}$$

或者

$$pv^{s+p-1}v_t+a(t)pv^{s+p-1}v_x+b(t)mpv^{s+mp-1}v_x+k(t)np(np-1)(np-2)v^{s+np-3}v_tv_x^2$$
$$+2k(t)np(np-1)v^{s+np-2}v_xv_{xt}+k(t)np(np-1)v^{s+np-2}v_tv_{xx}+k(t)npv^{s+np-1}v_{xxt}=0, \tag{3.9}$$

其中s是任意常数。若设系数有如下的关系：

$$\begin{cases} s+p-1=A, \\ s+mp-1=B, \\ s+np-3=C, \end{cases} \tag{3.10}$$

其中A，B 和C 都是实常数。因此通过(3.10)，就会有

$$p=\frac{B-A}{m-1}=\frac{C-A+2}{n-1}. \tag{3.11}$$

接下来，我们将分两种情况$m=n$ 和$m\neq n$来证明性质1 和2。

Case (1)若当$m=n$，即$B=C+2$，方程(3.9)就可以变为

$$pv^Av_t+a(t)pv^Av_x+b(t)mpv^Bv_x+k(t)np(np-1)(np-2)v^{B-2}v_tv_x^2$$
$$+2k(t)np(np-1)v^{B-1}v_xv_{xt}+k(t)np(np-1)v^{B-1}v_tv_{xx}+k(t)npv^Bv_{xxt}=0. \tag{3.12}$$

若当$B-2\geq A$，方程(3.12)也可以变为

$$pv_t+a(t)pv_x+b(t)mpv^{B-A}v_x+k(t)np(np-1)(np-2)v^{B-A-2}v_tv_x^2$$
$$+2k(t)np(np-1)v^{B-A-1}v_xv_{xt}+k(t)np(np-1)v^{B-A-1}v_tv_{xx}+k(t)npv^{B-A}v_{xxt}=0. \tag{3.13}$$

为了方便我们的讨论，取$B-A=\alpha$。然后方程(3.13) 能化简为方程(3.2)。

若当$B-2<A$，方程(3.12) 就可以变为

$$pv^{A-B+2}v_t+a(t)pv^{A-B+2}v_x+b(t)mpv^2v_x+k(t)np(np-1)(np-2)v_tv_x^2$$
$$+2k(t)np(np-1)vv_xv_{xt}+k(t)np(np-1)vv_tv_{xx}+k(t)npv^2v_{xxt}=0. \tag{3.14}$$

然后方程(3.14) 能够化简为方程(3.3)。于是我们就完成了性质1 的证明。

Case (2) 若当$m\neq n$，方程(3.9)就可以变为

$$pv_t+a(t)pv_x+b(t)mpv^{B-A}v_x+k(t)np(np-1)(np-2)v^{C-A}v_tv_x^2$$
$$+2k(t)np(np-1)v^{C-A+1}v_xv_{xt}+k(t)np(np-1)v^{C-A+1}v_tv_{xx}+k(t)npv^{C-A+2}v_{xxt}=0. \tag{3.15}$$

让$B-A=\alpha$ 和$C-A=\beta$，我们重新写方程(3.15)。到此为止，我们完成性质4 的证明。

注1: 当$m=n$ 且$\alpha=1$，Lv 等人 [25]应用变换$u=v^{\frac{1}{n-1}}$获得方程(1.4)的一些精确解。

接下来，我们将利用辅助方程法考虑方程(1.4)的三角函数周期波解。

4. 三角函数周期波解

若我们令$\alpha=2$，则方程(3.2)可以变为

$$v_t+a(t)v_x+nb(t)v^2v_x+\frac{n(n+1)k(t)}{n-1}\left(\frac{2v_tv_x^2}{n-1}+2vv_xv_{xt}+vv_tv_{xx}\right)+nk(t)v^2v_{xxt}=0. \tag{4.1}$$

通过将方程(2.1)代入到方程(4.1)中去，并设$x^sz^i(\xi)\sqrt{a_1+a_2z+a_3z^2}(s=0,1,i=0,1,2)$的系数为零。我们就得到关于$c_2$, $p(t)$ and $q(t)$ 的代数方程。求解这些代数方程我们得到下列式子

$$\begin{cases} c_2=\frac{2a_3c_1}{a_2}, \\ q'(t)=\pm\frac{n-1}{2\sqrt{2}n^{\frac{3}{2}}a_2}\sqrt{\frac{b(t)}{a_3k(t)}}N_1, \\ p(t)=\mp\sqrt{\frac{b(t)}{a_3k(t)}}\frac{(n-1)a_2}{\sqrt{2n}N_1}, \\ p'(t)=0, \end{cases} \tag{4.2}$$

其中$N_1 = \sqrt{2na(t)a_2^2 + (n+1)b(t)(a_2^2 - 4a_1a_3)c_1^2}$, $a_2a_3c_1 \neq 0$, a_1, a_2, a_3 和c_1 都是实常数。事实上，我们需要假设因子$\sqrt{\frac{b(t)}{a_3k(t)}}\frac{1}{N_1}$ 是一个常数，因为$p'(t) = 0$。然后我们就得到了方程(1.4)的三角函数周期波解如下：

(i) 若$a_3 < 0$, $\Delta > 0$ 和$\frac{b(t)}{k(t)} < 0$，我们得到如下三角函数周期波解

$$u_9(x,t) = \left[\frac{c_1\sqrt{\Delta}}{a_2}\sin\left(\frac{(n-1)a_2}{\sqrt{2n}N_1}\sqrt{-\frac{b(t)}{k(t)}}x - \sqrt{-a_3}q(t)\right)\right]^{\frac{2}{n-1}},$$

和

$$u_{10}(x,t) = \left[\frac{c_1\sqrt{\Delta}}{a_2}\cos\left(\frac{(n-1)a_2}{\sqrt{2n}N_1}\sqrt{-\frac{b(t)}{k(t)}}x - \sqrt{-a_3}q(t)\right)\right]^{\frac{2}{n-1}},$$

当$m \neq n$，我们考虑$\alpha = 1$和$\beta = -3$的情形。就会有$n = -m + 2$ 和$p = \frac{1}{m-1}$。因此方程(3.5)可以化为

$$\begin{aligned} &v^3v_t + a(t)v^3v_x + b(t)mv^4v_x + \tfrac{(2-m)(3-2m)(4-3m)}{(m-1)^2}k(t)v_tv_x^2 \\ &+2\tfrac{(2-m)(3-2m)}{m-1}k(t)vv_xv_{xt} + \tfrac{(2-m)(3-2m)}{m-1}k(t)vv_tv_{xx} + k(t)(2-m)v^2v_{xxt} = 0. \end{aligned} \tag{4.7}$$

将方程(2.1)代入方程(4.7)中并设$x^s z^j(\xi)\sqrt{a_1 + a_2z + a_3z^2}(s = 0, 1, i = 0, 1, 2, 3, 4)$ 的系数等于零我们就得到未知参数a_3, c_2, $p(t)$ 和$q(t)$ 的代数方程。通过求解这些代数方程，我们可以得到

$$\begin{cases} a_3 = \frac{a_2^2}{4a_1}, \\ c_2 = \frac{a_2c_1}{2a_1}, \\ p(t) = \pm\frac{2(m-1)}{(m-2)a_2}\sqrt{\frac{-mb(t)a_1}{(m-2)a(t)k(t)}}, \\ p'(t) = 0, \\ q'(t) = -p(t)a(t). \end{cases} \tag{4.8}$$

其中$a_1a_2c_1 \neq 0$, a_1, a_2 和c_1 都是实常数。因此我们就会获得一个如下的指数函数解

$$u_{14\pm}(x,t) = \left[\frac{c_1}{a_2}e^{p(t)(x-a(t))}\right]^{\frac{1}{m-1}}.$$

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