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过特征多边形始末端点的四次均匀B样条曲线及其应用

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摘 要

本文构造了过控制多边形始点和终点的四次均匀B样条曲线，并以此四次均匀B样条曲线为轴线的管道光滑拼接了轴线异面管道。该方法相较于过始末端点的三次均匀B样条曲线为轴线的管道光滑拼接轴线异面管道，通过给定位于曲线上的一些点，反算出B样条曲线的特征多边形的顶点构造的B样条曲线为轴线的管道光滑拼接轴线异面的管道有其不同的特性。具有一定的应用价值。

关键词

B样条，光滑拼接，轴线异面，拼接管道

Quadratic Uniform B-Spline Curve of Passing Start Point and End Point of the Characteristic Polygon and Its Application

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Abstract

In this paper, we construct a quartic uniform B-spline curve of passing start point and end point of the control polygon. Tubes whose axes are in non-coplaner are smoothly blended by tube taking a quartic uniform B-spline curve as its axis. Compared with the tube with cubic uniform B-spline curve as the axis passing through the start and end points, the tubes whose axes are in non-coplaner are smoothly blended by the tube. By giving some points located on the curve, this method can calculate the vertices of the characteristic polygon of the B-spline curve. The constructed B-spline curve is used as the tube with axis, and tubes are smoothly blended by the tube whose axis is in non-coplaner which has different characteristics. It has certain application value.

Keywords

B-Spline, Smooth Blending, Non-Coplaner, Blending Tube

1. 引言

我们在文[1]中，根据实际问题给定 B 样条曲线的控制多边形的顶点，构造轴线异面管道的拼接管道。在文[2] [3] [4]中，假定 B 样条曲线过某些固定的点，反过来求 B 样条曲线的实际控制顶点，构造轴线异面管道的拼接曲面。两种方法具有明显的优缺点：前者，拼接管道由多段光滑拼接的管道构成，光顺性较好，应用广泛，不仅能应用于轴线异面圆管道的拼接，还能应用于椭圆管道的拼接，实际应用时加工复杂。后者，拼接管道的构造可根据需要尽量简单，由一段或两段光滑拼接的管道构成，加工容易。但是，应用范围受限，不能应用于轴线异面椭圆管道的拼接。

本文研究四次均匀 B 样条曲线过控制多边形始末端点的条件，并讨论光滑拼接轴线异面管道拼接问题。

设

$$\Phi_1:\begin{cases}x = x_1 + aN_{11}\cos\varphi + aB_{11}\sin\varphi,\\ y = y_1 + b_1 s + aN_{12}\cos\varphi + aB_{12}\sin\varphi,\\ z = aN_{13}\cos\varphi + aB_{13}\sin\varphi.\end{cases} \text{和 } \Phi_2:\begin{cases}x = aN_{21}\cos\varphi + aB_{21}\sin\varphi,\\ y = y_2 + aN_{22}\cos\varphi + aB_{22}\sin\varphi,\\ z = z_2 + c_2 s + aN_{23}\cos\varphi + aB_{23}\sin\varphi.\end{cases} \quad s\in[0,1],\varphi\in[0,2\pi]$$

是两个待拼接的管道表达式，其中 a 是管道的半径，$N_i = (N_{i1}, N_{i2}, N_{i3}), B_i = (B_{i1}, B_{i2}, B_{i3}), i = 1,2$ 分别是 $s = 1$ 和 $s = 0$ 时的法矢和副法矢。管道轴线的表达式为

$$L_1:\begin{cases}x = x_1 + 0\cdot s,\\ y = y_1 + b_1 s,\\ z = 0 + 0\cdot s,\end{cases} \text{和 } L_2:\begin{cases}x = 0 + 0\cdot s,\\ y = y_2 + b_2\cdot s,\\ z = z_2 + c_2 s.\end{cases}$$

2. 过始末端点的四次均匀 B 样条曲线

定义 1 设

$$r_i(s) = \sum_{j=0}^{4} N_{j,4}(s) V_{i+j}$$

是四次均匀 B 样条曲线段。其中 $N_{0,4}(s),N_{1,4}(s),N_{2,4}(s),N_{3,4}(s),N_{4,4}(s)$ 为 B 样条基，$V_i,V_{i+1},V_{i+2},V_{i+3},V_{i+4}$ 为特征多边形的顶点。

为了使所求 B 样条曲线过特征多边形的始点 V_0 和终点 V_n，我们在 V_0V_1 的反向延长线上取 V_{-1}，在 $V_{n-1}V_n$ 的延长线上取 V_{n+1}。在(1)中，可令 $i=0,s=0$，得

$$r_0(0)=\frac{1}{24}V_{-1}+\frac{11}{24}V_0+\frac{11}{24}V_1+\frac{1}{24}V_2,$$

并令 $\frac{1}{24}V_{-1}+\frac{11}{24}V_0+\frac{11}{24}V_1+\frac{1}{24}V_2=V_0$，解得

$$V_{-1}=13V_0-11V_1-V_2. \tag{2}$$

同理，令 $i=n-2,s=0$，得

$$r_{n-2}(0)=\frac{1}{24}V_{n-2}+\frac{11}{24}V_{n-1}+\frac{11}{24}V_n+\frac{1}{24}V_{n+1},$$

并令 $\frac{1}{24}V_{n-2}+\frac{11}{24}V_{n-1}+\frac{11}{24}V_n+\frac{1}{24}V_{n+1}=V_{n+1}$，解得

$$V_{n+1}=13V_n-11V_{n-1}-V_n. \tag{3}$$

式(2)和(3)即为四次均匀 B 样条曲线过控制多边形的始末端点的条件。

先构造过始末端点的光滑拼接轴线的四次均匀 B 样条曲线。

例 1 设 $V_0=(5,-5,0),V_1=(5,0,0),V_3=(0,5,0),V_4=(0,5,5)$，分别取 $V_2=\left(\frac{5}{2},\frac{5}{2},0\right),V_2=\left(\frac{5}{2},\frac{5}{2},1\right)$ 和 $V_2=\left(\frac{5}{2},\frac{5}{2},-1\right)$。带入公式(2)和(3)，分别得 $V_{-11}=\left(\frac{15}{2},-\frac{135}{2},1\right)$，$V_{51}=\left(-\frac{5}{2},\frac{15}{2},64\right)$，$V_{-12}=\left(\frac{15}{2},-\frac{135}{2},0\right)$，$V_{52}=\left(-\frac{5}{2},\frac{15}{2},65\right)$，$V_{-13}=\left(\frac{15}{2},-\frac{135}{2},1\right)$，$V_{53}=\left(-\frac{5}{2},\frac{15}{2},66\right)$。用以上三组点序列为顶点 $V_{-1i}V_0,V_1,V_{2i},V_3,V_4$，$V_{5i}\,(i=1,2,3)$ 的四次均匀 B 样条曲线拼接轴线异面管道的表达式为

$$r_1(s):\begin{cases} r_{11}(s)=\begin{cases} x_{11}=\frac{5}{16}s^4-\frac{5}{6}s^3-\frac{5}{6}s+5,\\ y_{11}=-\frac{35}{16}s^4+\frac{55}{6}s^3-15s^2+\frac{85}{6}s-5,s\in[0,1],\\ z_{11}=\frac{5}{24}s^4-\frac{1}{3}s^3-\frac{1}{3}s. \end{cases}\\ r_{12}(s)=\begin{cases} x_{12}=\frac{5}{12}s^3-\frac{5}{8}s^2-\frac{25}{12}s+\frac{175}{48},\\ y_{12}=-\frac{5}{24}s^4+\frac{5}{12}s^3-\frac{5}{8}s^2+\frac{35}{12}s+\frac{55}{48},s\in[0,1],\\ z_{12}=-\frac{1}{24}s^4+\frac{1}{2}s^3+\frac{1}{4}s^2-\frac{1}{2}s. \end{cases}\\ r_{13}(s)=\begin{cases} x_{13}=-\frac{5}{16}s^4+\frac{5}{12}s^3+\frac{5}{8}s^2-\frac{25}{12}s+\frac{65}{48},\\ y_{13}=\frac{5}{16}s^4-\frac{5}{12}s^3-\frac{5}{8}s^2+\frac{25}{12}s+\frac{25}{12},s\in[0,1]\\ z_{13}=\frac{25}{12}s^4+\frac{1}{3}s^3+\frac{3}{2}s^2+\frac{4}{3}s. \end{cases} \end{cases} \tag{4}$$

$$r_2(s):\begin{cases} r_{21}(s)=\begin{cases} x_{21}=\dfrac{5}{16}s^4-\dfrac{5}{6}s^3-\dfrac{5}{6}s+5,\\ y_{21}=-\dfrac{35}{16}s^4+\dfrac{55}{6}s^3-15s^2+\dfrac{85}{6}s-5, s\in[0,1]\\ z_{21}=0. \end{cases}\\ r_{22}(s)=\begin{cases} x_{22}=\dfrac{5}{12}s^4-\dfrac{5}{8}s^3-\dfrac{25}{12}s+\dfrac{175}{48},\\ y_{22}=-\dfrac{5}{24}s^4+\dfrac{5}{12}s^3-\dfrac{5}{8}s^2+\dfrac{35}{12}s+\dfrac{55}{48}, s\in[0,1]\\ z_{22}=-\dfrac{5}{24}s^4. \end{cases}\\ r_{23}(s)=\begin{cases} x_{23}=-\dfrac{5}{16}s^4+\dfrac{5}{12}s^3+\dfrac{5}{8}s^2-\dfrac{25}{12}s+\dfrac{65}{48},\\ y_{23}=\dfrac{5}{16}s^4-\dfrac{5}{12}s^3-\dfrac{5}{8}s^2+\dfrac{25}{12}s+\dfrac{175}{48}, s\in[0,1]\\ z_{23}=\dfrac{15}{8}s^4+\dfrac{5}{8}s^3+\dfrac{5}{4}s^2+\dfrac{5}{6}s+\dfrac{5}{24}. \end{cases} \end{cases} \tag{5}$$

$$r_3(s):\begin{cases} r_{31}(s)=\begin{cases} x_{31}=\dfrac{5}{16}s^4-\dfrac{5}{6}s^3-\dfrac{5}{6}s+5,\\ y_{31}=-\dfrac{35}{16}s^4+\dfrac{55}{6}s^3-15s^2-\dfrac{85}{6}s-5, s\in[0,1]\\ z_{31}=-\dfrac{5}{24}s^4+\dfrac{1}{3}s^3+\dfrac{1}{3}s. \end{cases}\\ r_{32}(s)=\begin{cases} x_{32}=\dfrac{5}{12}s^4-\dfrac{5}{8}s^2-\dfrac{25}{12}s+\dfrac{175}{48},\\ y_{32}=-\dfrac{5}{24}s^4+\dfrac{5}{12}s^3-\dfrac{5}{8}s^2+\dfrac{35}{12}s+\dfrac{55}{48}, s\in[0,1]\\ y_{32}=\dfrac{11}{24}s^4-\dfrac{1}{2}s^3-\dfrac{1}{4}s^2-\dfrac{1}{2}s. \end{cases}\\ r_{33}(s)=\begin{cases} x_{33}=-\dfrac{5}{16}s^4+\dfrac{5}{12}s^3+\dfrac{5}{8}s^2-\dfrac{25}{12}s+\dfrac{65}{48},\\ y_{33}=\dfrac{5}{16}s^4-\dfrac{5}{12}s^3-\dfrac{5}{8}s^2+\dfrac{25}{12}s+\dfrac{175}{48}, s\in[0,1]\\ z_{33}=\dfrac{5}{3}s^4+\dfrac{4}{3}s^3+s^2+\dfrac{1}{3}s+\dfrac{2}{3}. \end{cases} \end{cases} \tag{6}$$

是三条三段连续的四次 B 样条曲线。$r_1(s),r_2(s),r_3(s)$ 分别与异面轴线拼接效果如图 1：

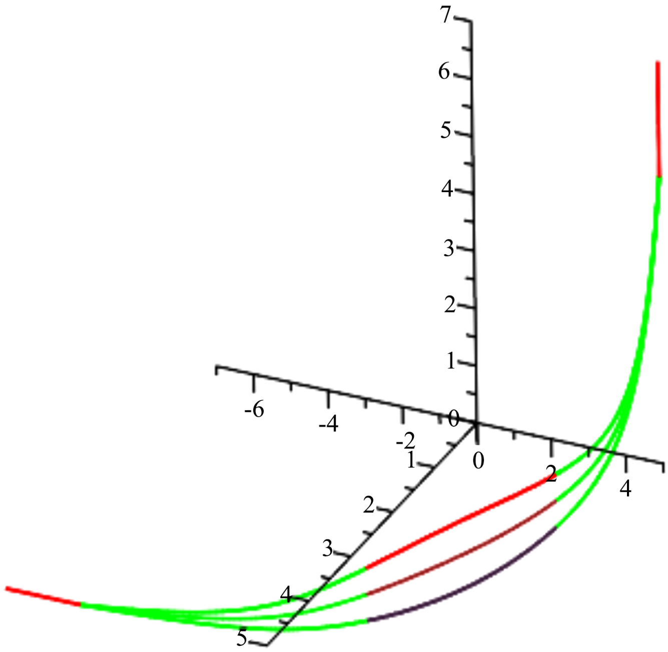

Figure 1. When V_2 takes $V_2=\left(\frac{5}{2},\frac{5}{2},1\right),V_2=\left(\frac{5}{2},\frac{5}{2},0\right)$ and $V_2=\left(\frac{5}{2},\frac{5}{2},-1\right)$ respectively, the effect diagram

图 1. V_2 **分别取** $V_2=\left(\frac{5}{2},\frac{5}{2},1\right),V_2=\left(\frac{5}{2},\frac{5}{2},0\right)$ **和** $V_2=\left(\frac{5}{2},\frac{5}{2},-1\right)$ **时，与异面轴线拼接效果图**

由例 1 可以看出：

1) 用过始末端点的四次均匀 B 样条曲线光滑拼接异面轴线，与[1]一样也需要三段曲线；

2) 曲线位置与 V_2 的位置相关，V_2 在上，拼接曲线位于上侧；V_2 居中，拼接曲线在中间；V_2 在下边，拼接曲线在下侧；

3) 从拼接效果看，轴线拼接效果没有太大差异，光滑度很好。

3. 过始末端点的四次均匀 B 样条曲线及其在轴线异面管道拼接中的应用

基于轴线光滑拼接的异面管道拼接方法，在[5] [6] [7] [8] [9]中进行了详细讨论。下面考察过控制多边形始末端点的四次均匀 B 样条曲线在管道拼接中的效果。

例 2 设

$$\Phi_1:\begin{cases}x=5+N_{11}\cos\varphi+B_{11}\sin\varphi,\\ y=-5+s+N_{12}\cos\varphi+B_{12}\sin\varphi,\\ z=N_{13}\cos\varphi+B_{13}\sin\varphi.\end{cases} \text{和 } \Phi_2:\begin{cases}x=N_{21}\cos\varphi+B_{21}\sin\varphi,\\ y=5+N_{22}\cos\varphi+B_{22}\sin\varphi,\\ z=5+s+N_{23}\cos\varphi+B_{23}\sin\varphi.\end{cases} \quad s\in[0,1],\varphi\in[0,2\pi]$$

构造以轴线 $r_2(s)$ 为轴线的圆管道 $\Phi_2(s,\varphi)$：

$$\Phi_{21}(s,\varphi):\begin{cases}x_1=x_{21}+N_{11}(s)\cos\varphi+B_{11}(s)\sin\varphi,\\ y_1=y_{21}+N_{12}(s)\cos\varphi+B_{12}(s)\sin\varphi, s\in[0,1],\varphi\in[0,2\pi]\\ z_1=z_{21}+N_{13}(s)\cos\varphi+B_{13}(s)\sin\varphi.\end{cases}$$

$$\Phi_{22}(s,\varphi):\begin{cases} x_2 = x_{22} + N_{21}(s)\cos\varphi + B_{21}(s)\sin\varphi, \\ y_2 = y_{22} + N_{22}(s)\cos\varphi + B_{22}(s)\sin\varphi, s\in[0,1], \varphi\in[0,2\pi] \\ z_2 = z_{22} + N_{23}(s)\cos\varphi + B_{23}(s)\sin\varphi. \end{cases}$$

$$\Phi_{23}(s,\varphi):\begin{cases} x_3 = x_{23} + N_{31}(s)\cos\varphi + B_{31}(s)\sin\varphi, \\ y_3 = y_{23} + N_{32}(s)\cos\varphi + B_{32}(s)\sin\varphi, s\in[0,1], \varphi\in[0,2\pi] \\ z_3 = z_{23} + N_{33}(s)\cos\varphi + B_{33}(s)\sin\varphi. \end{cases}$$

其中，$\Phi_{21}(s,\varphi),\Phi_{23}(s,\varphi),\Phi_{23}(s,\varphi)$ 是三段光滑拼接的圆管道，$N_i=(N_{i1}(s),N_{i2}(s),N_{i3}(s))$ 和 $B_i=(B_{i1}(s),B_{i2}(s),B_{i3}(s)), i=1,2,3$ 是构造的 B 样条曲线的主法矢和副法矢。其拼接效果如图 2：

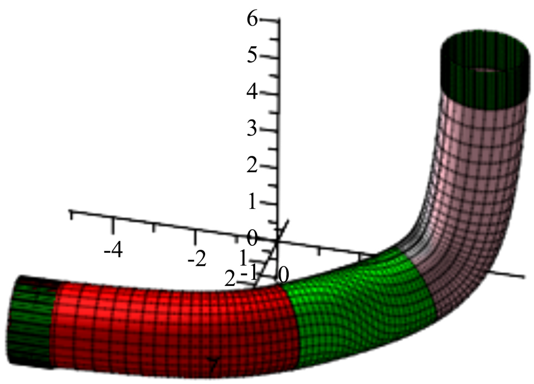

Figure 2. When $V_2=\left(\frac{5}{2},\frac{5}{2},0\right)$, circular pipe stitching effect drawing

图 2. $V_2=\left(\frac{5}{2},\frac{5}{2},0\right)$ 时，圆管道拼接效果图

例 3 设

$$\Phi_1:\begin{cases} x = 5 + N_{11}\cos\varphi + 0.6B_{11}\sin\varphi, \\ y = -5 + s + N_{12}\cos\varphi + 0.6B_{12}\sin\varphi, \\ z = N_{13}\cos\varphi + 0.6B_{13}\sin\varphi. \end{cases} \text{和}\ \Phi_2:\begin{cases} x = N_{21}\cos\varphi + 0.6B_{21}\sin\varphi, \\ y = 5 + N_{22}\cos\varphi + 0.6B_{22}\sin\varphi, \quad s\in[0,1], \varphi\in[0,2\pi] \\ z = 5 + s + N_{23}\cos\varphi + 0.6B_{23}\sin\varphi. \end{cases}$$

构造以轴线 $r_2(s)$ 为轴线的椭圆管道 $\Phi_2(s,\varphi)$：

$$\Phi_{21}(s,\varphi):\begin{cases} x_1 = x_{21} + N_{11}(s)\cos\varphi + 0.6B_{11}(s)\sin\varphi, \\ y_1 = y_{21} + N_{12}(s)\cos\varphi + 0.6B_{12}(s)\sin\varphi, s\in[0,1], \varphi\in[0,2\pi] \\ z_1 = z_{21} + N_{13}(s)\cos\varphi + 0.6B_{13}(s)\sin\varphi. \end{cases}$$

$$\Phi_{22}(s,\varphi):\begin{cases} x_2 = x_{22} + N_{21}(s)\cos\varphi + 0.6B_{21}(s)\sin\varphi, \\ y_2 = y_{22} + N_{22}(s)\cos\varphi + 0.6B_{22}(s)\sin\varphi, s\in[0,1], \varphi\in[0,2\pi] \\ z_2 = z_{22} + N_{23}(s)\cos\varphi + 0.6B_{23}(s)\sin\varphi. \end{cases}$$

$$\Phi_{23}(s,\varphi):\begin{cases} x_3 = x_{23} + N_{31}(s)\cos\varphi + 0.6B_{31}(s)\sin\varphi, \\ y_3 = y_{23} + N_{32}(s)\cos\varphi + 0.6B_{32}(s)\sin\varphi, s\in[0,1], \varphi\in[0,2\pi] \\ z_3 = z_{23} + N_{33}(s)\cos\varphi + 0.6B_{33}(s)\sin\varphi. \end{cases}$$

其中，$\Phi_{21}(s,\varphi),\Phi_{22}(s,\varphi),\Phi_{23}(s,\varphi)$ 是三段光滑拼接的管道，$N_i=(N_{i1}(s),N_{i2}(s),N_{i3}(s))$ 和 $B_i=(B_{i1}(s),B_{i2}(s),B_{i3}(s)), i=1,2,3$ 是构造的 B 样条曲线的主法矢和副法矢，$x_{2i},y_{2i},z_{2i}\,(i=1,2,3)$ 由(5)式给出。

其拼接效果如图 3：

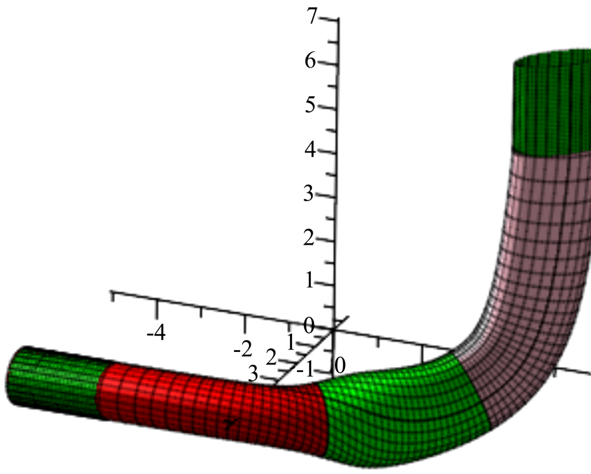

Figure 3. When $V_2=\left(\frac{5}{2},\frac{5}{2},0\right)$, the effect picture of elliptical pipeline splicing

图 3. $V_2=\left(\frac{5}{2},\frac{5}{2},0\right)$ 时，椭圆管道拼接效果图

例 4 设

$$\Phi_1:\begin{cases}x=5+N_{11}\cos\varphi+0.6B_{11}\sin\varphi,\\ y=-5+s+N_{12}\cos\varphi+0.6B_{12}\sin\varphi,\\ z=N_{13}\cos\varphi+0.6B_{13}\sin\varphi.\end{cases} \text{和 } \Phi_2:\begin{cases}x=N_{21}\cos\varphi+0.6B_{21}\sin\varphi,\\ y=5+N_{22}\cos\varphi+0.6B_{22}\sin\varphi,\\ z=5+s+N_{23}\cos\varphi+0.6B_{23}\sin\varphi.\end{cases} s\in[0,1],\varphi\in[0,2\pi]$$

构造以轴线 $r_3(s)$ 为轴线的椭圆管道 $\Phi_3(s,\varphi)$：

$$\Phi_{31}(s,\varphi):\begin{cases}x_1=x_{31}+N_{11}(s)\cos\varphi+0.6B_{11}(s)\sin\varphi,\\ y_1=y_{31}+N_{12}(s)\cos\varphi+0.6B_{12}(s)\sin\varphi, s\in[0,1],\varphi\in[0,2\pi]\\ z_1=z_{31}+N_{13}(s)\cos\varphi+0.6B_{13}(s)\sin\varphi.\end{cases}$$

$$\Phi_{32}(s,\varphi):\begin{cases}x_2=x_{32}+N_{21}(s)\cos\varphi+0.6B_{21}(s)\sin\varphi,\\ y_2=y_{32}+N_{22}(s)\cos\varphi+0.6B_{22}(s)\sin\varphi, s\in[0,1],\varphi\in[0,2\pi]\\ z_2=z_{32}+N_{23}(s)\cos\varphi+0.6B_{23}(s)\sin\varphi.\end{cases}$$

$$\Phi_{33}(s,\varphi):\begin{cases}x_3=x_{33}+N_{31}(s)\cos\varphi+0.6B_{31}(s)\sin\varphi,\\ y_3=y_{33}+N_{32}(s)\cos\varphi+0.6B_{32}(s)\sin\varphi, s\in[0,1],\varphi\in[0,2\pi]\\ z_3=z_{33}+N_{33}(s)\cos\varphi+0.6B_{33}(s)\sin\varphi.\end{cases}$$

其中，$\Phi_{31}(s,\varphi),\Phi_{32}(s,\varphi),\Phi_{33}(s,\varphi)$ 是三段光滑拼接的管道，$N_i=(N_{i1}(s),N_{i2}(s),N_{i3}(s))$ 和 $B_i=(B_{i1}(s),B_{i2}(s),B_{i3}(s)), i=1,2,3$ 是构造的 B 样条曲线的主法矢和副法矢，$x_{3i},y_{3i},z_{3i}\,(i=1,2,3)$ 由(6)式给出。

其拼接效果如图 4：

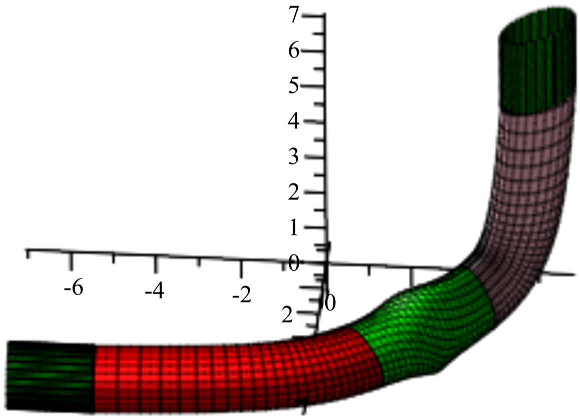

Figure 4. When $V_2=\left(\frac{5}{2},\frac{5}{2},-1\right)$, the effect picture of elliptical pipeline splicing

图 4. $V_2=\left(\frac{5}{2},\frac{5}{2},-1\right)$ 时，椭圆管道拼接效果图

由例 2、例 3 和例 4 可以看出，在 V_0,V_1,V_3,V_4 取定的情况下，用我们所构造的管道拼接轴线异面管道，还需要优化以下几点：

1) 拼接效果与 V_2 的选取有关，对于椭圆管道拼接需要优化 V_2 的选取方法；

2) 用三段连续的四次均匀 B 样条曲线为轴线的管道拼接轴线异面椭圆管道，段与段之间能够光滑拼接。要想取得视觉上的光顺效果，还需要用[8]中给出的方法，逐步旋转椭圆管道的长半轴与短半轴来实现；

3) 用三段连续的四次均匀 B 样条曲线为轴线的管道拼接轴线异面椭圆管道，对于异面椭圆管道的长半轴和短半轴的方向有特殊要求，也需要逐步旋转每段拼接椭圆管道的长半轴与短半轴来实现。

4. 结束语

过始末端点的四次 B 样条曲线为轴线的管道拼接轴异面管道需要三段管道，光顺性取决于控制顶点 V_2 的选取。对于椭圆管道拼接更为明显。需要优化 V_2 点的选取。尽管相较于插值于两端点的三次有理 B 样条曲线、插值于制定点的三次 B 样条曲线和四次 B 样条曲线段数多，但是段与段之间的光顺程度较好，且能够应用于椭圆管道的拼接。所以，比以上三种情形具有更好的应用前景。

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