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C^*-代数值度量空间中公共耦合不动点定理

辛巧玲[1*]，曹天庆[2]

[1]天津师范大学，数学科学学院，天津
[2]天津工业大学，数学科学学院，天津
Email: *xinqiaoling0923@163.com

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摘 要

本文给出了完备的C^*-代数值度量空间上不同压缩映射的公共耦合不动点定理.作为应用，证明了Fredholm非线性积分算子解的存在唯一性.

关键词

耦合重合点, 不动点, C^*-代数, 正元

Common Coupled Fixed Point Theorems in C^*-Algebra-Valued Metric Spaces

Qiaoling Xin[1*], Tianqing Cao[2]

[1]School of Mathematical Sciences, Tianjin Normal University, Tianjin
[2]School of Mathematical Sciences, Tiangong University, Tianjin
Email: *xinqiaoling0923@163.com

* 通讯作者。

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Abstract

In this paper, we prove some common coupled fixed point theorems for mappings satisfying different contractive conditions in the context of complete C^*-algebra-valued metric spaces. Moreover, the paper provides an application to prove the existence and uniqueness of a solution for Fredholm nonlinear integral equations.

Keywords

Coupled Coincidence Point, Fixed Point, C^*-Algebra, Positive Element

1. 介绍

不动点理论是数学研究中的活跃分支之一, 在矩阵方程、偏微分方程和积分方程解的存在性和唯一性中发挥着重要的作用 [1–5]. 1987年, 郭大钧和Lakshmikantham [6]首次引入了耦合不动点的概念. 随后, 许多学者开始研究不同度量空间不同压缩映射下的不动点理论. 2006年, Bhaskar和Lakshmikantham [2]引入混合单调性质并研究相应映射的耦合不动点定理, 进而讨论了周期边值问题的解的存在唯一性. 2011年, Luong和Thuan [7]研究了偏序度量空间中耦合不动点定理, 并讨论非线性积分方程解的存在唯一性. 2012年, Jleli和Samet [3]讨论了奇异非线性分数微分方程正解的存在唯一性. 总之, 许多学者研究了序空间上的耦合不动点和耦合重合点定理 [1, 8–10].

另一方面, 学者研究了不同空间上的不动点和耦合不动点定理, 例如b-度量空间 [11], 锥度量空间 [12], 模糊度量空间 [13], G-度量空间 [14], 拟Banach空间 [15], 非交换Banach空间 [16]等. 2007年, 黄龙光和张宪 [12]推广了度量空间的概念, 引入锥度量空间并且得到一些压缩映射下的各种不动点定理. 随后, 学者们研究了锥度量空间上相应的耦合不动点定理 [17–19]. 2014年, 麻振华等人 [20]首次介绍了C^*-代数值度量空间及相关性质, 并且证明了此空间上压缩映射与扩张映射的不动点定理.

受Abbas [17]和麻振华 [20]工作的启发, 本文将证明C^*-代数值空间上不同压缩映射的公共耦合不动点定理, 并且得到Fredholm非线性积分方程解的存在唯一性.

首先我们回顾C^*-代数的相关概念、记号和结论 [21]. 设$\mathcal{A}$是有单位元的代数. 若$\mathcal{A}$上的共轭线性映射$a \to a^*$满足: 对任意$a, b \in \mathcal{A}$, 都有$a^{**} = a$和$(ab)^* = b^*a^*$, 则称该映射为$\mathcal{A}$上的对合, 并称$(\mathcal{A}, *)$是$*$-代数. 如果$*$-代数$\mathcal{A}$上存在完备的次可乘范数满足$\|a^*\| = \|a\|$, 则称$\mathcal{A}$为Banach $*$-代数. 进而, 如果该范数还满足$\|a^*a\| = \|a\|^2$, 则称$\mathcal{A}$为C^*-代数. 设$a \in \mathcal{A}$, 若a是自伴的, 且$\sigma(a) \subseteq [0, +\infty)$, 其中$\sigma(a)$表示$a$的谱, 则称$a$是$C^*$-代数$\mathcal{A}$的正元, 记作$0_{\mathcal{A}} \preceq a$, 其中$0_{\mathcal{A}}$是$\mathcal{A}$的零元. 我们分别用$\mathcal{A}_+$, $\mathcal{A}_h$表示$\mathcal{A}$上正元和自伴元的全体. 很自然地可以诱导$\mathcal{A}_h$上的偏序: $a \preceq b$当且仅当$0_{\mathcal{A}} \preceq b - a$. 从现在起, 用$\mathcal{A}'$表示集合$\{a \in \mathcal{A} \colon ab = ba, \forall b \in \mathcal{A}\}$.

引理 1.1. *[21] 设$\mathcal{A}$是有单位元$1_{\mathcal{A}}$的C^*-代数.*

(1) 若$a \in \mathcal{A}_+$且$\|a\| < \frac{1}{2}$, 则$1_{\mathcal{A}} - a$是可逆的.

(2) 若$a, b \in \mathcal{A}_+$且$ab = ba$, 则$0_{\mathcal{A}} \preceq ab$.

(3) 若$a, b \in \mathcal{A}_h$且$c \in \mathcal{A}'_+$, 则$a \preceq b$可推出$ca \preceq cb$, 其中$\mathcal{A}'_+ = \mathcal{A}_+ \cap \mathcal{A}'$.

定义 1.2. 设X是集合. 若映射$d \colon X \times X \to \mathcal{A}$满足:

(1) $0_{\mathcal{A}} \preceq d(x, y)$, $x, y \in X$;

(2) $d(x, y) = 0_{\mathcal{A}}$当且仅当$x = y$;

(3) $d(x, y) = d(y, x)$, $x, y \in X$;

(4) $d(x, y) \preceq d(x, z) + d(z, y)$, $x, y, z \in X$;

则称d是X上的C^*-代数值度量, 并称$(X, \mathcal{A}, d)$是C^*-代数值度量空间.

定义 1.3. 设$(X, \mathcal{A}, d)$是C^*-代数值度量空间, $x \in X$, $\{x_n\}_{n=1}^{\infty}$是X中的序列. 若$d(x_n, x) \xrightarrow{\|\cdot\|_{\mathcal{A}}} 0_{\mathcal{A}}$ $(n \to \infty)$, 则称$\{x_n\}$收敛于x, 记作$\lim\limits_{n\to\infty} x_n = x$. 若对于任意正整数$p$, 有$d(x_{n+p}, x_n) \xrightarrow{\|\cdot\|_{\mathcal{A}}} 0_{\mathcal{A}}$ $(n \to \infty)$, 则称$\{x_n\}$是X中的Cauchy列.

如果X中的Cauchy列都是收敛的, 则称$(X, \mathcal{A}, d)$是完备的C^*-代数值度量空间.

显然, Banach空间是完备的C^*-代数值度量空间. 进而, C^*-代数值度量空间推广了赋范线性空间和度量空间. 一些非平凡的C^*-代数值度量空间的例子可参见文献 [20].

2. 主要结论

我们在这部分将给出C^*-代数值度量空间中不同压缩映射的公共耦合不动点定理. 首先给出一些必要的定义.

定义 2.1. [17] 设X是非空集合, $(x, y) \in X \times X$.

(1) 若映射$F \colon X \times X \to X$满足$F(x, y) = x$和$F(y, x) = y$, 则称$(x, y)$是映射$F$的耦合不动点.

(2) 若映射$F \colon X \times X \to X$和$g \colon X \to X$满足$F(x, y) = gx$和$F(y, x) = gy$, 则称$(x, y)$是映射$F$和$g$的耦合重合点. 此时, (gx, gy)称为映射F和g的重合耦合点.

(3) 若映射$F \colon X \times X \to X$和$g \colon X \to X$满足$F(x, y) = gx = x$和$F(y, x) = gy = y$, 则称$(x, y)$是映射$F$和$g$的公共耦合不动点.

定义 2.2. [17] 若映射$F\colon X\times X\to X$和$g\colon X\to X$满足：当$gx=F(x,y)$，$gy=F(y,x)$时，有$g(F(x,y))=F(gx,gy)$，则称映射F和g是w-相容的.

定理 2.3. *设$(X,\mathcal{A},d)$是完备的C^*-代数值度量空间. 映射$F\colon X\times X\to X$和$g\colon X\to X$满足:*

$$d(F(x,y),F(u,v))\preceq a^*d(gx,gu)a+a^*d(gy,gv)a,\ \forall x,y,u,v\in X, \tag{2.1}$$

其中$a\in\mathcal{A}$且$\|a\|<\frac{1}{\sqrt{2}}$. 若$F(X\times X)\subseteq g(X)$，$g(X)$在$X$中是完备的，则$F$和$g$有耦合重合点. 进而，若$F$和$g$是$w$-相容的，则$F$和$g$在$X$中有唯一的公共耦合不动点.

证明: X中取两点x_0，y_0，令$g(x_1)=F(x_0,y_0)$，$g(y_1)=F(y_0,x_0)$. 可得两个序列$\{x_n\}$和$\{y_n\}$满足$g(x_{n+1})=F(x_n,y_n)$和$g(y_{n+1})=F(y_n,x_n)$. 利用(2.1)，可知

$$\begin{aligned} d(gx_n,gx_{n+1}) &= d(F(x_{n-1},y_{n-1}),F(x_n,y_n)) \\ &\preceq a^*d(gx_{n-1},gx_n)a+a^*d(gy_{n-1},gy_n)a \\ &\preceq a^*(d(gx_{n-1},gx_n)+d(gy_{n-1},gy_n))a. \end{aligned} \tag{2.2}$$

同理

$$\begin{aligned} d(gy_n,gy_{n+1}) &= d(F(y_{n-1},x_{n-1}),F(y_n,x_n)) \\ &\preceq a^*d(gy_{n-1},gy_n)a+a^*d(gx_{n-1},gx_n)a \\ &\preceq a^*(d(gy_{n-1},gy_n)+d(gx_{n-1},gx_n))a. \end{aligned} \tag{2.3}$$

令

$$\delta_n=d(gx_n,gx_{n+1})+d(gy_n,gy_{n+1}),$$

结合(2.2)和(2.3)，可知

$$\begin{aligned} \delta_n &= d(gx_n,gx_{n+1})+d(gy_n,gy_{n+1}) \\ &\preceq a^*(d(gx_{n-1},gx_n)+d(gy_{n-1},gy_n))a+a^*(d(gy_{n-1},gy_n)+d(gx_{n-1},gx_n))a \\ &\preceq (\sqrt{2}a)^*(d(gx_{n-1},gx_n)+d(gy_{n-1},gy_n))(\sqrt{2}a) \\ &\preceq (\sqrt{2}a)^*\delta_{n-1}(\sqrt{2}a), \end{aligned}$$

又因为当$b,c\in\mathcal{A}_h$时，$b\preceq c$可推出$a^*ba\preceq a^*ca$ (定理2.2.5 [21])，故对任意$n\in\mathbb{N}$，有

$$0_{\mathcal{A}}\preceq\delta_n\preceq(\sqrt{2}a)^*\delta_{n-1}(\sqrt{2}a)\preceq\cdots\preceq[(\sqrt{2}a)^*]^n\delta_0(\sqrt{2}a)^n.$$

若$\delta_0=0_{\mathcal{A}}$，由定义2.1(2)可知F和g有耦合重合点(x_0,y_0). 若$0_{\mathcal{A}}\preceq\delta_0$，对于$n\in\mathbb{N}$，$p\in\mathbb{N}$，有

$$\begin{aligned} d(gx_{n+p},gx_n) &\preceq d(gx_{n+p},gx_{n+p-1})+d(gx_{n+p-1},gx_{n+p-2})+\cdots+d(gx_{n+1},gx_n), \\ d(gy_{n+p},gy_n) &\preceq d(gy_{n+p},gy_{n+p-1})+d(gy_{n+p-1},gy_{n+p-2})+\cdots+d(gy_{n+1},gy_n). \end{aligned}$$

因此

$$\begin{aligned} d(gx_{n+p},gx_n)+d(gy_{n+p},gy_n) &\preceq \delta_{n+p-1}+\delta_{n+p-2}+\cdots+\delta_n \\ &\preceq \sum_{k=n}^{n+p-1}[(\sqrt{2}a)^*]^k\delta_0(\sqrt{2}a)^k, \end{aligned}$$

从而

$$\|d(gx_{n+p},gx_n)+d(gy_{n+p},gy_n)\| \leq \sum_{k=n}^{n+p-1}\|\sqrt{2}a\|^{2k}\delta_0 \leq \sum_{k=n}^{\infty}\|\sqrt{2}a\|^{2k}\delta_0=\frac{\|\sqrt{2}a\|^{2n}}{1-\|\sqrt{2}a\|^2}\delta_0.$$

又$\|a\|<\frac{1}{\sqrt{2}}$, 故

$$\|d(gx_{n+p},gx_n)+d(gy_{n+p},gy_n)\| \leq \frac{\|\sqrt{2}a\|^{2n}}{1-\|\sqrt{2}a\|^2}\delta_0 \to 0,$$

结合$d(gx_{n+p},gx_n)\preceq d(gx_{n+p},gx_n)+d(gy_{n+p},gy_n)$, $d(gy_{n+p},gy_n)\preceq d(gx_{n+p},gx_n)+d(gy_{n+p},gy_n)$, 可知$\{gx_n\}$和$\{gy_n\}$都是$g(X)$的Cauchy列. 由$g(X)$的完备性知, 存在$x,y\in X$使得$\lim\limits_{n\to\infty}gx_n=gx$, $\lim\limits_{n\to\infty}gy_n=gy$. 下面证明$F(x,y)=gx$, $F(y,x)=gy$. 因为

$$\begin{aligned} d(F(x,y),gx) &\preceq d(F(x,y),gx_{n+1})+d(gx_{n+1},gx) \\ &\preceq d(F(x,y),F(x_n,y_n))+d(gx_{n+1},gx) \\ &\preceq a^*d(gx_n,gx)a+a^*d(gy_n,gy)a+d(gx_{n+1},gx). \end{aligned}$$

在上式中令$n\to\infty$, $d(F(x,y),gx)=0_{\mathcal{A}}$, 从而$F(x,y)=gx$. 同理$F(y,x)=gy$. 因此$F$和$g$有耦合重合点$(x,y)$.

设(x',y')也是F和g的耦合重合点, 则

$$d(gx,gx')=d(F(x,y),F(x',y'))\preceq a^*d(gx,gx')a+a^*d(gy,gy')a,$$

$$d(gy,gy')=d(F(y,x),F(y',x'))\preceq a^*d(gy,gy')a+a^*d(gx,gx')a,$$

两式相加, 得

$$d(gx,gx')+d(gy,gy')\preceq(\sqrt{2}a)^*(d(gx,gx')+d(gy,gy'))(\sqrt{2}a),$$

从而

$$\|d(gx,gx')+d(gy,gy')\|\leq\|\sqrt{2}a\|^2\|d(gx,gx')+d(gy,gy')\|.$$

又$\|\sqrt{2}a\|<1$, 故$\|d(gx,gx')+d(gy,gy')\|=0$. 从而$gx=gx'$, $gy=gy'$. 同理可证$gx=gy'$, $gy=gx'$. 说明F和g有唯一的重合耦合点(gx,gx). 令$v=gx$, 则$v=gx=F(x,x)$. 由于F和g是w-相容的,

$$gv = g(gx) = g(F(x,x)) = F(gx,gx) = F(v,v),$$

故(gv,gv)是F和g的重合耦合点. 由重合耦合点的唯一性, 可知$gv = gx$, 从而$v = gv = F(v,v)$. 因此F和g有唯一的公共耦合不动点(v,v). □

注 2.4. 设$X = \mathbb{R}$, $\mathcal{A} = M_2(\mathbb{C})$, 定义映射$d\colon X \times X \to \mathcal{A}$为

$$d(x,y) = \begin{bmatrix} |x-y| & 0 \\ 0 & k|x-y| \end{bmatrix},$$

其中k为正常数. 则$(X,\mathcal{A},d)$是完备的C^*-代数值度量空间. 映射$F\colon X \times X \to X$和$g\colon X \to X$分别定义为$F(x,y) = \frac{x+y}{2}$, $g(x) = 2x$. 设$\lambda \in \mathbb{C}$且$|\lambda| < \frac{1}{\sqrt{2}}$, 令$a = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, 则$a \in \mathcal{A}$且$\|a\|_\infty = |\lambda|$. 可以验证$F$满足下面的压缩条件:

$$d(F(x,y),F(u,v)) \preceq a^*d(x,u)a + a^*d(y,v)a,\ \forall x,y,u,v \in X.$$

此时, $(0,0)$是F和g的耦合重合点. 又由于F和g是w-相容的, 故F和g的耦合重合点是唯一的.

推论 2.5. *设$(X,\mathcal{A},d)$是完备的C^*-代数值度量空间. 若映射$F\colon X \times X \to X$满足:*

$$d(F(x,y),F(u,v)) \preceq a^*d(x,u)a + a^*d(y,v)a,\ \forall x,y,u,v \in X,$$

其中$a \in \mathcal{A}$且$\|a\| < \frac{1}{\sqrt{2}}$. 则$F$有唯一的耦合不动点.

定理 2.6. *设$(X,\mathcal{A},d)$是完备的C^*-代数值度量空间. 映射$F\colon X \times X \to X$和$g\colon X \to X$满足:*

$$d(F(x,y),F(u,v)) \preceq ad(F(x,y),gx) + bd(F(u,v),gu),\ \forall x,y,u,v \in X, \tag{2.4}$$

其中$a,b \in \mathcal{A}'_+$且$\|a\| + \|b\| < 1$. 若$F(X \times X) \subseteq g(X)$, $g(X)$在X中完备, 则F和g有耦合重合点. 进而, 若F和g是w-相容的, 则F和g的耦合重合点是唯一的.

证明: 类似于定理2.3, 构造X 中的序列$\{x_n\}$和$\{y_n\}$满足$gx_{n+1} = F(x_n,y_n)$, $gy_{n+1} = F(y_n,x_n)$. 利用(2.4)式, 可得

$$\begin{aligned}(1_\mathcal{A} - b)d(gx_n,gx_{n+1}) &\preceq ad(gx_n,gx_{n-1}),\\ (1_\mathcal{A} - b)d(gy_n,gy_{n+1}) &\preceq ad(gy_n,gy_{n-1}).\end{aligned}$$

由于$a,b \in \mathcal{A}'_+$ 且$\|a\| + \|b\| < 1$, 故$1_\mathcal{A} - b$是可逆的, $(1_\mathcal{A} - b)^{-1}a \in \mathcal{A}'_+$. 因此

$$\begin{aligned}d(gx_n,gx_{n+1}) &\preceq (1_\mathcal{A} - b)^{-1}ad(gx_n,gx_{n-1}),\\ d(gy_n,gy_{n+1}) &\preceq (1_\mathcal{A} - b)^{-1}ad(gy_n,gy_{n-1}).\end{aligned}$$

从而

$$\begin{aligned}\|d(gx_n,gx_{n+1})\| &\preceq \|(1_\mathcal{A} - b)^{-1}a\|\|d(gx_n,gx_{n-1})\|,\\ \|d(gy_n,gy_{n+1})\| &\preceq \|(1_\mathcal{A} - b)^{-1}a\|\|d(gy_n,gy_{n-1})\|.\end{aligned}$$

结合

$$\|(1_{\mathcal{A}}-b)^{-1}a\| \leq \|(1_{\mathcal{A}}-b)^{-1}\|\|a\| \leq \sum_{k=0}^{\infty}\|b\|^k\|a\| = \frac{\|a\|}{1-\|b\|} < 1$$

可知$\{gx_n\}$和$\{gy_n\}$都是$g(X)$中的Cauchy列. 再由$g(X)$的完备性, 存在$x, y \in X$使得$\lim_{n\to\infty} gx_n = gx$, $\lim_{n\to\infty} gy_n = gy$. 因为

$$\begin{aligned} d(F(x,y),gx) &\preceq d(gx_{n+1},F(x,y)) + d(gx_{n+1},gx) \\ &= d(F(x_n,y_n),F(x,y)) + d(gx_{n+1},gx) \\ &\preceq ad(F(x_n,y_n),gx_n) + bd(F(x,y),gx) + d(gx_{n+1},gx) \\ &\preceq ad(gx_{n+1},gx_n) + bd(F(x,y),gx) + d(gx_{n+1},gx), \end{aligned}$$

所以

$$d(F(x,y),gx) \preceq (1-b)^{-1}ad(gx_{n+1},gx_n) + (1-b)^{-1}d(gx_{n+1},gx).$$

从而$d(F(x,y),gx) = 0_{\mathcal{A}}$, 即$F(x,y) = gx$. 类似可证$F(y,x) = gy$.

设(x', y')也是F和g的耦合重合点, 由(2.4)式可知

$$\begin{aligned} d(gx',gx) &\preceq d(F(x',y'),F(x,y)) \\ &\preceq ad(F(x',y'),gx') + bd(F(x,y),gx) = 0_{\mathcal{A}}, \end{aligned}$$

和

$$\begin{aligned} d(gy',gy) &\preceq d(F(y',x'),F(y,x)) \\ &\preceq ad(F(y',x'),gy') + bd(F(y,x),gy) = 0_{\mathcal{A}}, \end{aligned}$$

说明$gx' = gx$, $gy' = gy$. 类似可得$gx' = gy$, $gy' = gx$. 因此F和g有唯一的重合耦合点(gx, gx). 进而说明F和g有唯一的公共耦合不动点. □

推论 2.7. 设$(X, \mathcal{A}, d)$是完备的C^*-代数值度量空间. 映射$F\colon X\times X\to X$和$g\colon X\to X$满足:

$$d(F(x,y),F(u,v)) \preceq ad(F(x,y),gx) + ad(F(u,v),gu),\ \forall x,y,u,v \in X,$$

其中$a \in \mathcal{A}'_+$且$\|a\| < \frac{1}{2}$. 若$F(X\times X) \subseteq g(X)$, $g(X)$在X中是完备的, 则F和g有耦合重合点.

推论 2.8. 设$(X, \mathcal{A}, d)$是完备的C^*-代数值度量空间. 若映射$F\colon X\times X\to X$满足:

$$d(F(x,y),F(u,v)) \preceq ad(F(x,y),x) + bd(F(u,v),u),\ \forall x,y,u,v \in X,$$

其中$a, b \in \mathcal{A}'_+$且$\|a\| + \|b\| < 1$. 则F有唯一的耦合不动点.

定理 2.9. 设$(X,\mathcal{A},d)$是完备的C^*-代数值度量空间. 映射$F\colon X\times X\to X$和$g\colon X\to X$满足:

$$d(F(x,y),F(u,v))\preceq ad(F(x,y),gu)+bd(F(u,v),gx),\ \forall x,y,u,v\in X, \tag{2.5}$$

其中$a,b\in\mathcal{A}'_+$且$\|a\|+\|b\|<1$. 若$F(X\times X)\subseteq g(X)$, $g(X)$在X中是完备的, 则F和g有耦合重合点. 进而, 若F和g是w-相容, 则F和g的耦合重合点是唯一的.

证明: 类似于定理2.3, 构造X中的序列$\{gx_n\}$和$\{gy_n\}$满足$gx_{n+1}=F(x_n,y_n)$, $gy_{n+1}=F(y_n,x_n)$. 由(2.5)式, 可知

$$\begin{aligned}
d(gx_n,gx_{n+1}) &= d(F(x_{n-1},y_{n-1}),F(x_n,y_n))\\
&\preceq ad(F(x_{n-1},y_{n-1}),gx_n)+bd(F(x_n,y_n),gx_{n-1})\\
&\preceq bd(gx_{n+1},gx_{n-1})\\
&\preceq bd(gx_{n+1},gx_n)+bd(gx_n,gx_{n-1}),
\end{aligned}$$

从而

$$(1_{\mathcal{A}}-b)d(gx_n,gx_{n+1})\preceq bd(gx_n,gx_{n-1}). \tag{2.6}$$

由(2.5)式的对称性可得

$$\begin{aligned}
d(gx_{n+1},gx_n) &= d(F(x_n,y_n),F(x_{n-1},y_{n-1}))\\
&\preceq ad(F(x_n,y_n),gx_{n-1})+bd(F(x_{n-1},y_{n-1}),gx_n)\\
&\preceq ad(gx_{n+1},gx_{n-1})\\
&\preceq ad(gx_{n+1},gx_n)+ad(gx_n,gx_{n-1}),
\end{aligned}$$

即

$$(1_{\mathcal{A}}-a)d(gx_n,gx_{n+1})\preceq ad(gx_n,gx_{n-1}). \tag{2.7}$$

结合(2.6)和(2.7)可知

$$\left(1_{\mathcal{A}}-\frac{a+b}{2}\right)d(gx_n,gx_{n+1})\preceq\frac{a+b}{2}d(gx_n,gx_{n-1}).$$

由于$a,b\in\mathcal{A}'_+$且$\|a+b\|\le\|a\|+\|b\|<1$, 故$\left(1_{\mathcal{A}}-\frac{a+b}{2}\right)^{-1}\in\mathcal{A}'_+$, 再结合引理1.1(3)可知

$$d(gx_n,gx_{n+1})\preceq\left(1_{\mathcal{A}}-\frac{a+b}{2}\right)^{-1}\frac{a+b}{2}d(gx_n,gx_{n-1}).$$

令$t=\left(1_{\mathcal{A}}-\frac{a+b}{2}\right)^{-1}\frac{a+b}{2}$, 则$\|t\|=\|\left(1_{\mathcal{A}}-\frac{a+b}{2}\right)^{-1}\frac{a+b}{2}\|<1$. 类似于定理2.6可证$\{gx_n\}$是$g(X)$中的Cauchy列. 同理$\{gy_n\}$也是$g(X)$中的Cauchy列. 由$g(X)$的完备性, 可知存在$x,y\in X$, 使得$\lim\limits_{n\to\infty}gx_n=$

$gx, \lim_{n\to\infty} gy_n = gy$. 下证$F(x,y) = gx$, $F(y,x) = gy$. 由于

$$\begin{aligned} d(F(x,y),gx) &\preceq d(gx_{n+1},F(x,y)) + d(gx_{n+1},gx) \\ &= d(F(x_n,y_n),F(x,y)) + d(gx_{n+1},gx) \\ &\preceq ad(F(x_n,y_n),gx) + bd(F(x,y),gx_n) + d(gx_{n+1},gx) \\ &\preceq ad(gx_{n+1},gx) + bd(F(x,y),gx_n) + d(gx_{n+1},gx), \end{aligned}$$

从而

$$\|d(F(x,y),gx)\| \leq \|a\|\|d(gx_{n+1},gx)\| + \|b\|\|d(F(x,y),gx_n)\| + \|d(gx_{n+1},gx)\|.$$

由范数和度量的连续性, 可知

$$\|d(F(x,y),gx)\| \leq \|b\|\|d(F(x,y),gx)\|.$$

再由$\|b\| < 1$可知$\|d(F(x,y),gx)\| = 0$. 因此$F(x,y) = gx$. 同理$F(y,x) = gy$. 从而(x,y)是F和g的耦合重合点. 类似于定理2.6的证明, 可得F和g的公共耦合不动点是唯一的. □

推论 2.10. *设$(X,\mathcal{A},d)$是完备的C^*-代数值度量空间. 映射$F\colon X\times X\to X$满足:*

$$d(F(x,y),F(u,v)) \preceq ad(F(x,y),gu) + ad(F(u,v),gx),\ \forall x,y,u,v\in X,$$

其中$a\in\mathcal{A}'_+$且$\|a\| < \frac{1}{2}$. 若$F(X\times X)\subseteq g(X)$, $g(X)$在X中是完备的, 则F和g有耦合重合点.

推论 2.11. *设$(X,\mathcal{A},d)$是完备的C^*-代数值度量空间. 若映射$F\colon X\times X\to X$满足:*

$$d(F(x,y),F(u,v)) \preceq ad(F(x,y),u) + bd(F(u,v),x),\ \forall x,y,u,v\in X,$$

其中$a,b\in\mathcal{A}'_+$且$\|a\| + \|b\| < 1$. 则F有唯一的耦合不动点.

偏序度量空间上的耦合不动点定理在积分方程、周期边值问题等方面有着广泛的应用, 因此受到了许多学者的关注, 可参见文献 [2,9,15]. 下面我们将考虑Fredholm非线性积分方程解的存在唯一性问题.

设E是Lebesgue可测集且$m(E) < \infty$, $X = L^\infty(E)$表示可测集E上本性有界可测函数的全体. 考虑积分方程

$$x(t) = \int_E \Big(K_1(t,s) + K_2(t,s)\Big)\Big(f(s,x(s)) + g(s,x(s))\Big)ds + h(t),\quad t\in E. \tag{2.8}$$

定理 2.12. *设函数K_1, K_2, f和g满足:*

(i) $K_1\colon E\times E\to[0,+\infty)$, $K_2\colon E\times E\to(-\infty,0]$, $f,g\colon E\times\mathbb{R}\to\mathbb{R}$*是可测的*, $h\in L^\infty(E)$*;*

(ii) *存在*$k\in(0,\frac{1}{2})$, *使得当*$t\in E$, $x,y\in\mathbb{R}$*时, 有*

$$0 \le f(t,x) - f(t,y) \le k(x-y),$$

$$-k(x-y) \le g(t,x) - g(t,y) \le 0;$$

(iii) $\sup\limits_{t\in E} \int_E (K_1(t,s) - K_2(t,s))ds \le 1$.

则积分方程(2.8)在$L^\infty(E)$中有唯一解.

证明： 设$B(L^2(E))$是Hilbert空间$L^2(E)$上有界线性算子的全体. 定义X上的锥度量d: $X \times X \to B(L^2(E))$为

$$d(f,g) = M_{|f-g|},$$

其中$M_{|f-g|}$ 表示$L^2(E)$上的可乘算子. 显然, $(X, B(L^2(E)), d)$是完备的C^*-代数值度量空间.

对任意$x, y \in X$, $t \in E$, 定义映射F: $X \times X \to X$为

$$F(x,y)(t) = \int_E K_1(t,s)\Big(f(s,x(s)) + g(s,y(s))\Big)ds + K_2(t,s)\Big(f(s,y(s)) + g(s,x(s))\Big)ds + h(t).$$

则

$$d(F(x,y), F(u,v)) = M_{|F(x,y)-F(u,v)|}.$$

又由于

$$\begin{aligned}
& |(F(x,y) - F(u,v))(t)| \\
= & \ |\textstyle\int_E K_1(t,s)\Big(f(s,x(s)) + g(s,y(s))\Big)ds + \int_E K_2(t,s)\Big(f(s,y(s)) + g(s,x(s))\Big)ds \\
& -\textstyle\int_E K_1(t,s)\Big(f(s,u(s)) + g(s,v(s))\Big)ds - \int_E K_2(t,s)\Big(f(s,v(s)) + g(s,u(s))\Big)ds| \\
= & \ |\textstyle\int_E K_1(t,s)\Big(f(s,x(s)) - f(s,u(s)) + g(s,y(s)) - g(s,v(s))\Big)ds| \\
& +|\textstyle\int_E K_2(t,s)\Big(f(s,y(s)) - f(s,v(s)) + g(s,x(s)) - g(s,u(s))\Big)ds| \\
\le & \ \textstyle\int_E K_1(t,s)|f(s,x(s)) - f(s,u(s)) + g(s,y(s)) - g(s,v(s))|ds \\
& -\textstyle\int_E K_2(t,s)|f(s,y(s)) - f(s,v(s)) + g(s,x(s)) - g(s,u(s))|ds \\
\le & \ \sup_{s\in E}[k|x(s) - u(s)| + k|y(s) - v(s)|] \textstyle\int_E (K_1(t,s) - K_2(t,s))ds \\
\le & \ [k\|x-u\|_\infty + k\|y-v\|_\infty] \sup_{t\in E} \textstyle\int_E (K_1(t,s) - K_2(t,s))ds \\
\le & \ k\|x-u\|_\infty + k\|y-v\|_\infty.
\end{aligned}$$

因此

$$\begin{aligned}
\|d(F(x,y), F(u,v))\| &= \|M_{|F(x,y)-F(u,v)|}\| \\
&= \sup_{\|\varphi\|=1} (M_{|F(x,y)-F(u,v)|}\varphi, \varphi) \\
&= \sup_{\|\varphi\|=1} \textstyle\int_E |(F(x,y) - F(u,v))(t)|\varphi(t)\overline{\varphi(t)}dt \\
&\le \sup_{\|\varphi\|=1} \textstyle\int_E |\varphi(t)|^2 dt (k\|x-u\|_\infty + k\|y-v\|_\infty) \\
&\le k\|x-u\|_\infty + k\|y-v\|_\infty.
\end{aligned}$$

令$a = \sqrt{k}1_{B(L^2(E))}$, 那么$a \in B(L^2(E))$且$\|a\| = |\sqrt{k}| < \frac{1}{\sqrt{2}}$. 由推论2.5可得结论. □

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参考文献

[1] Aydi, H., Samet, B. and Vetro, C. (2011) Coupled Fixed Point Results in Cone Metric Spaces for Compatible Mappings. *Fixed Point Theory and Applications*, **2011**, Article No. 27. https://doi.org/10.1186/1687-1812-2011-27

[2] Bhaskar, T.G. and Lakshmikantham, V. (2006) Fixed Point Theorems in Partially Ordered Metric Spaces and Applications. *Nonlinear Analysis: Theory, Methods and Applications*, **65**, 1379-1393. https://doi.org/10.1016/j.na.2005.10.017

[3] Jleli, M. and Samet, B. (2012) On Positive Solutions for a Class of Singular Nonlinear Fractional Differential Equations. *Boundary Value Problems*, **2012**, Article No. 73. https://doi.org/10.1186/1687-2770-2012-73

[4] Nieto, J.J. and Rodriguez-Lopez, R. (2007) Existence and Uniqueness of Fixed Point in Partially Ordered Sets and Applications to Ordinary Differential Equations. *Acta Mathematica Sinica, English Series*, **23**, 2205-2212. https://doi.org/10.1007/s10114-005-0769-0

[5] Ran, A.C.M. and Reurings, M.C.B. (2004) A Fixed Point Theorem in Partially Ordered Sets and Some Applications to Matrix Equations. *Proceedings of the American Mathematical Society*, **132**, 1435-1443. https://doi.org/10.1090/S0002-9939-03-07220-4

[6] Guo, D. and Lakshmikantham, V. (1987) Coupled Fixed Points of Nonlinear Operators with Applications. *Nonlinear Analysis: Theory, Methods and Applications*, **11**, 623-632. https://doi.org/10.1016/0362-546X(87)90077-0

[7] Luong, N.V. and Thuan, N.X. (2011) Coupled Fixed Points in Partially Ordered Metric Spaces and Application. *Nonlinear Analysis: Theory, Methods and Applications*, **74**, 983-992. https://doi.org/10.1016/j.na.2010.09.055

[8] Beg, I. and Butt, A.R. (2010) Coupled Fixed Points of Set Valued Mappings in Partially Ordered Metric Spaces. *Journal of Nonlinear Sciences and Applications*, **3**, 179-185. https://doi.org/10.22436/jnsa.003.03.03

[9] Berinde, V. (2012) Coupled Fixed Point Theorems for ϕ-Contractive Mixed Monotone Mappings in Partially Ordered Metric Spaces. *Nonlinear Analysis: Theory, Methods and Applications*, **75**, 3218-3228. https://doi.org/10.1016/j.na.2011.12.021

[10] Samet, B. and Vetro, C. (2011) Coupled Fixed Point Theorems for Multi-Valued Nonlinear Contraction Mappings in Partially Ordered Metric Spaces. *Nonlinear Analysis: Theory, Methods and Applications*, **74**, 4260-4268. https://doi.org/10.1016/j.na.2011.04.007

[11] Malhotra, N. and Bansal, B. (2015) Some Common Coupled Fixed Point Theorems for Generalized Contraction in b-Metric Spaces. *Journal of Nonlinear Sciences and Applications*, **8**, 8-16. https://doi.org/10.22436/jnsa.008.01.02

[12] Huang, L. and Zhang, X. (2007) Cone Metric Spaces and Fixed Point Theorems of Contractive Mappings. *Journal of Mathematical Analysis and Applications*, **332**, 1468-1476. https://doi.org/10.1016/j.jmaa.2005.03.087

[13] Gupta, V., Shatanawi, W. and Kanwar, A. (2020) Coupled Fixed Point Theorems Employing CLR-Property on V-Fuzzy Metric Spaces. *Mathematics*, **8**, 404. https://doi.org/10.3390/math8030404

[14] Chen, J. and Huang, X. (2015) Coupled Fixed Point Theorems for Compatible Mappings in Partially Ordered G-Metric Spaces. *Journal of Nonlinear Sciences and Applications*, **8**, 130-141. https://doi.org/10.22436/jnsa.008.02.05

[15] Hussain, N., Salimi, P. and Al-Mezel, S. (2013) Coupled Fixed Point Results on Quasi-Banach Spaces with Application to a System of Integral Equations. *Fixed Point Theory and Applications*, **2013**, Article No. 261. https://doi.org/10.1186/1687-1812-2013-261

[16] Xin, Q.L. and Jiang, L.N. (2014) Fixed-Point Theorems for Mappings Satisfying the Ordered Contractive Condition on Noncommutative Spaces. *Fixed Point Theory and Applications*, **2014**, Article No. 30. https://doi.org/10.1186/1687-1812-2014-30

[17] Abbas, M., Khan, M.A. and Radenovic, S. (2010) Common Coupled Fixed Point Theorems in Cone Metric Spaces for w-Compatible Mappings. *Applied Mathematics and Computation*, **217**, 195-202. https://doi.org/10.1016/j.amc.2010.05.042

[18] Karapinar, E. (2010) Couple Fixed Point Theorems for Nonlinear Contractions in Cone Metric Spaces. *Computers and Mathematics with Applications*, **59**, 3656-3668. https://doi.org/10.1016/j.camwa.2010.03.062

[19] Sabetghadam, F., Masiha, H.P. and Sanatpour, A.H. (2009) Some Coupled Fixed Point Theorems in Cone Metric Spaces. *Fixed Point Theory and Applications*, **2009**, Article No. 125426. https://doi.org/10.1155/2009/125426

[20] Ma, Z.H., Jiang, L.N. and Sun, H.K. (2014) C^*-Algebra-Valued Metric Spaces and Related Fixed Point Theorems. *Fixed Point Theory and Applications*, **2014**, Article No. 206. https://doi.org/10.1186/1687-1812-2014-206

[21] Murphy, G.J. (1990) C^*-Algebras and Operator Theory. Academic Press, London.