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从双倍权Bergman空间A^p_ω到Bloch型空间的Volterra型积分算子

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摘 要

本文研究了从双倍权Bergman空间A^p_ω到Bloch型空间的Volterra型积分复合算子T^φ_g 和S^φ_g 的有界性和紧性问题，给出了Volterra 型积分复合算子的有界性和紧性刻画。

关键词

Volterra积分复合算子，双倍权Bergman空间，Bloch型空间，有界性，紧性

Volterra Type Operators from Bergman Spaces A^p_ω with Doubling Weights to the Bloch Type Spaces

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Abstract

We consider the boundedness and compactness of Volterra type operators from the Bergman spaces A^p_ω with exponential weights to the Bloch Space. We obtain the characterizations of the boundedness and compactness of Volterra type integral composition operators.

Keywords

Volterra Type Integral-Composition Operators, Bergman Spaces with Doubling Weights, Blcoh Type Spaces, Boundedness, Compactness

1. 引言

设$\mathbb{D}$是复平面$\mathbb{C}$上的单位圆盘, $H(\mathbb{D})$是$\mathbb{D}$上解析函数的全体.设φ是$\mathbb{D}$上的解析自映射,定义其在$H(\mathbb{D})$上诱导的复合算子C_φ为:

$$(C_\varphi f)(z) = f(\varphi(z)).$$

设$g \in H(\mathbb{D})$,Volterra型积分算子T_g和S_g定义为

$$(T_g f)(z) = \int_0^z f(\zeta)g'(\zeta)d\zeta,$$

和

$$(S_g f)(z) = \int_0^z f'(\zeta)g(\zeta)d\zeta.$$

Pommerenke [1]在1977年首先研究了Volterra积分算子T_g在H^2上的有界性问题,证明了T_g在H^2有界当且仅当$g \in BMOA$.此后,许多学者研究了其他不同空间之间的解析函数空间.另外一些学者研究了Volterra型积分复合算子T_g^φ和S_g^φ.Li在 [2]中首次引入定义,并研究了从Bergman 空间到Bloch型空

间的Volterra型积分复合算子的有界性和紧性问题.

设$g \in H(\mathbb{D})$,φ是$\mathbb{D}$上的解析自映射,两种Volterra型积分复合算子定义如下:

$$(T_g^\varphi f)(z) = \int_0^z (f \circ \varphi)(\zeta)(g \circ \varphi)'(\zeta)d\zeta,$$

和

$$(S_g^\varphi f)(z) = \int_0^z (f \circ \varphi)'(\zeta)(g \circ \varphi)(\zeta)d\zeta.$$

当$\varphi(z) = z$ 时, $T_g^\varphi = T_g$, $S_g^\varphi = S_g$; 当$g = 1$ 时, $S_g^\varphi = C_\varphi - C_\psi$, 这里$\psi(z) = \varphi(0)$.

我们称 ω 是权函数,如果ω 是$\mathbb{D}$ 上的非负可积函数. 当 $0 < p < \infty$ 时,加权Bergman 空间A_ω^p 定义为:

$$A_\omega^p = \{f \in H(\mathbb{D}) : \|f\|_{A_\omega^p}^p := \int_{\mathbb{D}} |f(z)|^p \omega(z)dA(z) < \infty\}.$$

我们称 A_ω^p 是双倍权Bergman空间, 是指对径向权 ω, 满足双倍不等式$\hat{\omega}(r) \le C\hat{\omega}(\frac{1+r}{2})$, 这里 $\hat{\omega}(r) = \int_r^1 \omega(s)ds$. 此时我们记 $\omega \in \widehat{\mathcal{D}}$. 双倍权Bergman空间的相关知识可以见 [3,4].

对于$0 < \alpha < \infty$,Bloch型空间$\mathcal{B}^\alpha$定义为由$\mathbb{D}$ 上满足条件

$$||f||_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

的解析函数f组成的集合.

本文主要研究了从双倍权Bergman空间 A_ω^p 到 Bloch 型空间 $\mathcal{B}^\alpha$ 的 Volterra 型积分复合算子T_g^φ 和 S_g^φ 的有界性和紧性刻画. 文中,符号C表示与讨论的量无关的正常数,每次出现未必相同.对于给定的两个量A和B,如果存在正常数C, 使得$\frac{1}{C}B \le A \le CB$, 我们称$A$与$B$等价,记作$A \asymp B$.

2. 预备知识

首先, 我们给出下面的结论.

引理2.1 设$0 < p < \infty$, $\omega \in \omega \in \widehat{\mathcal{D}}$. 若$f \in A_\omega^p$,则

$$|f(z)| \le C\omega(S(z))^{-\frac{1}{p}} \|f\|_{A_\omega^p}.$$

证明 设$f \in A_\omega^p$,由$|f(z)|^p$次调和性质,我们有

$$\begin{aligned} |f(z)|^p &\le \frac{\int_{\frac{1-|z|}{2}}^1 \frac{1}{2\pi}\int_0^{2\pi} |f(z+se^{i\theta})|^p d\theta\omega(s)sds}{\int_{\frac{1-|z|}{2}}^1 \omega(s)sds} \\ &\le \frac{1}{\int_{\frac{1-|z|}{2}}^1 \omega(s)sds}\|f\|_{A_\omega^p}^p \end{aligned}$$

另外一方面,

$$\begin{aligned}\int_{\frac{1-|z|}{2}}^{1}\omega(s)sds &\geq \frac{1-|z|}{2}\int_{\frac{1-|z|}{2}}^{1}\omega(s)ds\\ &\geq \frac{1-|z|}{2}\int_{\frac{1+|z|}{2}}^{1}\omega(s)ds\\ &\gtrsim \frac{1-|z|}{2}\int_{|z|}^{1}\omega(s)ds\\ &\gtrsim \omega(S(z))\end{aligned}$$

证毕.

下面引理为文献 [3]的引理2.4.

引理2.2 设$0 < p < \infty$, $\omega \in \omega \in \widehat{\mathcal{D}}$, 则存在$\lambda_0(\omega) > 0$ 使得当$\lambda \geq \lambda_0$ 时,对任意$a \in \mathbb{D}$, 解析函数$F_{a,p}(z) = \left(\frac{1-|a|^2}{1-\bar{a}z}\right)^{\frac{\lambda+1}{p}}$ 满足

$$|F_{a,p}(z)| \asymp 1,\ \ z \in S(a), a \in \mathbb{D}.$$

且

$$\|F_{a,p}(z)\|_{A^p_\omega}^p \asymp \omega(S(a)), a \in \mathbb{D}.$$

由引理2.2,我们有如下引理.

引理2.3 设$0 < p < \infty$, $\omega \in \omega \in \widehat{\mathcal{D}}$,记$f_{a,p}(z) = \omega(S(a))^{-\frac{1}{p}}\left(\frac{1-|a|^2}{1-\bar{a}z}\right)^{\frac{\lambda+1}{p}}$, 则

$$|f_{a,p}(z)| \asymp \omega(S(a))^{-\frac{1}{p}},\ \ z \in S(a), a \in \mathbb{D}.$$

且

$$\|f_{a,p}\|_{A^p_\omega} \asymp 1,$$

且$|z| \to 1$时,$f_{a,p}$在$\mathbb{D}$上的紧子集一致收敛于0.

引理2.4 设$T_g^\varphi : A^p_\omega \to \mathcal{B}^\alpha$是有界算子,则$T_g^\varphi : A^p_\omega \to \mathcal{B}^\alpha$ 是紧算子等价于对任意有界序列$\{f_n\} \subset A^p_\omega$且在$\mathbb{D}$上的任意紧子集一致收敛于0, 则

$$\lim_{n\to\infty}\|T_g^\varphi f_n\|_{\mathcal{B}^\alpha} = 0.$$

证明 充分性. 设$\{f_n\}$为A^p_ω中的任意有界序列,由引理2.1,有

$$|f_n(z)| \lesssim \omega(S(z))^{-\frac{1}{p}}\|f_n\|_{A^p_\omega}.$$

因此,$\{f_n\}$在$\mathbb{D}$的任意紧子集上一致收敛. 由Montel定理,存在$\{f_n\}$ 的子序列,不妨仍记为$\{f_n\}$,及解

析函数f,使得$\{f_n\}$在$\mathbb{D}$的紧子集一致收敛于f. 根据Fatou引理, 易得$f \in A^p_\omega$. 从而

$$\|T^\varphi_g f_n - f\|_{\mathcal{B}^\alpha} \to 0(n \to \infty).$$

故$T^\varphi_g : A^p_\omega \to \mathcal{B}^\alpha$是紧算子.

必要性. 设有界序列$\{f_n\} \subset A^p_\omega$满足在$\mathbb{D}$上的任意紧子集一致收敛于0. 由于$T^\varphi_g : A^p_\omega \to \mathcal{B}^\alpha$是紧算子,故存在$\{f_n\}$的子序列,仍记为$\{f_n\}$ 及$f \in \mathcal{B}^\alpha$, 使得

$$\lim_{n\to\infty} \|T^\varphi_g f_n - f\|_{\mathcal{B}^\alpha} = 0.$$

所以, $f(0) = 0$ 且

$$\sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f_n(\varphi(z))(g \circ \varphi)'(z) - f(z)| \to 0(n \to \infty).$$

因为$f_n(\varphi(z))(g \circ \varphi)'(z)$在$\mathbb{D}$上一致收敛于$f'(z)$. 而$\{f_n\}$在$\mathbb{D}$ 一致收敛于0,$g \in H(\mathbb{D})$, 所以有$f'(z) = 0$.故$f(z) \equiv 0$.因此,我们证明了

$$\lim_{n\to\infty} \|T_g f_n\|_{\mathcal{B}^\alpha} = 0.$$

引理2.5 设$0 < p < \infty$, $\omega \in \omega \in \widehat{\mathcal{D}}$. 若$f \in A^p_\omega$,则

$$|f'(z)| \lesssim \frac{\|f\|_{A^p_\omega}}{\omega(S(z))^{\frac{1}{p}}(1 - |z|^2)}.$$

证明 由柯西积分公式

$$f'(z) = \frac{1}{2\pi i} \int_{|\xi - z| = \frac{1-|z|}{2}} \frac{f(\xi)d\xi}{(\xi - z)^2}.$$

利用引理2.1, 并结合$\omega(S(\xi)) \geq \omega(S(\frac{1+|z|}{2})) \asymp \omega(S(z))$,即得

$$|f'(z)| \lesssim \frac{\|f\|_{A^p_\omega}}{\omega(S(z))^{\frac{1}{p}}(1 - |z|^2)}.$$

3. 从A^p_ω空间到Bloch型空间的Volterra型积分复合算子

结合以上引理得到本文的主要定理

定理3.1 假设$g \in H(\mathbb{D})$, $\omega \in \widehat{\mathcal{D}}$,则$T^\varphi_g : A^p_\omega \to \mathcal{B}^\alpha$是有界算子当且仅当

$$\sup_{z\in\mathbb{D}} (1 - |z|)^\alpha |(g \circ \varphi)'(z)| \omega(S(\varphi(z))^{-\frac{1}{p}} < \infty.$$

证明 充分性: 假设

$$\sup_{z\in\mathbb{D}} (1 - |z|)^\alpha |(g \circ \varphi)'(z)| \omega(S(\varphi(z))^{-\frac{1}{p}} < \infty.$$

令$f \in A^p_\omega$, 由引理2.1,有

$$\begin{aligned}(1-|z|^2)^\alpha|(g\circ\varphi)'(z)|f(\varphi(z))| &\lesssim (1-|z|^2)^\alpha|(g\circ\varphi)'(z)|\omega(S(\varphi(z))^{-\frac{1}{p}}\|f\|_{A^p_\omega}\\ &\lesssim (1-|z|^2)^\alpha|(g\circ\varphi)'(z)|\omega(S(\varphi(z))^{-\frac{1}{p}}.\end{aligned}$$

所以$\|T^\alpha_g f\|_{\mathcal{B}^\alpha} < \infty$. 故$T^\varphi_g : A^p_\psi \to \mathcal{B}^\alpha$是有界算子.

必要性: 假设$T^\varphi_g : A^p_\omega \to \mathcal{B}^\alpha$是有界算子. 由引理2.4和引理2.3,我们有

$$(1-|z|^2)^\alpha|(g\circ\varphi)'(z)|\omega(S(\varphi(z))^{-\frac{1}{p}} \lesssim \|T_g f_{a,p}\|_{\mathcal{B}} < \infty.$$

证毕.

推论3.1 假设$g \in H(\mathbb{D})$, $\omega \in \widehat{\mathcal{D}}$,则$T_g : A^p_\omega \to \mathcal{B}^\alpha$是有界算子当且仅当

$$\sup_{z\in\mathbb{D}}(1-|z|)^\alpha|g'(z)|\omega(S(z)^{-\frac{1}{p}} < \infty.$$

定理3.2 假设$g \in H(\mathbb{D})$, $\psi \in \mathcal{W}_0$,则$T^\varphi_g : A^p_\omega \to \mathcal{B}^\alpha$是有紧算子当且仅当

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha|(g\circ\varphi)'(z)|\omega(S(\varphi(z))^{-\frac{1}{p}} = 0.$$

证明 充分性: 假设$T^\varphi_g : A^p_\omega \to \mathcal{B}^\alpha$是紧算子. 记$f_{z,p}(w) = \omega(S(z))^{-\frac{1}{p}}\left(\frac{1-|z|^2}{1-\bar{z}w}\right)^{\frac{\lambda+1}{p}}$, 则$\|f_{z,p}\|_{A^p_\omega} \asymp 1$, 并且当$|z| \to 1$时,$f_{z,p}$在$\mathbb{D}$上的紧子集一致收敛于0. 故

$$\begin{aligned}0 = \lim_{|\varphi(z)|\to 1}\|T^\varphi_g f_{\varphi(z),p}\|_{\mathcal{B}^\alpha} &\gtrsim \lim_{|\varphi(z)|\to 1}\sup_{w\in\mathbb{D}}(1-|w|^2)^\alpha|(g\circ\varphi)'(w)||f_{\varphi(z),p}(w)|\\ &\gtrsim \lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha|(g\circ\varphi)'(z)||f_{\varphi(z),p}(\varphi(z))|\\ &\gtrsim \lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha|(g\circ\varphi)'(z)|\omega(S(\varphi(z))^{-\frac{1}{p}}.\end{aligned}$$

必要性: 假设$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha|(g\circ\varphi)'(z)|\omega(S(\varphi(z))^{-\frac{1}{p}}$, 则对任意的$\epsilon > 0$,存在$r_0 \in (0,1)$,使得$|\varphi(z)| > r_0$时,有

$$(1-|z|^2)^\alpha|(g\circ\varphi)'(z)|\omega(S(\varphi(z))^{-\frac{1}{p}} < \epsilon.$$

又因为$|\varphi(z)| \le r_0$时,显然有$(1-|z|^2)^\alpha|(g\circ\varphi)'(z)|\omega(S(\varphi(z))^{-\frac{1}{p}} < \infty$有界.从而由定理3.1, $T^\varphi_g : A^p_\omega \to \mathcal{B}^\alpha$是界算子.注意到$g\circ\varphi \in \mathcal{B}^\alpha$, 故存在正数$M$,使得

$$M := \sup_{|\varphi(z)|\le r_0}(1-|z|^2)^\alpha|(g\circ\varphi)'(z)| < \infty.$$

设$\{f_n\}$为A^p_ω中的有界序列,且在$\mathbb{D}$的紧子集上一致收敛于0.

$$\begin{aligned}\|T^\varphi_g f_n\|_{\mathcal{B}^\alpha} &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|(g\circ\varphi)'(z)||f_n(\varphi(z))|\\ &= \sup_{|\varphi(z)|\le r_0}(1-|z|^2)^\alpha|(g\circ\varphi)'(z)||f_n(\varphi(z))| + \sup_{|\varphi(z))|>r_0}(1-|z|^2)^\alpha|(g\circ\varphi)'(z)||f_n(\varphi(z))|\\ &\lesssim M\sup_{|w|\le r_0}|f_n(w)| + \sup_{|\varphi(z))|>r_0}(1-|z|^2)^\alpha|(g\circ\varphi)'(z)|S(\varphi(z))^{-\frac{1}{p}}\|f_n\|_{A^p_\omega}\\ &\lesssim M\sup_{|w|\le r_0}|f_n(w)|+\epsilon.\end{aligned}$$

因此,当$n\to\infty$时,$\|T_g f_n\|_{\mathcal{B}^\alpha}\to 0$. 所以$T^\varphi_g: A^p_\omega\to\mathcal{B}^\alpha$是有紧算子.

证毕.

推论3.2 假设$g\in H(\mathbb{D})$, $\omega\in\widehat{\mathcal{D}}$,则$T_g: A^p_\omega\to\mathcal{B}^\alpha$是紧算子当且仅当

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha|g'(z)|\omega(S(z)^{-\frac{1}{p}}<\infty.$$

定理3.3 设$0<p<\infty$,$0<\alpha<\infty$,$g\in H(\mathbb{D})$ 且φ 是$\mathbb{D}$ 上的解析自映射, $\omega\in\widehat{\mathcal{D}}$, 则$S^\varphi_g: A^p_\omega\to\mathcal{B}^\alpha$是有界算子当且仅当

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\alpha|\varphi'(z)|}{(1-|\varphi(z)|^2)\omega(S(\varphi(z)))^{\frac{1}{p}}}|g(\varphi(z))|<\infty.$$

证明 充分性: 假设

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\alpha|\varphi'(z)|}{(1-|\varphi(z)|^2)\omega(S(\varphi(z)))^{\frac{1}{p}}}|g(\varphi(z))|<\infty.$$

令$f\in A^p_\omega$, 由引理2.5,有

$$\begin{aligned}(1-|z|^2)^\alpha|g(\varphi(z))||(f\circ\varphi)'(z)| &= (1-|z|^2)^\alpha|\varphi'(z)||g(\varphi(z))||f'(\varphi(z))|\\ &\lesssim \frac{(1-|z|^2)^\alpha|\varphi'(z)||g(\varphi(z))|}{(1-|\varphi(z)|^2)\omega(S(\varphi(z)))^{\frac{1}{p}}}\|f\|_{A^p_\omega}\\ &\lesssim \frac{(1-|z|^2)^\alpha|\varphi'(z)||g(\varphi(z))|}{(1-|\varphi(z)|^2)\omega(S(\varphi(z)))^{\frac{1}{p}}}.\end{aligned}$$

所以$\|S^\varphi_g f\|_{\mathcal{B}^\alpha}<\infty$. 故$S^\varphi_g: A^p_\omega\to\mathcal{B}^\alpha$是有界算子.

必要性: 假设$S^\varphi_g: A^p_\omega\to\mathcal{B}^\alpha$是有界算子.
当$|\varphi(z)|\ge\frac{1}{2}$ 时,

$$\begin{aligned}\|S_g f_{\varphi(z),p}\|_{\mathcal{B}^\alpha} &= \sup_{w\in\mathbb{D}}(1-|w|^2)^\alpha|g(\varphi(w))||(f_{\varphi(z),p}\circ\varphi)'(w)|\\ &\ge (1-|z|^2)^\alpha|\varphi'(z)||g(\varphi(z))|f'_{\varphi(z),p}(\varphi(z))|\\ &\gtrsim \frac{(1-|z|^2)^\alpha|\varphi'(z)||g(\varphi(z))|}{(1-|\varphi(z)|^2)\omega(S(\varphi(z)))^{\frac{1}{p}}}.\end{aligned}$$

当$|\varphi(z)| < \frac{1}{2}$

$$\begin{aligned}\frac{(1-|z|^2)^\alpha|\varphi'(z)||g(\varphi(z))|}{(1-|\varphi(z)|^2)\omega(S(\varphi(z)))^{\frac{1}{p}}} &\lesssim (1-|z|^2)^\alpha|\varphi'(z)||g(\varphi(z))| \\ &\lesssim \|S_g^\varphi(z)\|_{\mathcal{B}^\alpha} < \infty.\end{aligned}$$

证毕.

推论3.3 设$0 < p < \infty$,$0 < \alpha < \infty$,且φ 是$\mathbb{D}$ 上的解析自映射, $\omega \in \widehat{\mathcal{D}}$, 则$C_\varphi : A^p_\omega \to \mathcal{B}^\alpha$ 是有界算子当且仅当

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha|\varphi'(z)|}{(1-|\varphi(z)|^2)\omega(S(\varphi(z)))^{\frac{1}{p}}} < \infty.$$

推论3.4 设$0 < p < \infty$,$0 < \alpha < \infty$,$g \in H(\mathbb{D})$, $\omega \in \widehat{\mathcal{D}}$, 则$S_g : A^p_\omega \to \mathcal{B}^\alpha$ 是有界算子当且仅当

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha-1}\omega(S(z))^{-\frac{1}{p}}|g(z)| < \infty.$$

定理3.4 设$0 < p < \infty$,$0 < \alpha < \infty$,$g \in H(\mathbb{D})$ 且φ 是$\mathbb{D}$ 上的解析自映射, $\omega \in \widehat{\mathcal{D}}$, 则$S_g^\varphi : A^p_\omega \to \mathcal{B}^\alpha$ 是紧算子当且仅当

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha|\varphi'(z)|}{(1-|\varphi(z)|^2)\omega(S(\varphi(z)))^{\frac{1}{p}}}|g(\varphi(z))| = 0.$$

证明 充分性: 假设

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha|\varphi'(z)|}{(1-|\varphi(z)|^2)\omega(S(\varphi(z)))^{\frac{1}{p}}}|g(\varphi(z))| = 0.$$

设$\{f_n\}$为A^p_ω中的有界序列,且在$\mathbb{D}$的紧子集上一致收敛于0. 对于任意固定的$0 < r < 1$. 由引理2.4和引理2.5,有

$$\begin{aligned}\|S_g^\varphi f_n\|_{\mathcal{B}^\alpha} &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|g(\varphi(z))||(f_n\circ\varphi)'(z)| \\ &= \sup_{|\varphi(z)|\le r}(1-|z|^2)^\alpha|\varphi'(z)||g(\varphi(z))||f_n'(\varphi(z))| \\ &\quad + \sup_{|\varphi(z)|> r}(1-|z|^2)^\alpha|\varphi'(z)||g(\varphi(z))||f_n'(\varphi(z))| \\ &\lesssim \sup_{|\varphi(z)|\le r}(1-|z|^2)^\alpha|\varphi'(z)||g(\varphi(z))||f_n'(\varphi(z))| \\ &\quad + \sup_{|\varphi(z)|> r}\frac{(1-|z|^2)^\alpha|\varphi'(z)||g(\varphi(z))|}{(1-|\varphi(z)|^2)\omega(S(\varphi(z)))^{\frac{1}{p}}}\|f_n\|_{A^p_\omega}.\end{aligned}$$

因为当$|\varphi(z)| \le r$时, $(1-|z|^2)^\alpha|\varphi'(z)||g(\varphi(z))| \lesssim \|S_g^\varphi(z)\|_{\mathcal{B}^\alpha} < \infty$ 且

$$\lim_{n\to\infty}\sup_{|\varphi(z)|\le r}|f_n(\varphi(z))| = 0.$$

所以令$n \to \infty$,$r \to 1$,有

$$\lim_{n\to\infty} \|S_g^{\varphi} f_n\|_{\mathcal{B}^\alpha} \lesssim \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |\varphi'(z)||g(\varphi(z))|}{(1-|\varphi(z)|^2)\omega(S(\varphi(z)))^{\frac{1}{p}}} = 0.$$

故$S_g^{\varphi}: A_\omega^p \to \mathcal{B}^\alpha$是紧算子.

必要性: 假设$S_g^{\varphi}: A_\omega^p \to \mathcal{B}^\alpha$是紧算子. 由引理2.4,可得

$$\begin{aligned} 0 &= \lim_{|\varphi(z)|\to 1} \|S_g f_{\varphi(z),p}\|_{\mathcal{B}^\alpha} \\ &= \lim_{|\varphi(z)|\to 1} \sup_{w\in\mathbb{D}} (1-|w|^2)^\alpha |g(\varphi(w))||(f_{\varphi(z),p}\circ\varphi)'(w)| \\ &\geq \lim_{|\varphi(z)|\to 1} (1-|z|^2)^\alpha |\varphi'(z)||g(\varphi(z))||f'_{\varphi(z),p}(\varphi(z))| \\ &\gtrsim \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |\varphi'(z)||g(\varphi(z))|}{(1-|\varphi(z)|^2)\omega(S(\varphi(z)))^{\frac{1}{p}}}. \end{aligned}$$

证毕.

推论3.5 设$0 < p < \infty$,$0 < \alpha < \infty$,且φ 是$\mathbb{D}$ 上的解析自映射, $\omega \in \widehat{\mathcal{D}}$, 则$C_\varphi : A_\omega^p \to \mathcal{B}^\alpha$ 是紧算子当且仅当

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |\varphi'(z)|}{(1-|\varphi(z)|^2)\omega(S(\varphi(z)))^{\frac{1}{p}}} = 0.$$

推论3.6 设$0 < p < \infty$,$0 < \alpha < \infty$,$g \in H(\mathbb{D})$, $\omega \in \widehat{\mathcal{D}}$, 则$S_g : A_\omega^p \to \mathcal{B}^\alpha$ 是紧算子当且仅当

$$\lim_{|z|\to 1} (1-|z|^2)^{\alpha-1} \omega(S(z))^{-\frac{1}{p}} |g(z)| = 0.$$

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