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Dirichlet空间上H-Toeplitz算子的交换性

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摘 要

本文基于Bergman空间上H-Toeplitz算子的研究，在这篇文章中主要研究了Dirichlet空间上H-Toeplitz算子的交换性。

关键词

H-Toeplitz算子，Dirichlet空间，交换性

Commutants of H-Toeplitz Operators on Dirichlet Space

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Abstract

Based on the research of H-Toeplitz operators on Bergman space, this article mainly studies the commutants of H-Toeplitz operators on Dirichlet space.

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Keywords

H-Toeplitz Operators, Dirichlet Space, Commutant

1. 介绍

令$\mathbb{D}$表示复平面$\mathbb{C}$上的单位圆盘, 即$\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$, Dirichlet空间$\mathcal{D}$是由所有在单位圆盘上的解析函数$f$组成, 其中$f$要满足$D(f) < \infty$, $D(f) =: \int_D |f'(w)|^2 dA(w)$是Dirichlet积分,其中$dA(w)$表正则化的勒贝格面积测度.

对于任意的$f(z) = \sum_{k=1}^{\infty} = a_k z^k, g(z) = \sum_{j=1}^{\infty} a_j z^j \in \mathcal{D}$, 空间中的内积满足:

$$\langle f, g \rangle = \int_D f'(w)\overline{g'(w)} dA(w),$$

且Dirichlet空间$\mathcal{D}$是满足以下范数

$$\|f\|^2 = D(f) = \sum_{k=1}^{\infty} k|a_k|^2$$

的解析函数Hilbert空间. 此外, $\mathcal{D}$也是一个再生核Hilbert空间, 其核为

$$K_z(w) = \log \frac{1}{1 - \overline{z}w}, \qquad w, z \in \mathbb{D},$$

即

$$f(z) = \langle f, K_z \rangle, \qquad f \in \mathcal{D}.$$

Dirichlet空间$\mathcal{D}$是Sobolev空间$L^{2,1}(\mathbb{D})$的闭子空间, 对$\forall f \in L^{2,1}(\mathbb{D})$,

$$\|f\|^2 = |\int f dA|^2 + \int_{\mathbb{D}} |\frac{\partial f}{\partial w}|^2 dA + \int_{\mathbb{D}} |\frac{\partial f}{\partial \overline{w}}|^2 dA < \infty,$$

其中求导是在分布意义下的. 而且, $L^{2,1}(\mathbb{D})$满足内积

$$\langle f,g\rangle=\int_{\mathbb{D}}fdA+\overline{\int_{\mathbb{D}}gdA}+\left\langle\frac{\partial f}{\partial w},\frac{\partial g}{\partial w}\right\rangle_2+\left\langle\frac{\partial f}{\partial \overline{w}},\frac{\partial g}{\partial \overline{w}}\right\rangle_2,$$

其中$\langle f,g\rangle=\int_{\mathbb{D}}f\overline{g}dA$.

在算子理论的研究中, 数学家们研究了多种算子, 如Toeplitz算子、Hankel算子等, 当然算子在不同的空间中也有各种各样不同的性质, 研究得最多也是最为广泛的即Hardy空间上算子的基本性质, 这也是数学家们最开始所研究的问题. 接着是在此基础上进行延伸, 比如Bergman空间, Dirichlet空间等等, 这也是被研究得最多的空间. 当然算子的性质也有很多, 比如Toeplitz算子和Hankel算子在Bergman空间中的紧性([1] [2]), 此外加权Dirichlet空间中Toeplitz算子的紧性也有被研究([3]), 还有有界性, 这两种性质是研究得较为广泛的, 随后, 也有数学家研究算子的交换性, S.Axler等人, Zhao Lian kuo分别在2000年 [4], 2008年 [5]研究了Bergman空间中解析的Toeplitz算子的交换性以及调和的Dirichlet空间中Toeplitz算子的交换性, 更多的研究结果也可参阅文献 [6] [7] [8]. 由Toeplitz算子和Hankel算子的性质可以研究H-Toeplitz 算子的一些相似性质, S.C.Arora等人在 [9]中也研究了H-Toeplitz算子的一部分内容.

对任意的正整数n, 令$e_n(z)=\frac{1}{\sqrt{n}}z^n, z\in\mathbb{D}$,则$\{e_n(z)\}_{n>0}$是Dirichlet空间中的标准正交基. 令$P:L^{2,1}(\mathbb{D})\to\mathcal{D}$表示$L^{2,1}(\mathbb{D})$到$\mathcal{D}$上的正交投影. 空间$L^{\infty}(\mathbb{D})$表示单位圆盘上所有的本质有界勒贝格可测函数$f$的$Banach$空间, 且范数满足

$$\|f\|_{\infty}=esssup\{|f(z)|:z\in\mathbb{D}\}.$$

$L^{\infty,1}(\mathbb{D})$表示$\{f\in L^{\infty,1}(\mathbb{D})|f,\frac{\partial f}{\partial z},\frac{\partial f}{\partial \overline{z}}\in L^{\infty}(\mathbb{D})\}$. 令$\mathcal{D}_{harm}$表示调和的Dirichlet空间, 则$P_{harm}:L^{2,1}(\mathbb{D})\to\mathcal{D}_{harm}$表示$L^{2,1}(\mathbb{D})$到$\mathcal{D}_{harm}$上的正交投影. 令$C(\overline{\mathbb{D}})$表示$\overline{\mathbb{D}}$上的所有连续函数, 取

$$C^1(\overline{\mathbb{D}})=\{f:f,\frac{\partial f}{\partial w},\frac{\partial f}{\partial \overline{w}}\in C(\overline{\mathbb{D}})\}$$

对任意的$\phi\in C^1(\overline{\mathbb{D}})$,定义Dirichlet空间$\mathcal{D}$上的Toeplitz算子$T_\phi$为

$$T_\phi(f)=P(\phi f)(z)=\int_{\mathbb{D}}\frac{\partial(\phi f)}{\partial w}\frac{\overline{\partial K_z}}{\partial w}dA(w),f\in\mathcal{D},$$

显然T_ϕ是有界的. 对于$\phi\in C^1(\overline{\mathbb{D}})$,Hankel算子$H_\phi$定义为

$$H_\phi(f)=PM_\phi J(f),f\in\mathcal{D},$$

其中算子$J:\mathcal{D}\to\overline{\mathcal{D}}$定义为$J(e_n(z))=\overline{e_{n+1}(z)}$, n为任意的正整数.

小Hankel算子$h_\phi(f)=P(\phi Jf),f\in\mathcal{D}$.

在这篇文章中，我们主要研究Dirichlet空间$\mathcal{D}$上的H-Toeplitz算子，在文章开始我们也找到了H-Toeplitz算子关于$\mathcal{D}$中标准正交基的矩阵，这也与Toeplitz算子和Hankel算子的矩阵有着密切的关联. 在2019年，A.Gupta等人在 [10]中介绍了Slant H-Toeplitz 算子的符号，并且研究了H-Toeplitz算子在Hardy空间中的一些性质，包括最广泛的紧性和交换性. 基于这些研究，我们在这篇文章中便来研究Dirichlet空间中H-Toeplitz算子的交换性，其符号是$L^{\infty,1}(\mathbb{D})$中的调和函数.

2. Dirichlet空间中的H-Toeplitz算子

在开始之前，我们需要给出一些引理为后面的结果做准备.

引理2.1 在调和的Dirichlet空间中，对于任意的非负整数m和n，下列式子成立:

$$P_{harm}(\overline{z}^m z^n)=\begin{cases} z^{n-m} & n>m \\ \overline{z}^{m-n} & n<m \\ \frac{1}{n+1} & m=n \end{cases}$$

证明 对$n>m$，有

$$\begin{aligned}\langle P_{harm}(\overline{z}^m z^n),z^k\rangle &= \langle \overline{z}^m z^n),z^k\rangle \\ &= \int_{\mathbb{D}}\overline{z}^m z^n dA(z)\int \overline{z}^k dA(z)+\int_{\mathbb{D}} nk\overline{z}^m z^{n-1}\overline{z}^{k-1}dA(z) \\ &= \langle z^n,z^m\rangle_2+nk\int_{\mathbb{D}} z^{n-1}\overline{z}^{m+k-1}dA(z) \\ &= nk\int_{\mathbb{D}} z^{n-1}\overline{z}^{m+k-1}dA(z) \\ &= nk\frac{1}{\pi}\int_0^{2\pi}\int_0^1 r^{m+k+n-1}e^{i(n-m-k)\theta}drd\theta \\ &= \begin{cases} n-m & k=n-m \\ 0 & 其它 \end{cases} \\ &= \begin{cases} \frac{n-m}{k}<z^k,z^k> & k=n-m \\ 0 & 其它 \end{cases} \\ &= \langle z^{n-m},z^k\rangle\end{aligned}$$

另一方面, 对$n<m$, 有

$$
\begin{aligned}
\left\langle P_{harm}(\overline{z}^m z^n), \overline{z}^k \right\rangle &= \left\langle \overline{z}^m z^n), \overline{z}^k \right\rangle \\
&= \int_{\mathbb{D}} \overline{z}^m z^n dA(z) \int z^k dA(z) + \int_{\mathbb{D}} mk\overline{z}^{m-1} z^n z^{k-1} dA(z) \\
&= \langle z^n, z^m \rangle_2 + mk \int_{\mathbb{D}} z^{n+k-1} \overline{z}^{m-1} dA(z) \\
&= mk\frac{1}{\pi}\int_0^{2\pi}\int_0^1 r^{m+k+n-1} e^{i(n+k-m)\theta} dr d\theta \\
&= \begin{cases} m-n & k=m-n \\ 0 & \text{其它} \end{cases} \\
&= \begin{cases} \frac{m-n}{k}\left\langle \overline{z}^k, \overline{z}^k \right\rangle & k=m-n \\ 0 & \text{其它} \end{cases} \\
&= \left\langle \overline{z}^{m-n}, \overline{z}^k \right\rangle
\end{aligned}
$$

除此之外, 当$m=n$时,

$$
\left\langle P_{harm}(\overline{z}^m z^n), z^k + \overline{z}^k \right\rangle = \frac{1}{n+1}
$$

综上, 可以得到

$$
P_{harm}(\overline{z}^m z^n) = \begin{cases} z^{n-m} & n>m \\ \overline{z}^{m-n} & n<m \\ \frac{1}{n+1} & m=n \end{cases}
$$

那么对于Dirichlet 空间, 由引理2.1可以直接得到.

引理2.2 　在Dirichlet空间中, 对于任意的正整数m和n, 下列式子成立

$$
(a) \quad \langle z^m, z^n \rangle = \begin{cases} m & m=n \\ 0 & \text{其它} \end{cases}
$$

$$
(b) \quad P(\overline{z}^n z^m) = \begin{cases} z^{m-n} & m>n \\ 0 & m<n \\ \frac{1}{m+1} & m=n \end{cases}
$$

下面我们去找Dirichlet空间中Toeplitz算子T_ϕ和Hankel算子H_ϕ的矩阵, 其中ϕ是调和函数. 在这里我们考虑调和符号$\phi(z)=\sum_{i=1}^{\infty} a_i z^i + \sum_{j=1}^{\infty} b_j \overline{z}^j \in L^{\infty,1}(\mathbb{D})$, 则$T_\phi$关于正交基$\{e_n\}_{n>0}$的矩阵

的(m,n)次项为

$$\begin{aligned}\langle T_\phi e_n, e_m\rangle &= \langle P(\phi e_n), e_m\rangle = \langle \phi e_n, e_m\rangle \\ &= \frac{1}{\sqrt{mn}}\langle \phi z^n, z^m\rangle \\ &= \frac{1}{\sqrt{mn}}\left\langle (\sum_{i=1}^{\infty} a_i z^i + \sum_{j=1}^{\infty} b_j \overline{z}^j) z^n, z^m\right\rangle\end{aligned}$$

(1) 当$m \geq n$时,

$$\langle T_\phi e_n, e_m\rangle = \frac{1}{\sqrt{mn}}\sum_{i=1}^{\infty}\langle a_i z^{i+n}, z^m\rangle = \sqrt{\frac{m}{n}}a_{m-n}.$$

(2) 当$m < n$时,

$$\langle T_\phi e_n, e_m\rangle = \frac{1}{\sqrt{mn}}\left\langle \sum_{j=1}^{\infty} b_j \overline{z}^j z^n, z^m\right\rangle = \sum_{j=1}^{\infty} b_j nm\left\langle \overline{z}^j z^{n-1}, z^{m-1}\right\rangle_2 = \sqrt{\frac{m}{n}}b_{n-m}.$$

因此,

$$\langle T_\phi e_n, e_m\rangle = \begin{cases}\sqrt{\frac{m}{n}}a_{m-n} & m \geq n \\ \sqrt{\frac{m}{n}}b_{n-m} & m < n\end{cases}$$

其中m和n都是正整数.
所以T_ϕ的矩阵可以由以下给出

$$T_\phi = \begin{pmatrix} a_0 & \sqrt{\frac{1}{2}}b_1 & \sqrt{\frac{1}{3}}b_2 & \sqrt{\frac{1}{4}}b_3 & \cdots \\ \sqrt{2}a_1 & a_0 & \sqrt{\frac{2}{3}}b_1 & \sqrt{\frac{1}{2}}b_2 & \cdots \\ \sqrt{3}a_2 & \sqrt{\frac{3}{2}}a_1 & a_0 & \sqrt{\frac{3}{4}}b_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

接下来我们要找有调和符号的Hankel算子的矩阵, 令$\phi(z) = \sum_{i=1}^{\infty} a_i z^i + \sum_{j=1}^{\infty} b_j \overline{z}^j \in L^{\infty,1}(\mathbb{D})$,则$H_\phi$关于$\mathcal{D}$中的标准正交基$\{e_n\}_{n>0}$的矩阵的$(m,n)$次项为

$$\begin{aligned}\langle H_\phi e_n, e_m\rangle &= \langle PM_\phi J e_n, e_m\rangle = \langle PM_\phi \overline{e_{n+1}}, e_m\rangle = \langle M_\phi \overline{e_{n+1}}, e_m\rangle \\ &= \frac{1}{\sqrt{m(n+1)}}\left\langle (\sum_{i=1}^{\infty} a_i z^i + \sum_{j=1}^{\infty} b_j \overline{z}^j)\overline{z}^{n+1}, z^m\right\rangle \\ &= \frac{1}{\sqrt{m(n+1)}}(\sum_{i=1}^{\infty} a_i\left\langle z^i \overline{z}^{n+1}, z^m\right\rangle + \sum_{j=1}^{\infty} b_j\left\langle \overline{z}^{n+j+1}, z^m\right\rangle) \\ &= \frac{1}{\sqrt{m(n+1)}}\sum_{i=1}^{\infty} a_i im\left\langle z^{i-1}, z^{m+n}\right\rangle_2 \\ &= \frac{m}{\sqrt{m(n+1)}}a_{m+n+1},\end{aligned}$$

其中m和n都为正整数.

则H_ϕ关于$\mathcal{D}$中的标准正交基$\{e_n\}_{n>0}$的矩阵为

$$H_\phi = \begin{pmatrix} \frac{1}{\sqrt{2}}a_3 & \frac{1}{\sqrt{3}}a_4 & \frac{1}{\sqrt{4}}a_5 & \cdots \\ a_4 & \frac{2}{\sqrt{6}}a_5 & \frac{1}{\sqrt{2}}a_6 & \cdots \\ \frac{3}{\sqrt{6}}a_5 & a_6 & \frac{3}{2\sqrt{3}}a_7 & \cdots \\ \sqrt{2}a_6 & \frac{3}{\sqrt{3}}a_7 & a_8 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

为了找到H-Toeplitz算子B_ϕ的定义, 我们首先考虑算子$K:\mathcal{D}\to\mathcal{D}_h$, 定义如下:

$$K(e_{2n}(z)) = e_n(z) = \frac{1}{\sqrt{n}}z^n, \qquad K(e_{2n+1}(z)) = \overline{e_{n+1}(z)} = \frac{1}{\sqrt{n+1}}\overline{z}^{n+1}, n>0, z\in\mathbb{D}.$$

算子K在$\mathcal{D}$中是有界线性的, 且$\|K\|=1$, 同样, 算子K^*定义为

$$K^*(e_n(z)) = e_{2n}(z), \qquad K^*(\overline{e_{n+1}(z)}) = e_{2n+1}(z), n>0.$$

下面我们将利用算子K的定义来定义Dirichlet空间$\mathcal{D}$中的H-Toeplitz算子.

定义　对$\phi\in C^1(\overline{\mathbb{D}})$, H-Toeplitz算子定义如下:$B_\phi:\mathcal{D}\to\mathcal{D}$, 且有$B_\phi(f) = PM_\phi K(f), \forall f\in\mathcal{D}$.

接着我们找H-Toeplitz算子B_ϕ在Dirichlet空间$\mathcal{D}$中在标准正交基下的矩阵, 其中ϕ为调和函数. 我们取$\phi\in L^{\infty,1}(\mathbb{D})$, 利用引理2.2, 对每个正整数$n$, 有

$$B_\phi(e_{2n}) = PM_\phi K(e_{2n}) = PM_\phi e_n = T_\phi(e_n)$$

$$B_\phi(e_{2n+1}) = PM_\phi K(e_{2n+1}) = PM_\phi \overline{e_{n+1}} = PM_\phi J e_n = h_\phi(e_n).$$

因此,

$$\langle B_\phi e_{2n}, e_m\rangle = \langle T_\phi e_n, e_m\rangle = \begin{cases} \sqrt{\frac{m}{n}}a_{m-n}, & m\geq n \\ \sqrt{\frac{m}{n}}b_{n-m}, & m<n \end{cases},$$

其中m,n都是正整数.

$$\langle B_\phi e_{2n+1}, e_m\rangle = \langle H_\phi e_n, e_m\rangle = \frac{m}{\sqrt{m(n+1)}}a_{m+n+1}, \qquad m>0, n>0,$$

则B_ϕ关于$\mathcal{D}$中的标准正交基$\{e_n\}_{n>0}$的矩阵为

$$B_\phi = \begin{pmatrix} a_2 & a_0 & \frac{1}{\sqrt{2}}a_3 & \frac{1}{\sqrt{2}}b_1 & \frac{1}{\sqrt{3}}a_4 & \sqrt{3}b_2 & \frac{1}{\sqrt{4}}a_5 & \cdots \\ \sqrt{2}a_3 & \frac{1}{\sqrt{2}}a_1 & a_4 & a_0 & \frac{2}{\sqrt{6}}a_5 & \sqrt{\frac{3}{2}}b_1 & \frac{1}{\sqrt{2}}a_6 & \cdots \\ \sqrt{3}a_4 & \sqrt{3}a_2 & \frac{3}{\sqrt{6}}a_5 & \sqrt{\frac{3}{2}}a_1 & a_6 & a_0 & \frac{3}{2\sqrt{3}}a_7 & \cdots \\ 2a_5 & 2a_3 & \sqrt{2}a_6 & \sqrt{2}a_2 & \frac{2}{\sqrt{3}}a_7 & \sqrt{\frac{4}{3}}a_1 & a_8 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

B_ϕ的伴随矩阵为

$$B_\phi^*=\begin{pmatrix} \overline{a_2} & \sqrt{2}\overline{a_3} & \sqrt{3}\overline{a_4} & 2\overline{a_5} & \cdots \\ \overline{a_0} & \frac{1}{\sqrt{2}}\overline{a_1} & \sqrt{3}\overline{a_2} & 2\overline{a_3} & \cdots \\ \frac{1}{\sqrt{2}}\overline{a_3} & \overline{a_4} & \frac{3}{\sqrt{6}}\overline{a_5} & \sqrt{2}\overline{a_6} & \cdots \\ \frac{1}{\sqrt{2}}\overline{b_1} & \overline{a_0} & \sqrt{\frac{3}{2}}\overline{a_1} & \sqrt{2}\overline{a_2} & \cdots \\ \frac{1}{\sqrt{3}}\overline{a_4} & \frac{2}{\sqrt{6}}\overline{a_5} & \overline{a_6} & \frac{2}{\sqrt{3}}\overline{a_7} & \cdots \\ \sqrt{3}\overline{b_2} & \sqrt{\frac{3}{2}}\overline{b_1} & \overline{a_0} & \sqrt{\frac{4}{3}}\overline{a_1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

由B_ϕ的矩阵可得到B_ϕ的矩阵表达式中的$2n+1(n=1,2,...)$列恰恰为Hankel算子的矩阵表达式, 偶数列为Toeplitz算子的矩阵表达式.

接下来我们将给出具体的Dirichlet H-Toeplitz矩阵:

定义　对于$\phi(z)=\sum_{i=1}^{\infty}a_iz^i+\sum_{j=1}^{\infty}b_j\overline{z}^j\in L^\infty(\mathbb{D})$,若一个矩阵$(C_{m,n})$的$(m,n)$次项满足下列关系:

$$C_{m,n}=\begin{cases} \sqrt{\frac{m}{j}}a_{m-j} & n=2jm\geq j \\ \sqrt{\frac{j}{m}}b_{j-m} & n=2j,m<j \\ \frac{m+j+1}{\sqrt{m(j+1)}}a_{m+j+1} & n+2j+1 \end{cases}$$

其中m,n,j都是正整数, 则可将无限矩阵$(C_{m,n})$定义为一个Dirichlet H-Toeplitz矩阵. 又由$(C_{m,n})$的矩阵可得$(C_{m,n})$满足$C_{1,2}=C_{j,2j},j\geq 1$.

用$\mathcal{B}(\mathcal{D})$表示$\mathcal{D}$中的有界线性算子集合, 下面我们证明对于映射$\gamma:\mathcal{G}\to\mathcal{B}(\mathcal{D})$, 若定义域要么是$\{\phi\in L^\infty(\mathbb{D}):\phi$是调和的$\}$要么是$\{\phi\in C^1(\overline{\mathbb{D}}):\phi$是调和的$\}$, 则$\gamma$是一一的.

命题2.5　定义为$\gamma(\phi)=B_\phi$的函数$\gamma:\mathcal{G}\to\mathcal{B}(\mathcal{D})$总是一一的, 其中$\mathcal{G}$要么是空间$\{\phi\in L^\infty(\mathbb{D}):\phi$是调和的$\}$要么是空间$\{\phi\in C^1(\overline{\mathbb{D}}):\phi$是调和的$\}$

证明　**情况1**　令$\mathcal{G}=\{\phi\in L^\infty(\mathbb{D}):\phi$是调和的$\}$,且$\phi,\psi\in L^\infty(\mathbb{D})$是$\mathbb{D}$上的调和函数，定义为$\phi(z)=\sum_{i=1}^{\infty}a_iz^i+\sum_{j=1}^{\infty}b_j\overline{z}^j,\psi(z)=\sum_{j=1}^{\infty}a_i^{'}z^i+\sum_{j=1}^{\infty}b_j^{'}\overline{z}^j$, 若$B_\phi=B_\psi$,则有$(B_\phi-B_\psi)(e_n)=0,n>0$,特别地, $(B_\phi-B_\psi)(1)=B_{\phi-\psi}(1)=0$, 利用定义,

$$B_{\phi-\psi}(1)=PM_{\phi-\psi}K(1)=p(\sum_{i=1}^{\infty}(a_i-a_i^{'})z^i+\sum_{j=1}^{\infty}(b_j-b_j^{'})\overline{z}^j)=\sum_{i=1}^{\infty}(a_i-a_i^{'})z^i=0$$

因此可得$a_i-a_i^{'}=0,i\geq 1$, 即$a_i=a_i^{'},i\geq 1$

且$B_{\phi-\psi}(e_2)=0$. 则

$$\begin{aligned}
B_{\phi-\psi}(e_2) &= PM_{\phi-\psi}K(e_2)=PM_{\phi-\psi}e_1=PM_{\phi-\psi}z \\
&= P(\sum_{i=1}^{\infty}(a_i-a_i^{'})z^i+\sum_{j=1}^{\infty}(b_j-b_j^{'})\overline{z}^j)z \\
&= P\sum_{i=1}^{\infty}(a_i-a_i^{'})z^{i+1}+\sum_{j=1}^{\infty}(b_j-b_j^{'})\overline{z}^j z) \\
&= \frac{1}{2}(b_1-b_1^{'})=0
\end{aligned}$$

则有$b_1-b_1^{'}=0$, 故$b_1=b_1^{'}$.
接着, $B_{\phi-\psi}(e_4)=0$, 则

$$\begin{aligned}
B_{\phi-\psi}(e_4) &= PM_{\phi-\psi}K(e_4)=PM_{\phi-\psi}e_2=PM_{\phi-\psi}\frac{1}{\sqrt{2}}z^2 \\
&= P\frac{1}{\sqrt{2}}(\sum_{i=1}^{\infty}(a_i-a_i^{'})z^{i+2}+\sum_{j=1}^{\infty}(b_j-b_j^{'})\overline{z}^j)z^2 \\
&= P\frac{1}{\sqrt{2}}\sum_{j=1}^{\infty}(b_j-b_j^{'})\overline{z}^j z^2)=\frac{1}{\sqrt{2}}(b_2-b_2^{'})P(\overline{z}^2z^2) \\
&= 0
\end{aligned}$$

所以必有$b_2-b_2^{'}=0$, 即$b_2=b_2^{'}$.

依次用类似的方法将B_ϕ作用到$e_6,e_8,\cdots$上, 可得到$b_j=b_j^{'},j\geq 1$.因此$\phi=\psi$,所以γ是一一的.

情况2 令$\mathcal{G}=\{\phi\in C^1(\overline{\mathbb{D}}):\phi$是调和的$\}$, 假定$\gamma(\phi)=0$, 即$B_\phi=0,\phi\in C^1(\overline{\mathbb{D}})$,则对任意的正整数$m,n$, 有

$$\begin{aligned}
0=\left\langle B_\phi z^{2m},z^n\right\rangle &= \sqrt{2mn}\left\langle PM_\phi e_m,e_n\right\rangle \\
&= \sqrt{2}\left\langle T_\phi z^m,z^n\right\rangle \\
&= \sqrt{2}n\left\langle \phi z^m,z^n\right\rangle_{H^2}
\end{aligned}$$

因此$\left\langle \phi z^m,z^n\right\rangle_{H^2}=0$, 所以$\phi$在边界上为0, 即$\phi|_{\partial\mathbb{D}}=0$. $\because$ ϕ是调和的, $\therefore$ 可由泊松积分公式$\phi(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r^2)\phi(Re^{i\theta})}{R^2-2Rr\cos(\theta-\varphi)+r^2}d\theta,z\in\mathbb{D}$. 当$\phi(Re^{i\theta})=0$时, 则$\phi(z)=0,z\in\mathbb{D}$, 即$\phi$在$\mathbb{D}$内恒为0. 故$\gamma$是一一的.

3. H-Toeplitz算子的交换性

在这一部分我们研究H-Toeplitz算子的交换性, 其中H-Toeplitz算子的符号为解析的和共轭解析的, 一般地, 两个符号都为解析且符号次数不同的H-Toeplitz算子是不可以交换的, 下面的例子可以予以证明.

例子3.1 令$\phi(z)=z,\psi(z)=z^3$, 则$B_z(e_4(z))=PM_zK(e_4(z))=PM_ze_2(z)=PM_z\frac{1}{\sqrt{2}}z^2=$

$\frac{1}{\sqrt{2}}z^3$, $B_{z^3}(e_4(z)) = PM_{z^3}K(e_4(z)) = PM_{z^3}e_2(z) = PM_{z^3}\frac{1}{\sqrt{2}}z^2 = \frac{1}{\sqrt{2}}z^5$, 因此,

$$B_zB_{z^3}(e_4(z)) = B_z(\frac{1}{\sqrt{2}}z^5) = \frac{1}{\sqrt{2}}PM_zK(z^5) = \frac{1}{\sqrt{2}}PM_z\sqrt{5}\overline{e_3} = 0,$$

$$B_{z^3}B_z(e_4(z)) = B_{z^3}(\frac{1}{\sqrt{2}}z^3) = \frac{1}{\sqrt{2}}PM_{z^3}K(z^3) = \frac{\sqrt{2}}{\sqrt{3}}PM_{z^3}\overline{e_2} = \frac{\sqrt{3}}{2}P(z^3\overline{z}^2) = \frac{\sqrt{3}}{2}z.$$

所以$B_zB_{z^3} \neq B_{z^3}B_z$.

当两个H-Toeplitz算子的符号分别为解析的和共轭解析的时, 例如令$\phi(z) = z, \quad \psi(z) = \overline{z}$, 则$B_zB_{\overline{z}} \neq B_{\overline{z}}B_z$, 因为

$$B_zB_{\overline{z}}(z^2) = B_zPM_{\overline{z}}\sqrt{2}z = \frac{\sqrt{2}}{2}z,$$

$$B_{\overline{z}}B_z(z^2) = B_{\overline{z}}PM_z\sqrt{2}z = PM_{\overline{z}}K(\sqrt{2}z_2) = 1.$$

通过上面的例子, 我们发现对于两个都是解析符号且符号次数不同的H-Toeplitz算子以及两个分别是解析的和共轭解析的H-Toeplitz算子一般是不能交换的, 所以下面我们用一个定理来刻画两个H-Toeplitz算子在Dirichlet空间中可以交换的条件.

定理3.2 令$\phi(z) = \sum_{n=1}^{\infty} a_nz^n$和$\psi(z) = \sum_{m=1}^{\infty} b_m\overline{z}^m$, 其中$a_n, b_m, n, m \geq 1$都是非负常数, 且$\phi(0) = 0, \psi(0) = 0$, 若$B_\phi$与$B_\psi$可交换当且仅当$\phi = 0$或$\psi = 0$.

证 首先充分性是显然的, 若$\phi = 0$或$\psi = 0$,则$B_\phi B_\psi = B_\psi B_\phi$.其次是必要性, 若$B_\phi B_\psi = B_\psi B_\phi$, 则一定有$B_\phi B_\psi(1) = B_\psi B_\phi(1)$, 利用定义,

$$B_\phi B_\psi(1) = PM_\phi KPM_\psi K(1) = PM_\phi KPM_\psi(1) = 0,$$

$$B_\psi B_\phi(1) = PM_\psi KPM_\phi K(1) = PM_\psi KPM_\phi(1)$$
$$= PM_\psi KP(\sum_{n=1}^{\infty} a_nz^n) = PM_\psi(\sum_{n=1}^{\infty} a_n\sqrt{n}K(e_n))$$

当$n = 2k+1$时

$$B_\psi B_\phi(1) = PM_\psi(\sum_{k=1}^{\infty} a_{2k+1}\sqrt{2k+1}K(e_{2k+1})) = 0$$

当$n = 2k$时

$$B_\psi B_\phi(1) = PM_\psi KPM_\phi K(1) = PM_\psi KPM_\phi(1)$$

$$= PM_\psi(\sum_{k=1}^{\infty} a_{2k}\sqrt{2k}K(e_{2k})) = P(\sum_{m=1}^{\infty} b_m\overline{z}^m)(\sqrt{2}\sum_{k=1}^{\infty} a_{2k}z^k)$$

$$= \frac{1}{2}a_1b_1 + \sum_{t=1}^{\infty}\sqrt{2}(\sum_{j=1}^{\infty} a_{j+t}b_j)z^t$$

因此, $\frac{1}{2}a_1b_1 + \sum_{t=1}^{\infty}\sqrt{2}(\sum_{j=1}^{\infty} a_{j+t}b_j)z^t = 0$, 由于$a_n, b_m, n, m \geq 1$ 都是非负的, 则$a_1b_1 = 0, a_{j+t}b_j = 0, j \geq 1, t \geq 1$. 因此$b_j = 0, j \geq 1$,或$a_{i+t} = 0, i, t \geq 1$,即$b_j = 0, j \geq 1$,或$a_i = 0, i \geq 1$, 这

表明$\phi=0$或$\psi=0$.

利用该定理我们也可以研究当两个H-Toeplitz算子的符号都是调和的时候的交换性.

定理3.3 令$\phi(z)=\sum_{i=1}^{\infty}a_iz^i+\sum_{j=1}^{\infty}b_j\overline{z}^j$, 且$\psi(z)=\sum_{m=1}^{\infty}c_mz^m+\sum_{n=1}^{\infty}d_n\overline{z}^n$是$L^{\infty,1}(\mathbb{D})$中的两个调和函数, 且有$\phi(0)=0=\psi(0)$, 其中$a_i,b_j,c_m,d_n,i,j,m,n\geq 1$都是非零标量, 且假定$a_ic_{k-2i}\geq c_ia_{k-2i},a_ic_{2i-j}\geq c_ic_{2i-j}$ 以及$a_ld_l\geq c_lb_l,b_jc_{2j+h}\geq d_ja_{2j+h}$,其中$k,j,h$均为偶数, $l\geq 1$. 则下列条件等价:

(1) $B_\phi B_\psi=B_\psi B_\phi$,

(2) ϕ与ψ是线性相关的.

证 若ϕ与ψ线性相关, 显然B_ϕ与B_ψ可交换. 若B_ϕ与B_ψ可交换, 则一定有$B_\phi B_\psi(z^4)=B_\psi B_\phi(z^4)$,

$$\begin{aligned}
&B_\phi B_\psi(z^4)=PM_\phi KPM_\psi K(z^4)=PM_\phi KPM_\psi(2e_2)\\
&=PM_\phi KPM_\psi\sqrt{2}z^2=\sqrt{2}PM_\phi KP[(\sum_{m=1}^{\infty}c_mz^m+\sum_{n=1}^{\infty}d_n\overline{z}^n)z^2]\\
&=\sqrt{2}PM_\phi KP(\sum_{m=1}^{\infty}c_mz^{m+2}+\sum_{n=1}^{\infty}d_n\overline{z}^nz^2)\\
&=\sqrt{2}PM_\phi K(\sum_{m=1}^{\infty}c_mz^{m+2}+d_1z+\frac{1}{3}d_2)\\
&=\sqrt{2}PM_\phi(\sum_{m=1}^{\infty}c_m\sqrt{2}z^{\frac{m+2}{2}}+\sum_{m=1}^{\infty}c_m\sqrt{\frac{2(m+2)}{m+3}}\overline{z}^{\frac{m+3}{2}}+d_1\overline{z}+\frac{1}{3}d_2)\\
&=\sqrt{2}P(\sum_{i=1}^{\infty}a_iz^i+\sum_{j=1}^{\infty}b_j\overline{z}^j)(\sum_{m=1}^{\infty}c_m\sqrt{2}z^{\frac{m+2}{2}}+\sum_{m=1}^{\infty}c_m\sqrt{\frac{2(m+2)}{m+3}}\overline{z}^{\frac{m+3}{2}}+d_1\overline{z}+\frac{1}{3}d_2)\\
&=\sqrt{2}(\sum_{i=1}^{\infty}a_i\sum_{m=1}^{\infty}c_m\sqrt{2}z^{\frac{m+2i+2}{2}}+\sum_{i=1}^{\infty}a_i\sum_{m=1}^{\infty}c_m\sqrt{\frac{2(m+2)}{m+3}}z^{\frac{2i-m-3}{2}}+\frac{1}{2}a_1d_1\\
&+\sum_{i=2}^{\infty}a_id_1z^{i-1}+\frac{1}{3}d_2\sum_{i=1}^{\infty}a_iz^i+\sum_{j=1}^{\infty}b_j\sum_{m=1}^{\infty}c_m\sqrt{2}z^{\frac{m-2j+2}{2}})\\
&B_\psi B_\phi(z^4)=PM_\psi KPM_\phi K(z^4)=PM_\psi KPM_\phi(2e_2)\\
&=PM_\psi KPM_\phi\sqrt{2}z^2=\sqrt{2}PM_\psi KP[(\sum_{i=1}^{\infty}a_iz^i+\sum_{j=1}^{\infty}b_j\overline{z}^j)z^2]\\
&=\sqrt{2}PM_\psi KP(\sum_{i=1}^{\infty}a_iz^{i+2}+\sum_{j=1}^{\infty}b_j\overline{z}^jz^2)\\
&=\sqrt{2}PM_\psi K(\sum_{i=1}^{\infty}a_iz^{i+2}+b_1z+\frac{1}{3}b_2)\\
&=\sqrt{2}PM_\psi(\sum_{i=1}^{\infty}a_i\sqrt{i+2}K(e_{i+2})+b_1\overline{z}+\frac{1}{3}b_2)\\
&=\sqrt{2}PM_\psi(\sum_{i=1}^{\infty}a_i\sqrt{i+2}e_{\frac{i+2}{2}}+\sum_{i=1}^{\infty}a_i\sqrt{i+2}\overline{e_{\frac{i+3}{2}}}+b_1\overline{z}+\frac{1}{3}b_2)
\end{aligned}$$

$$= \sqrt{2}P(\sum_{m=1}^{\infty} c_m z^m + \sum_{n=1}^{\infty} d_n \overline{z}^n)(\sum_{i=1}^{\infty} a_i \sqrt{2} z^{\frac{i+2}{2}} + \sum_{i=1}^{\infty} a_i \sqrt{\frac{2(i+2)}{i+3}} \overline{z}^{\frac{i+3}{2}} + b_1 \overline{z} + \frac{1}{3} b_2)$$

$$= \sqrt{2}(\sum_{m=1}^{\infty} c_m \sum_{i=1}^{\infty} a_i \sqrt{2} z^{\frac{2m+i+2}{2}} + \sum_{m=1}^{\infty} c_m \sum_{i=1}^{\infty} a_i \sqrt{\frac{2(i+2)}{i+3}} z^{\frac{2m-i-3}{2}} + \frac{1}{2} c_1 b_1$$

$$+ \sum_{m=2}^{\infty} c_m b_1 z^{m-1} + \frac{1}{3} b_2 \sum_{m=1}^{\infty} c_m z^m + \sum_{n=1}^{\infty} d_n \sum_{i=1}^{\infty} a_i \sqrt{2} z^{\frac{i-2n+2}{2}})$$

比较等式两边z项的系数, 可得

$$\sqrt{2}(\sum_{i=1}^{\infty} a_i c_{2i-5} \sqrt{\frac{2i-3}{i-1}} + \frac{1}{2} a_1 d_1 + a_2 d_1 + \frac{1}{3} a_1 d_2 + \sqrt{2} \sum_{j=1}^{\infty} b_j c_{2j})$$

$$= \sqrt{2}(\sum_{i=1}^{\infty} c_m a_{2m-5} \sqrt{\frac{2m-3}{m-1}} + \frac{1}{2} c_1 b_1 + c_2 b_1 + \frac{1}{3} b_2 c_1 + \sqrt{2} \sum_{n=1}^{\infty} d_n c_{2n}).$$

或者等价于$\frac{a_{2m-5}}{c_{2m-5}} = \frac{a_m}{c_m}, m \geq 1, \frac{b_1}{d_1} = \frac{a_1}{c_1} = \frac{a_2}{c_2} = \frac{b_2}{d_2}$ 且$\frac{a_{2n}}{c_{2n}} = \frac{b_n}{d_n}, n \geq 1$, 进而$\frac{a_i}{c_i} = \frac{a_{i+1}}{c_{i+1}}, i = 1, 2, 3, \frac{b_1}{d_1} = \frac{b_2}{d_2}$.

比较等式两边z^2项的系数, 可得

$$\sqrt{2}(\sum_{i=1}^{\infty} a_i c_{2i-7} \sqrt{\frac{2i-5}{i-1}} + \frac{1}{2} a_1 d_1 + a_3 d_1 + \frac{1}{3} d_2 a_2 + \sqrt{2} \sum_{j=1}^{\infty} b_j c_{2j+2})$$

$$= \sqrt{2}(\sum_{m=1}^{\infty} c_m a_{2m-7} \sqrt{\frac{2m-5}{m-2}} + \frac{1}{2} c_1 b_1 + c_3 b_1 + \frac{1}{3} b_2 c_2 + \sqrt{2} \sum_{n=1}^{\infty} d_n c_{2n+2})$$

或者等价于$\frac{a_{2m-7}}{c_{2m-7}} = \frac{a_m}{c_m}, m \geq 1, \frac{b_1}{d_1} = \frac{a_1}{c_1} = \frac{a_3}{c_3}$ 且$\frac{a_{2n+2}}{c_{2n+2}} = \frac{b_n}{d_n}, n \geq 1$, 进而$\frac{a_i}{c_i} = \frac{a_{i+1}}{c_{i+1}}, i = 1, 2, 3, 4, 5, \frac{b_{j+1}}{d_{j+1}} = \frac{b_j}{d_j}, j = 1, 2$.

比较等式两边z^3项的系数, 可得

$$\sqrt{2}(\sqrt{2} \sum_{i=1}^{\infty} a_i c_{4-2i} + \sum_{i=1}^{\infty} a_i c_{2i-9} \sqrt{\frac{2i-7}{i-3}} + \frac{1}{2} a_1 d_1 + a_4 d_1 + \frac{1}{3} d_2 a_3 + \sqrt{2} \sum_{j=1}^{\infty} b_j c_{2j+4})$$

$$= \sqrt{2}(\sum_{m=1}^{\infty} c_m a_{4-2m} \sqrt{2} + \sum_{m=1}^{\infty} c_m a_{2m-9} \sqrt{\frac{2m-7}{m-3}} + \frac{1}{2} c_1 b_1 + c_4 b_1 + \frac{1}{3} b_2 c_3 + \sqrt{2} \sum_{n=1}^{\infty} d_n c_{2n+4})$$

或者等价于$\frac{a_{2m-9}}{c_{2m-9}} = \frac{a_m}{c_m} = \frac{a_{4-2m}}{c_{4-2m}}, m \geq 1, \frac{b_1}{d_1} = \frac{a_1}{c_1} = \frac{a_4}{c_4}$ 且$\frac{a_{2n+4}}{c_{2n+4}} = \frac{b_n}{d_n}, n \geq 1$, 进而$\frac{a_i}{c_i} = \frac{a_{i+1}}{c_{i+1}}, i = 1, 2, 3, 4, 5, 6, \frac{b_{j+1}}{d_{j+1}} = \frac{b_j}{d_j}, j = 1, 2, 3$.

接着依次比较下去, 我们可以得到$\frac{a_k}{c_k} = \frac{a_{k+1}}{c_{k+1}}, k \geq 1, \frac{b_{j+1}}{d_{j+1}} = \frac{b_j}{d_j}, j \geq 1$, 由于$\frac{b_1}{d_1} = \frac{a_1}{c_1} = \frac{a_2}{c_2} = \frac{b_2}{d_2}$, 所以可得$\frac{a_t}{c_t} = \frac{b_t}{d_t} = \lambda$, 其中$\lambda = \frac{b_1}{d_1}$,也表明$\phi = \lambda\psi$.

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