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具有左右分数阶导数和时滞的非瞬时脉冲微分方程非线性边值问题

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摘 要

本文研究了一类特殊的具有左右分数阶导数和时滞的非瞬时脉冲微分方程，该方程具有交叉时滞，且带有非线性边界条件。并基于上下解方法得到多个正解存在性定理。

关键词

左右分数阶导数，时滞，非瞬时脉冲微分方程，非线性边界条件，上下解方法

Nonlinear Boundary Value Problems for Non-Instantaneous Pulse Differential Equations with Left-Right Fractional Derivatives and Delays

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Abstract

In this paper, we study a class of special non-instantaneous impulsive differential equations with left and right fractional derivatives and delays. The equations have cross delays and nonlinear

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boundary conditions. Based on the upper and lower solution method, we obtain the existence theorems of multiple positive solutions.

Keywords

Left-Right Fractional Derivatives, Time Delay, Non-Instantaneous Pulse, Nonlinear Boundary Conditions, Upper and Lower Solution Method

1. 引言

分数阶微分方程非常适合刻画具有记忆和遗传性质的材料及过程，其对复杂系统的描述具有建模简单、描述准确、参数物理意义清楚等优势，因此也是复杂力学、物理过程数学建模的重要工具，如利用分数阶微积分在不同粘弹性流体的本构关系，在非牛顿流体中进行应用；分数阶 SEIR 传染病模型可以准确研究传染病、社交网络信息传播等方面的问题；在含未知参数的情况下，利用非线性分数阶系统状态估计分别含有分数阶有色过程噪声和有色测量噪声的连续时间问题。

在分数阶微分方程边值问题的研究[1] [2] [3] [4] [5]中，有时需要考虑左侧和右侧不同的定义，而同时带有左侧和右侧分数阶导数的微分方程相对于只含有分数阶右导数或左导数的分数阶微分方程，它的应用范围[6] [7] [8] [9]更加广泛，其在机械力学、生物工程、物理学、经济学等自然科学领域建立的数学模型中经常出现，并且具有很重要的作用，如用来分析空气中充满粒状材料时的室内外的温度数据等。

文献[10]研究了带有左右分数阶导数的微分方程边值问题：

$$\begin{cases} {}^cD_{b^-}^{\alpha}\, {}^cD_{a^+}^{\alpha} T(t) + \lambda T(t) = 0, \\ T(a) = T_0, T(b) = T_1, \end{cases}$$

其中，$T \in C[0,1]$，${}^cD_{b^-}^{\alpha}$ 为 Caputo 分数阶右导数，${}^cD_{a^+}^{\alpha}$ 为 Caputo 分数阶左导数。作者利用分数阶微分方程的数值解针对实际问题进行分析。

文献[3]研究了带有左右阶导数的耦合微分方程边值问题：

$$\begin{cases} -\left({}_{0^+}D_t^{\alpha} u(t)\right) = f(t, v(t)), \quad t \in (0,T), \\ -\left({}_tD_{T^-}^{\beta} v(t)\right) = g(t, u(t)), \quad t \in (0,T), \\ u(0) = 0, \quad {}_{0^+}D_t^{\alpha-1} u(T) = r_1\, {}_tD_{T^-}^{\beta-1} v(\xi), \\ v(T) = 0, \quad {}_tD_{T^-}^{\beta-1} v(0) = r_2\, {}_{0^+}D_t^{\alpha-1} u(\xi), \end{cases}$$

其中，${}_{0^+}D_t^{\alpha}$，${}_tD_{T^-}^{\beta}$ 分别是 α 阶 R-L 分数阶左导数和 β 阶 R-L 分数阶右导数，$0 < \alpha, \beta \le 2$，$\xi \in [0,T]$。作者运用上下解方法获得了边值问题解的存在性定理。

基于以上启发，本文研究含有左右分数阶导数和时滞的非瞬时脉冲微分方程非线性边值问题：

$$\begin{cases} {}_{t}^{c}D_{\xi^-}^{\alpha}u(t)=f_1\left(t,u(t),u(t+\tau_1)\right), \quad t\in[0,\xi], \\ {}_{\xi^+}^{\;\;c}D_t^{\beta}u(t)=f_2\left(t,u(t),u(t-\tau_2)\right), \quad t\in(\xi,1], \\ \Delta u(\xi)=I\left(\xi,u(\xi)\right),\ \Delta u'(\xi)=Q\left(\xi,u(\xi)\right), \\ h_0\left(u(0),u(1)\right)=0, \quad h_1\left({}_{t}^{c}D_{\xi^-}^{\alpha-1}u(0),\ {}_{\xi^+}^{\;\;c}D_t^{\beta-1}u(1)\right)=0 \end{cases} \tag{1}$$

解的存在性与多解性。其中，${}_{t}^{c}D_{\xi^-}^{\alpha}$ 是右侧 Caputo 分数阶导数，${}_{\xi^+}^{\;\;c}D_t^{\beta}$ 是左侧 Caputo 分数阶导数，$1<\alpha,\beta\le 2$，$\xi\in(0,1)$，$u(\xi^+)=\lim\limits_{\varepsilon\to 0^+}u(\xi+\varepsilon)$，$u(\xi^-)=\lim\limits_{\varepsilon\to 0^-}u(\xi+\varepsilon)$，$\tau_1\in(0,1-\xi)$，$\tau_2\in(0,\xi)$，$f_1\in C\left([0,\xi]\times\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+\right)$，$f_2\in C\left((\xi,1]\times\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+\right)$，$I,Q\in C\left(\mathbb{R},\mathbb{R}^+\right)$，$h_0,h_1\in C\left(\mathbb{R}^2,\mathbb{R}\right)$ 为给定的非线性函数。

2. 线性边值问题

定义 1 [11]：若 $\alpha>0$，$a<b\in\mathbb{R}$ 则

$$ {}_{a^+}I_t^{\alpha}\ {}_{a^+}^{\;\;c}D_t^{\alpha}u(t)=u(t)+c_1+c_2(t-a)+c_3(t-a)^2+\cdots+c_n(t-a)^{n-1}, $$

$$ {}_{t}I_{b^-}^{\alpha}\ {}_{t}^{c}D_{b^-}^{\alpha}u(t)=u(t)+d_1+d_2(b-t)+d_3(b-t)^2+\cdots+d_n(b-t)^{n-1}, $$

其中 $c_i,d_i\in\mathbb{R},i=1,2,\cdots,n,n\in\mathbb{N}$。

引理 1 [12]：令 E 为 Banach 空间，且 $P\subset E$ 是一个正规体锥。如果存在 α_1，β_1，α_2，$\beta_2\in P$ 使

$$\alpha_1\prec\beta_1\prec\alpha_2\prec\beta_2,$$

且 $A:[\alpha_1,\beta_2]\to E$ 是全连续算子，且为强增算子，使

$$\alpha_1\prec A\alpha_1,\ \ A\beta_1\prec\beta_1,\ \ \alpha_2\prec A\alpha_2,\ \ A\beta_2\preceq\beta_2 .$$

则算子 A 至少有三个不动点 x_1,x_2,x_3 使得

$$\alpha_1\preceq x_1\prec\prec\beta_1,\ \ \alpha_2\prec\prec x_2\preceq\beta_2,\ \ \alpha_2\npreceq x_2\npreceq\beta_2 .$$

令 $J=[0,1]$，$J_0=J\setminus\xi$，

$E=PC[J,\mathbb{R}]=\left\{u:J\to\mathbb{R}:u\text{在}J_0\text{上是连续的},u(\xi^+)\text{与}u(\xi^-)\text{存在且}u(\xi^-)=u(\xi)\right\}$。显然 E 是 Banach 空间且定义其范数为

$$\|u\|=\sup_{t\in[0,1]}|u(t)| .$$

令 $\|u\|_{[0,\xi]}=\sup\limits_{t\in[0,\xi]}|u(t)|$，$\|u\|_{(\xi,1]}=\sup\limits_{t\in(\xi,1]}|u(t)|$，则 $\|u\|=\max\left\{\|u\|_{[0,\xi]},\|u\|_{(\xi,1]}\right\}$。

引理 2：令 $h\in C\left([0,\xi],\mathbb{R}^+\right),y\in C\left((\xi,1],\mathbb{R}^+\right)$，对任意 $m_i,n_i\in\mathbb{R},i=1,2$，且 $\Delta_1\neq 0$。则边值问题

$$\begin{cases} {}_{t}^{c}D_{\xi^-}^{\alpha}u(t)=h(t),\ \ t\in(0,\xi), \\ {}_{\xi^+}^{\;\;c}D_t^{\beta}u(t)=y(t),t\in(\xi,1), \\ \Delta u(\xi)=I,\ \ \Delta u'(\xi)=Q, \\ m_1u(0)+n_1u(1)=\gamma_0,\ \ m_2\,{}_{t}^{c}D_{\xi^-}^{\alpha-1}u(0)+n_2\,{}_{\xi^+}^{\;\;c}D_t^{\beta-1}u(1)=\gamma_1 \end{cases} \tag{2}$$

在 E 中存在唯一解

$$u(t)=\begin{cases}\int_0^{\xi} G_1(t,s)h(s)\mathrm{d}s+\int_{\xi}^{1} g_2(t,s)y(s)\mathrm{d}s+\Delta_2+\dfrac{t}{\Delta_1}\left(-\dfrac{Qn_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)}-\gamma_1\right), t\in[0,\xi],\\ \int_{\xi}^{1} G_2(t,s)y(s)\mathrm{d}s+\int_0^{\xi} g_1(t,s)h(s)\mathrm{d}s+\Delta_3+\dfrac{t}{\Delta_1}\left(\dfrac{Qm_2}{\Gamma(3-\beta)}\xi^{2-\alpha}-\gamma_1\right), t\in(\xi,1].\end{cases} \tag{3}$$

其中

$$G_1(t,s)=\begin{cases} g_1(t,s),\ 0\le s\le t\le\xi,\\ g_1(t,s)+\dfrac{1}{\Gamma(\alpha)}(s-t)^{\alpha-1},\ 0\le t\le s\le\xi;\end{cases} \tag{4}$$

$$G_2(t,s)=\begin{cases} g_2(t,s)+\dfrac{1}{\Gamma(\beta)}(t-s)^{\beta-1},\ \xi\le s\le t\le 1,\\ g_2(t,s),\ \xi\le t\le s\le 1;\end{cases} \tag{5}$$

记

$$\Delta_1=\frac{m_2\xi^{2-\alpha}}{\Gamma(3-\alpha)}-\frac{n_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)};\Delta_2=-\frac{1}{m_1+n_1}\left(n_1I+n_1\left(1-\xi+\frac{n_2(1-\xi)^{2-\beta}}{\Delta_1\Gamma(3-\beta)}\right)Q-\frac{n_1}{\Delta_1}\gamma_1-\gamma_0\right);$$

$$\Delta_3=-\frac{1}{m_1+n_1}\left(-m_1I+\left(m_1\xi+n_1+\frac{n_1n_2(1-\xi)^{2-\beta}}{\Delta_1\Gamma(3-\beta)}\right)Q-\frac{n_1}{\Delta_1}\gamma_1-\gamma_0\right);$$

$$g_1(t,s)=-\frac{1}{m_1+n_1}\left(\frac{m_1}{\Gamma(\alpha)}s^{\alpha-1}+\frac{n_1m_2}{\Delta_1}\right)+\frac{tm_2}{\Delta_1};g_2(t,s)=-\frac{n_1}{m_1+n_1}\left(\frac{1}{\Gamma(\beta)}(1-s)^{\beta-1}+\frac{n_2}{\Delta_1}\right)-\frac{tn_2}{\Delta_1}.$$

证明：设 $u\in E$ 是边值问题(2)的解，则由定义 1 可知存在常数 $c_i\in\mathbb{R}$，$i=0,1,2,3$ 使 ${}_t^CD_{\xi^-}^{\alpha}u(t)=h(t)$ 的解为：

$$u(t)={}_tI_{\xi^-}^{\alpha}h(t)+c_0+c_1t=\frac{1}{\Gamma(\alpha)}\int_t^{\xi}(s-t)^{\alpha-1}h(s)\mathrm{d}s+c_0+c_1t,$$

$$u'(t)=-\frac{1}{\Gamma(\alpha-1)}\int_t^{\xi}(s-t)^{\alpha-2}h(s)\mathrm{d}s+c_1,$$

$${}_t^cD_{\xi^-}^{\alpha-1}u(t)=\int_t^{\xi}h(s)\mathrm{d}s-\frac{c_1}{\Gamma(3-\alpha)}(\xi-t)^{2-\alpha}$$

${}_{\xi^+}^{\ c}D_t^{\beta}u(t)=y(t)$ 的解为：

$$u(t)={}_{\xi^+}I_t^{\beta}y(t)+c_2+c_3t=\frac{1}{\Gamma(\beta)}\int_{\xi}^{t}(t-s)^{\beta-1}y(s)\mathrm{d}s+c_2+c_3t,$$

$$u'(t)=\frac{1}{\Gamma(\beta-1)}\int_{\xi}^{t}(t-s)^{\beta-2}y(s)\mathrm{d}s+c_3,$$

$${}_{\xi^+}^{\ c}D_t^{\beta-1}u(t)=\int_{\xi}^{t}y(s)\mathrm{d}s+\frac{c_3}{\Gamma(3-\beta)}(t-\xi)^{2-\beta}$$

由边值条件 $\Delta u(\xi)=I,\Delta u'(\xi)=Q$ 得

$$\begin{cases} c_2 - c_0 = I - Q\xi, \\ c_3 - c_1 = Q. \end{cases}$$

再由边值条件 $m_1 u(0) + n_1 u(1) = \gamma_0$， $m_2 \, {}_t^c D_{\xi^-}^{\alpha-1} u(0) + n_2 \, {}_{\xi^+}^{\;c} D_t^{\beta-1} u(1) = \gamma_1$，可得

$$\begin{cases} c_0 = -\dfrac{1}{m_1+n_1}\left(\displaystyle\int_0^\xi \left(\dfrac{m_1}{\Gamma(\alpha)} s^{\alpha-1} + \dfrac{n_1 m_2}{\Delta_1} \right) h(s)\,\mathrm{d}s + n_1 \int_\xi^1 \left(\dfrac{1}{\Gamma(\beta)} (1-s)^{\beta-1} - \dfrac{n_2}{\Delta_1} \right) y(s)\,\mathrm{d}s \right. \\ \qquad \left. + n_1 I + n_1 \left(1 - \xi + \dfrac{n_2 (1-\xi)^{2-\beta}}{\Delta_1 \Gamma(3-\beta)} \right) Q - \dfrac{n_1}{\Delta_1} \gamma_1 - \gamma_0 \right), \\ c_1 = \dfrac{1}{\Delta_1} \left(m_2 \displaystyle\int_0^\xi h(s)\,\mathrm{d}s + n_2 \int_\xi^1 y(s)\,\mathrm{d}s + \dfrac{Q n_2 (1-\xi)^{2-\beta}}{\Gamma(3-\beta)} - \gamma_1 \right), \\ c_2 = -\dfrac{1}{m_1+n_1}\left(\displaystyle\int_0^\xi \left(\dfrac{m_1 s^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{n_1 m_2}{\Delta_1} \right) h(s)\,\mathrm{d}s + n_1 \int_\xi^1 \left(\dfrac{(1-s)^{\beta-1}}{\Gamma(\beta)} - \dfrac{n_2}{\Delta_1} \right) y(s)\,\mathrm{d}s \right. \\ \qquad \left. - m_1 I + \left(m_1 \xi + n_1 + \dfrac{n_1 n_2 (1-\xi)^{2-\beta}}{\Delta_1 \Gamma(3-\beta)} \right) Q - \dfrac{n_1}{\Delta_1} \gamma_1 - \gamma_0 \right), \\ c_3 = \dfrac{1}{\Delta_1} \left(m_2 \displaystyle\int_0^\xi h(s)\,\mathrm{d}s + n_2 \int_\xi^1 y(s)\,\mathrm{d}s + \dfrac{Q m_2}{\Gamma(3-\beta)} \xi^{2-\alpha} - \gamma_1 \right). \end{cases}$$

因此，当 $t \in [0,\xi]$ 时，

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_t^\xi (s-t)^{\alpha-1} h(s)\,\mathrm{d}s - \frac{1}{m_1+n_1} \left(\int_0^\xi \left(\frac{m_1}{\Gamma(\alpha)} s^{\alpha-1} + \frac{n_1 m_2}{\Delta_1} \right) h(s)\,\mathrm{d}s \right. \\ &\quad \left. + n_1 \int_\xi^1 \left(\frac{1}{\Gamma(\beta)} (1-s)^{\beta-1} - \frac{n_2}{\Delta_1} \right) y(s)\,\mathrm{d}s + n_1 I + n_1 \left(1 - \xi + \frac{n_2 (1-\xi)^{2-\beta}}{\Delta_1 \Gamma(3-\beta)} \right) Q - \frac{n_1}{\Delta_1} \gamma_1 - \gamma_0 \right) \\ &\quad + \frac{t}{\Delta_1} \left(m_2 \int_0^\xi h(s)\,\mathrm{d}s + n_2 \int_\xi^1 y(s)\,\mathrm{d}s + \frac{Q n_2 (1-\xi)^{2-\beta}}{\Gamma(3-\beta)} - \gamma_1 \right) \\ &= \int_0^\xi G_1(t,s) h(s)\,\mathrm{d}s + \int_\xi^1 g_2(t,s) y(s)\,\mathrm{d}s + \Delta_2 + \frac{t}{\Delta_1} \left(\frac{Q n_2 (1-\xi)^{2-\beta}}{\Gamma(3-\beta)} - \gamma_1 \right). \end{aligned}$$

当 $t \in (\xi,1]$ 时，

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\beta)} \int_\xi^t (t-s)^{\beta-1} y(s)\,\mathrm{d}s - \frac{1}{m_1+n_1} \left(\int_0^\xi \left(\frac{m_1}{\Gamma(\alpha)} s^{\alpha-1} + \frac{n_1 m_2}{\Delta_1} \right) h(s)\,\mathrm{d}s \right. \\ &\quad \left. + n_1 \int_\xi^1 \left(\frac{1}{\Gamma(\beta)} (1-s)^{\beta-1} + \frac{n_2}{\Delta_1} \right) y(s)\,\mathrm{d}s - m_1 I + \left(m_1 \xi + n_1 + \frac{n_1 n_2 (1-\xi)^{2-\beta}}{\Delta_1 \Gamma(3-\beta)} \right) Q - \frac{n_1}{\Delta_1} \gamma_1 - \gamma_0 \right) \\ &\quad + \frac{t}{\Delta_1} \left(m_2 \int_0^\xi h(s)\,\mathrm{d}s + n_2 \int_\xi^1 y(s)\,\mathrm{d}s + \frac{Q m_2}{\Gamma(3-\beta)} \xi^{2-\alpha} - \gamma_1 \right) \\ &= \int_\xi^1 G_2(t,s) y(s)\,\mathrm{d}s + \int_0^\xi g_1(t,s) h(s)\,\mathrm{d}s + \Delta_3 + \frac{t}{\Delta_1} \left(\frac{Q m_2}{\Gamma(3-\beta)} \xi^{2-\alpha} - \gamma_1 \right). \end{aligned}$$

易证(3)是方程(2)的解，反之亦然。

证毕。

为了以后证明，我们给出如下假设：

(H1) $m_i, n_i \in \mathbb{R}(i=1,2)$，$n_1>0, n_2>0, -n_1>m_1>-\frac{n_1}{\xi}, \gamma_1\le 0, \gamma_0\le 0$，

$$m_2>\max\left\{\frac{\Gamma(3-\alpha)\xi^{\alpha-1}n_2}{\Gamma(3-\beta)(\Gamma(3-\alpha)\Gamma(\alpha)-\xi)},-\frac{\Gamma(3-\alpha)n_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)\xi^{2-\alpha}}\right\}.$$

引理 3：假设(H1)成立，则由式(4)、(5)定义的函数 $G_i(t,s), i=1,2$ 满足以下性质：

1) $0<G_1(0,s)\le G_1(t,s)\le G_1(\xi,s)$，对任意 $(t,s)\in[0,\xi]\times[0,\xi]$；

2) $0<G_2(\xi,s)\le G_2(t,s)\le G_2(1,s)$，对任意 $(t,s)\in[\xi,1]\times[\xi,1]$。

证明：1) 显然 $G_i(t,s), i=1,2$ 为连续函数。 由(H1)知，$\Delta_1>0$，$m_2>\frac{\xi^{\alpha-1}}{\Gamma(\alpha)}\Delta_1$，则对于 $t\in[0,\xi]$，当 $0\le s\le t\le\xi$ 时，

$$G_1(t,s)=g_1(t,s)=-\frac{1}{m_1+n_1}\left(\frac{m_1}{\Gamma(\alpha)}s^{\alpha-1}+\frac{n_1m_2}{\Delta_1}\right)+\frac{tm_2}{\Delta_1},\frac{\partial g_1(t,s)}{\partial t}=\frac{m_2}{\Delta_1}>0;$$

当 $0\le s\le t\le\xi$ 时，由于 $G_1(t,s)=g_1(t,s)+\frac{1}{\Gamma(\alpha)}(s-t)^{\alpha-1}$，

$$\frac{\partial G_1(t,s)}{\partial t}=-\frac{1}{\Gamma(\alpha-1)}(s-t)^{\alpha-2}+\frac{m_2}{\Delta_1},\frac{\partial^2G_1(t,s)}{\partial t^2}=\frac{\alpha-2}{\Gamma(\alpha-1)}(s-t)^{\alpha-3}<0,$$

则 $\frac{\partial G_1(t,s)}{\partial t}\ge\frac{\partial G_1(s,s)}{\partial t}=\frac{m_2}{\Delta_1}>0$。因此，$G_1(t,s)$ 是关于 t 的单调递增函数，且

$$G_1(0,s)\le G_1(t,s)\le G_1(\xi,s).$$

又

$$G_1(0,s)=g_1(0,s)+\frac{1}{\Gamma(\alpha)}s^{\alpha-1}=-\frac{1}{m_1+n_1}\left(\frac{m_1}{\Gamma(\alpha)}s^{\alpha-1}+\frac{n_1m_2}{\Delta_1}\right)+\frac{1}{\Gamma(\alpha)}s^{\alpha-1}$$
$$=-\frac{1}{m_1+n_1}\left(\frac{-n_1}{\Gamma(\alpha)}s^{\alpha-1}+\frac{n_1m_2}{\Delta_1}\right)>-\frac{n_1}{m_1+n_1}\left(\frac{-1}{\Gamma(\alpha)}\xi^{\alpha-1}+\frac{m_2}{\Delta_1}\right)>0,$$

则 $0<G_1(0,s)\le G_1(t,s)\le G_1(\xi,s)$ 成立。

2) 对于 $t\in[\xi,1]$，当 $\xi\le t\le s\le 1$ 时，

$$G_2(t,s)=g_2(t,s)=-\frac{n_1}{m_1+n_1}\left(\frac{1}{\Gamma(\beta)}(1-s)^{\beta-1}+\frac{n_2}{\Delta_1}\right)+\frac{tn_2}{\Delta_1},\frac{\partial g_2(t,s)}{\partial t}=\frac{n_2}{\Delta_1}>0;$$

当 $\xi\le s\le t\le 1$ 时，

$$G_2(t,s)=g_2(t,s)+\frac{1}{\Gamma(\beta)}(t-s)^{\beta-1},\frac{\partial G_2(t,s)}{\partial t}=\frac{n_2}{\Delta_1}+\frac{1}{\Gamma(\beta-1)}(t-s)^{\beta-2}>0,$$

则 $G_2(t,s)$ 是关于 t 的单调递增函数，那么

$$G_2(\xi,s)\le G_2(t,s)\le G_2(1,s).$$

又

$$G_2(\xi,s)=-\frac{n_1}{m_1+n_1}\left(\frac{1}{\Gamma(\beta)}(1-s)^{\beta-1}+\frac{n_2}{\Delta_1}\right)+\frac{n_2}{\Delta_1}=\frac{1}{m_1+n_1}\left(\frac{n_1}{\Gamma(\beta)}(1-s)^{\beta-1}+\frac{n_2\left(1-(m_1+n_1)\xi\right)}{\Delta_1}\right)>0,$$

因此，$0<G_2(\xi,s)\le G_2(t,s)\le G_2(1,s)$ 成立。

证毕。

引理 4： 若(H1)成立，$I,Q\in\mathbb{R}^+$，$\Delta_1\neq 0$。若 u 满足

$$\begin{cases} {}_t^cD_{\xi^-}^{\alpha}u(t)\ge 0,\ t\in(0,\xi),\\ {}_{\xi^+}^{\ c}D_t^{\beta}u(t)\ge 0,\ t\in(\xi,1),\\ \Delta u(\xi)=I,\ \Delta u'(\xi)=Q,\\ m_1u(0)+n_1u(1)\le 0,\ m_2\,{}_t^cD_{\xi^-}^{\alpha-1}u(0)+n_2\,{}_{\xi^+}^{\ c}D_t^{\beta-1}u(1)\ge 0,\end{cases}\tag{4}$$

则 $u(t)\ge 0$，$t\in[0,1]$。

证明：对任意 $h\in C([0,\xi],\mathbb{R}^+)$，$y\in C((\xi,1],\mathbb{R}^+)$，由于 $\gamma_1\le 0$，$\gamma_0\le 0$ 为常数。考虑以下边值问题:

$$\begin{cases} {}_t^cD_{\xi^-}^{\alpha}u(t)=h(t),\ t\in[0,\xi],\\ {}_{\xi^+}^{\ c}D_t^{\beta}u(t)=y(t),\ t\in(\xi,1],\\ \Delta u(\xi)=I,\ \Delta u'(\xi)=Q,\\ m_1u(0)+n_1u(1)=\gamma_0,\ m_2\,{}_t^cD_{\xi^-}^{\alpha-1}u(0)+n_2\,{}_{\xi^+}^{\ c}D_t^{\beta-1}u(1)=\gamma_1.\end{cases}$$

由引理 2 可得

$$u(t)=\begin{cases}\int_0^{\xi}G_1(t,s)h(s)\mathrm{d}s+\int_{\xi}^{1}g_2(t,s)y(s)\mathrm{d}s+\Delta_2+\frac{t}{\Delta_1}\left(\frac{Qn_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)}-\gamma_1\right),t\in[0,\xi],\\ \int_{\xi}^{1}G_2(t,s)y(s)\mathrm{d}s+\int_0^{\xi}g_1(t,s)h(s)\mathrm{d}s+\Delta_2+\frac{t}{\Delta_1}\left(\frac{Qm_2}{\Gamma(3-\beta)}\xi^{2-\alpha}-\gamma_1\right),t\in(\xi,1].\end{cases}$$

由(H1)可得，$\Delta_1,\Delta_2,\Delta_3>0$，$\frac{Qn_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)}-\gamma_1>0$，$\frac{Qm_2\xi^{2-\alpha}}{\Gamma(3-\beta)}-\gamma_1>0$，$t\in[0,1]$。再由引理 3 可得 $u(t)\ge 0$，$t\in[0,1]$ 显然成立。

证毕。

3. 分数阶微分方程的上下解方法

为方便叙述，我们假设下文满足以下假设：

(H2) 对任意 $u_1\le u_2,v_1\le v_2$，$f(t,u_1,v_1)\le f(t,u_2,v_2),t\in[0,\xi]$，

$$g(t,u_1,v_1)\le g(t,u_2,v_2),t\in(\xi,1],\ \ I(u_1)\le I(u_2),Q(u_1)\le Q(u_2),t\in[0,1].$$

对任意 $u_1<u_2,v_1<v_2$，$f(t,u_1,v_1)<f(t,u_2,v_2),t\in[0,\xi]$，

$$g(t,u_1,v_1)<g(t,u_2,v_2),t\in(\xi,1],\ \ I(u_1)<I(u_2),Q(u_1)<Q(u_2),t\in[0,1].$$

(H3) 若(H1)成立，且 $x_1,x_2,y_1,y_2\in\mathbb{R}$，当 $x_1\le x_2,y_1\le y_2$ 时，

$$h_0(x_2,y_2)-h_0(x_1,y_1)\le -m_1(x_2-x_1)-n_1(y_2-y_1),$$

$$h_1(x_2,y_2)-h_0(x_1,y_1)\ge -m_2(x_2-x_1)-n_2(y_2-y_1).$$

记 $P=\{u\in E:u(t)\ge 0,t\in[0,1]\}$，显然 P 为 E 中的正规体锥。且若 $u(t)\le v(t)$，$t\in[0,1]$，则 $u\preceq v\in P$。对任意 $u\in P$，考虑如下边值问题：

$$\begin{cases}{}_t^cD_{\xi^-}^{\alpha}u(t)=f_1(t,x(t),x(t+\tau_1)), \quad t\in(0,\xi),\\ {}_{\xi^+}^{\ c}D_t^{\beta}u(t)=f_2(t,x(t),x(t-\tau_2)), \quad t\in(\xi,1),\\ \Delta u(\xi)=I(\xi,x(\xi)), \quad \Delta u'(\xi)=Q(\xi,x(\xi)),\\ m_1u(0)+n_1u(1)=h_0(x(0),x(1))+m_1x(0)+n_1x(1):=\gamma_0(x),\\ m_2\,{}_t^cD_{\xi^-}^{\alpha-1}u(0)+n_2\,{}_{\xi^+}^{\ c}D_t^{\beta-1}u(1)\\ =h_1\left({}_t^cD_{\xi^-}^{\alpha-1}x(0),\,{}_{\xi^+}^{\ c}D_t^{\beta-1}x(1)\right)+m_2\,{}_t^cD_{\xi^-}^{\alpha-1}x(0)+n_2\,{}_{\xi^+}^{\ c}D_t^{\beta-1}x(1):=\gamma_1(x).\end{cases} \tag{5}$$

由引理 2 知，边值问题(5)有唯一解

$$u(t)=\begin{cases}\int_0^{\xi}G_1(t,s)f_1(s,x(s),x(s+\tau_1))\mathrm{d}s+\int_{\xi}^1 g_2(t,s)f_2(s,x(s),x(s-\tau_2))\mathrm{d}s\\ \quad +\Delta_2^x+\dfrac{t}{\Delta_1}\left(\dfrac{Q(\xi,x(\xi))n_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)}-\gamma_1(x)\right), \ t\in[0,\xi],\\ \int_{\xi}^1G_2(t,s)f_2(s,x(s),x(s-\tau_2))\mathrm{d}s+\int_0^{\xi}g_1(t,s)f_1(s,x(s),x(s+\tau_1))\mathrm{d}s\\ \quad +\Delta_3^x+\dfrac{t}{\Delta_1}\left(\dfrac{Q(\xi,x(\xi))m_2}{\Gamma(3-\beta)}\xi^{2-\alpha}-\gamma_1(x)\right), \ t\in(\xi,1].\end{cases}$$

其中

$$\Delta_2^x=-\frac{1}{m_1+n_1}\left(n_1I(\xi,x(\xi))+n_1\left(1-\xi+\frac{n_2(1-\xi)^{2-\beta}}{\Delta_1\Gamma(3-\beta)}\right)Q(\xi,x(\xi))-\frac{n_1}{\Delta_1}\gamma_1(x)-\gamma_0(x)\right);$$

$$\Delta_3^x=-\frac{1}{m_1+n_1}\left(-m_1I(\xi,x(\xi))+\left(m_1\xi+n_1+\frac{n_1n_2}{\Delta_1\Gamma(3-\beta)}\right)Q(\xi,x(\xi))-\frac{n_1}{\Delta_1}\gamma_1(x)-\gamma_0(x)\right).$$

定义算子

$$Tx(t)=\begin{cases}\int_0^{\xi}G_1(t,s)f_1(s,x(s),x(s+\tau_1))\mathrm{d}s+\int_{\xi}^1 g_2(t,s)f_2(s,x(s),x(s-\tau_2))\mathrm{d}s\\ \quad +\Delta_2^x+\dfrac{t}{\Delta_1}\left(\dfrac{Q(\xi,x(\xi))n_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)}-\gamma_1(x)\right), \ t\in[0,\xi],\\ \int_{\xi}^1G_2(t,s)f_2(s,x(s),x(s-\tau_2))\mathrm{d}s+\int_0^{\xi}g_1(t,s)f_1(s,x(s),x(s+\tau_1))\mathrm{d}s\\ \quad +\Delta_3^x+\dfrac{t}{\Delta_1}\left(\dfrac{Q(\xi,x(\xi))m_2\xi^{2-\alpha}}{\Gamma(3-\beta)}-\gamma_1(x)\right), \ t\in(\xi,1].\end{cases}$$

引理 5：若(H1)成立，则 T 为全连续算子。

证明：由引理 3，引理 4 知，对任意 $x\in P$，当 $t\in[0,1]$ 时，$Tx\geq 0$ 显然成立。

因此，$T:P\to P$ 是有意义的。

接下来，我们分两步证明：

第一步：T 是连续算子。

设对任意 $x_n\in P$，$n=1,2,\cdots$ 存在 $x\in P$ 使得当 $n\to\infty$ 时，$\|x_n-x\|\to 0$。则存在 $\bar{M}_0>0$，使得 $\|x_n\|\leq\bar{M}_0$，$\|x\|\leq\bar{M}_0$。又由于 f_1,f_2 连续，$I,Q\in C(\mathbb{R},\mathbb{R})$，且 $\gamma_0(x),\gamma_1(x)\in\mathbb{R}$，则

$$\lim_{n\to\infty}\left(f_1\left(t,x_n(t),x_n(t+\tau_1)\right)-f_1\left(t,x(t),x(t+\tau_1)\right)\right)=0,$$

$$\lim_{n\to\infty}\left(f_2\left(t,x_n(t),x_n(t-\tau_2)\right)-f_2\left(t,x(t),x(t-\tau_2)\right)\right)=0,$$

$$\lim_{n\to\infty}\left|I\left(x_n(t)\right)-I\left(x(t)\right)\right|=0,\ \lim_{n\to\infty}\left|Q\left(x_n(t)\right)-Q\left(x(t)\right)\right|=0,$$

$$\lim_{n\to\infty}\left(\gamma_0(x_n)-\gamma_0(x)\right)=0,\ \lim_{n\to\infty}\left(\gamma_1(x_n)-\gamma_1(x)\right)=0,$$

且存在常数 $\bar{M}_1>0$，使得 $\sup\limits_{(t,u,v)\in A}\left|f_1(t,u,v)\right|\leq\bar{M}_1$，$\sup\limits_{(t,u,v)\in B}\left|f_2(t,u,v)\right|\leq\bar{M}_1$，其中

$$A=[0,\xi]\times\left[-\bar{M}_0,\bar{M}_0\right]\times\left[-\bar{M}_0,\bar{M}_0\right],\ B=[\xi,1]\times\left[-\bar{M}_0,\bar{M}_0\right]\times\left[-\bar{M}_0,\bar{M}_0\right].$$

再由引理 3 可得，当 $t\in[0,\xi]$ 时，

$$\begin{aligned}&\left|T(x_n)-T(x)\right|\\&=\Bigg|\int_0^{\xi}G_1(t,s)\left(f_1\left(s,x_n(s),x_n(s+\tau_1)\right)-f_1\left(s,x(t),x(s+\tau_1)\right)\right)\mathrm{d}s\\&\quad+\int_{\xi}^{1}g_2(t,s)\left(f_2\left(s,x_n(s),x_n(s-\tau_2)\right)-f_2\left(s,x(s),x(s-\tau_2)\right)\right)\mathrm{d}s\\&\quad+\left(\Delta_2^{x_n}-\Delta_2^{x}\right)+\frac{t}{\Delta_1}\left(\frac{\left(Q\left(\xi,x_n(\xi)\right)-Q\left(\xi,x(\xi)\right)\right)n_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)}-\left(\gamma_1(x_n)-\gamma_1(x)\right)\right)\Bigg|\end{aligned}$$

$$\begin{aligned}&\left|T(x_n)-T(x)\right|\\&=\Bigg|\int_0^{\xi}G_1(t,s)\left(f_1\left(s,x_n(s),x_n(s+\tau_1)\right)-f_1\left(s,x(t),x(s+\tau_1)\right)\right)\mathrm{d}s\\&\quad+\int_{\xi}^{1}g_2(t,s)\left(f_2\left(s,x_n(s),x_n(s-\tau_2)\right)-f_2\left(s,x(s),x(s-\tau_2)\right)\right)\mathrm{d}s\\&\quad+\left(\Delta_2^{x_n}-\Delta_2^{x}\right)+\frac{t}{\Delta_1}\left(\frac{\left(Q\left(\xi,x_n(\xi)\right)-Q\left(\xi,x(\xi)\right)\right)n_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)}-\left(\gamma_1(x_n)-\gamma_1(x)\right)\right)\Bigg|\end{aligned}$$

则由 Lebesgue 控制收敛定理可知，$\lim\limits_{n\to\infty}\|Tu_n-Tu\|_{[0,\xi]}=0$。同理可得，$\lim\limits_{n\to\infty}\|Tu_n-Tu\|_{(\xi,1]}=0$。

因此，对任意 $t\in[0,1]$，有 $\lim\limits_{n\to\infty}\|Tu_n-Tu\|=0$，则算子 T 是连续算子。

第二步：T 是紧的。

令 $\Omega \subset P$ 为有界集，由 f_1，f_2，I，Q 的连续性得，存在 $\bar{M}_2 > 0$，使得对任意 $t \in [0,\xi]$，$u,v \in \Omega$，有 $|f_1(t,u,v)| \le \bar{M}_2$；对任意 $t \in (\xi,1]$，$u,v \in \Omega$ 有 $|f_2(t,u,v)| \le \bar{M}_2$，$|I| \le \bar{M}_2$，$|Q| \le \bar{M}_2$，$|\gamma_0(x)| \le \bar{M}_2$，$|\gamma_1(x)| \le \bar{M}_2$。

则

$$|\Delta_2^x| = -\frac{1}{m_1+n_1}\left(n_1\left(2-\xi+\frac{n_2(1-\xi)^{2-\beta}+\Gamma(3-\beta)}{\Delta_1\Gamma(3-\beta)}\right)-1\right)\bar{M}_2;$$

$$|\Delta_3^x| = -\frac{1}{m_1+n_1}\left(m_1(\xi-1)+n_1\left(1+\frac{n_2-\Gamma(3-\beta)}{\Delta_1\Gamma(3-\beta)}\right)-1\right)\bar{M}_2$$

$$\begin{aligned}\sup_{t\in[0,\xi]}|Tu(t)| \le &\left|\int_0^\xi G_1(\xi,s)f_1(s,x(s),x(s+\tau_1))\mathrm{d}s+\int_\xi^1 g_2(1,s)f_2(s,x(s),x(s-\tau_2))\mathrm{d}s\right.\\ &\left.+\Delta_2^x+\frac{\xi}{\Delta_1}\left(\frac{Q(\xi,x(\xi))n_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)}-\gamma_1(x)\right)\right|\\ \le &\left(\int_0^\xi G_1(\xi,s)\mathrm{d}s+\int_\xi^1 g_2(1,s)\mathrm{d}s-\frac{1}{m_1+n_1}\left(n_1\left(2-\xi+\frac{n_2(1-\xi)^{2-\beta}-\Gamma(3-\beta)}{\Delta_1\Gamma(3-\beta)}\right)-1\right)\right.\\ &\left.+\frac{\xi}{\Delta_1}\left(\frac{n_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)}-1\right)\right)\bar{M}_2,\end{aligned}$$

$$\begin{aligned}\sup_{t\in(\xi,1]}|Tu(t)| \le &\left|\int_\xi^1 G_2(1,s)f_2(s,x(s),x(s-\tau_2))\mathrm{d}s+\int_0^\xi g_1(\xi,s)f_1(s,x(s),x(s+\tau_1))\mathrm{d}s\right.\\ &\left.+\Delta_3^x+\frac{\xi}{\Delta_1}\left(\frac{Q(\xi,x(\xi))m_2}{\Gamma(3-\beta)}\xi^{2-\alpha}-\gamma_1(x)\right)\right|\\ \le &\left(\int_\xi^1 G_2(1,s)\mathrm{d}s+\int_0^\xi g_1(\xi,s)\mathrm{d}s-\frac{1}{m_1+n_1}\left(m_1(\xi-1)+n_1\left(1+\frac{n_2-\Gamma(3-\beta)}{\Delta_1\Gamma(3-\beta)}\right)-1\right)\right.\\ &\left.+\frac{1}{\Delta_1}\left(\frac{m_2\xi^{2-\alpha}}{\Gamma(3-\beta)}-1\right)\right)\bar{M}_2.\end{aligned}$$

因此，算子 $T(\Omega)$ 一致有界。

由于 $G_1(t,s)$，$g_2(t,s)$ 在 $[0,\xi]\times[0,\xi]$ 上连续，所以 $G_1(t,s)$，$g_2(t,s)$ 在 $[0,\xi]\times[0,\xi]$ 上一致连续。因此对任意 $\varepsilon > 0$，存在 $0 < \delta_1 < \dfrac{\varepsilon\Delta_1\Gamma(3-\beta)}{2\left|n_2(1-\xi)^{2-\beta}-\Gamma(3-\beta)(1-n_2)\right|\bar{M}_2}$，当 $|t_1-t_2| < \delta_1$ 时，有

$|G_1(t_1,s)-G_1(t_2,s)| < \dfrac{\varepsilon}{4\bar{M}_2}$，$|g_2(t_1,s)-g_2(t_2,s)| < \dfrac{\varepsilon}{4\bar{M}_2}$。因此，对任意的 $t_1,t_2 \in [0,\xi]$，$|t_1-t_2| < \delta_1$，$u \in \Omega$ 有

$$\begin{aligned}\left|Tu(t_2)-Tu(t_1)\right| &= \left|\int_0^{\xi}\left(G_1(t_1,s)-G_1(t_2,s)\right)f_1\left(s,u(s),u(s+\tau_1)\right)\mathrm{d}s\right.\\ &\quad+\int_{\xi}^{1}\left(g_2(t_1,s)-g_2(t_2,s)\right)f_2\left(s,u(s),u(s-\tau_2)\right)\mathrm{d}s\\ &\quad\left.+\frac{t_1-t_2}{\Delta_1}\left(\frac{Q\left(\xi,x(\xi)\right)n_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)}-\gamma_1(x)\right)\right|\\ &\le \bar{M}_2\left(\int_0^{\xi}\left|G_1(t_1,s)-G_1(t_2,s)\right|\mathrm{d}s+\int_{\xi}^{1}\left|g_2(t_1,s)-g_2(t_2,s)\right|\mathrm{d}s\right)\\ &\quad+\left|t_1-t_2\right|\frac{\left|n_2(1-\xi)^{2-\beta}-\Gamma(3-\beta)(1-n_2)\bar{M}_2\right.}{\Delta_1\Gamma(3-\beta)}\\ &<\varepsilon.\end{aligned}$$

又由于$G_2(t,s)$，$g_1(t,s)$在$[\xi,1]\times[\xi,1]$上连续，所以$G_2(t,s)$，$g_1(t,s)$在$[\xi,1]\times[\xi,1]$上一致连续。因此对上述$\varepsilon>0$，存在$0<\delta_2<\dfrac{\varepsilon\Delta_1\Gamma(3-\beta)}{2\left|m_2\xi^{2-\alpha}-(m_2+1)\Gamma(3-\beta)\right|}$，当$|t_3-t_4|<\delta_2$时，有

$\left|G_2(t_3,s)-G_2(t_4,s)\right|<\dfrac{\varepsilon}{4\bar{M}_2}$，$\left|g_1(t_3,s)-g_1(t_4,s)\right|<\dfrac{\varepsilon}{4\bar{M}_2}$。

因此，对任意的$t_3,t_4\in(\xi,1]$，$|t_3-t_4|<\delta_2$，$u\in\Omega$，有

$$\begin{aligned}\left|Tu(t_3)-Tu(t_4)\right| &= \left|\int_{\xi}^{1}\left(G_2(t_3,s)-G_1(t_4,s)\right)f_2\left(s,u(s),u(s-\tau_2)\right)\mathrm{d}s\right.\\ &\quad+\int_0^{\xi}\left(g_1(t_3,s)-g_1(t_4,s)\right)f_1\left(s,u(s),u(s+\tau_1)\right)\mathrm{d}s\\ &\quad\left.+\frac{t_3-t_4}{\Delta_1}\left(\frac{Q\left(\xi,x(\xi)\right)m_2}{\Gamma(3-\beta)}\xi^{2-\alpha}-\gamma_1(x)\right)\right|\\ &\le \bar{M}_2\left(\int_{\xi}^{1}\left|G_2(t_3,s)-G_1(t_4,s)\right|\mathrm{d}s+\int_0^{\xi}\left|g_1(t_3,s)-g_1(t_4,s)\right|\mathrm{d}s\right)\\ &\quad+\left|t_3-t_4\right|\frac{\left|m_2\xi^{2-\alpha}-(m_2+1)\Gamma(3-\beta)\right|}{\Delta_1\Gamma(3-\beta)}\bar{M}_2\\ &<\varepsilon.\end{aligned}$$

因此，$T(\Omega)$在$[0,\xi],(\xi,1]$上等度连续，易知，当$t\in[0,1]$时，对任意$\varepsilon>0$，存在$\delta_3>0$，当$|t_5-t_6|<\delta_3$时$\left|Tu(t_5)-Tu(t_6)\right|<\varepsilon$，因此，$T(\Omega)$是等度连续的。

由 Arzela-Ascoli 定理知$T(\Omega)$相对列紧。又因为算子T是连续算子，所以算子T是全连续的。

证毕。

引理 6：T为强增算子。

证明：对任意$x_1,x_2\in E,x_1\prec x_2$，即$x_1(t)\le x_2(t)$且$x_1(t)\not\equiv x_2(t)$，由(H2)可得，

$$f_1\left(t,x_2(t),x_2(t+\tau_1)\right)-f_1\left(t,x_1(t),x_1(t+\tau_1)\right)\ge 0,t\in[0,\xi],$$

$$f_2\left(t,x_2(t),x_2(t-\tau_2)\right)-f_2\left(t,x_1(t),x_1(t-\tau_2)\right)\ge 0,t\in(\xi,1],$$

$$\left(I\left(\xi,x_2(\xi)\right)-I\left(\xi,x_1(\xi)\right)\right)\ge 0,\left(Q\left(\xi,x_2(\xi)\right)-Q\left(\xi,x_1(\xi)\right)\right)\ge 0,\ t\in[0,1].$$

由于 $x_1(t)\not\equiv x_2(t)$，则存在区间 $[a,b]\subset[0,\xi]$ 或 $[a,b]\subset(\xi,1]$ 使得当 $t\in[a,b]$ 时， $x_1(t)<x_2(t)$。

因此，当 $[a,b]\subset[0,\xi]$ 时，

$$f_1(t,x_2(t),x_2(t+\tau_1))-f_1(t,x_1(t),x_1(t+\tau_1))>0,$$

$$\left(I(\xi,x_2(\xi))-I(\xi,x_1(\xi))\right)>0,\left(Q(\xi,x_2(\xi))-Q(\xi,x_1(\xi))\right)>0,$$

且由(H3)可得

$$\begin{aligned}\gamma_0(x_2)-\gamma_0(x_1)&=h_0(x_2(0),x_2(1))-h_0(x_1(0),x_1(1))\\&\quad+(m_1x_2(0)+n_1x_2(1))-\left(\begin{array}{l}m_1x_1(0)+n_1x_1(1)\\ \le 0,\end{array}\right)\end{aligned}$$

$$\begin{aligned}\gamma_1(x_2)-\gamma_1(x_1)&=h_1\left({}_t^cD_{\xi^-}^{\alpha-1}x_2(0),{}_{\xi^+}^{\ c}D_t^{\beta-1}x_2(1)\right)-h_1\left({}_t^cD_{\xi^-}^{\alpha-1}x_2(0),{}_{\xi^+}^{\ c}D_t^{\beta-1}x_2(1)\right)\\&\quad+m_2\,{}_t^cD_{\xi^-}^{\alpha-1}x_2(0)+n_2\,{}_{\xi^+}^{\ c}D_t^{\beta-1}x_2(1)-\left(m_2\,{}_t^cD_{\xi^-}^{\alpha-1}x_1(0)+n_2\,{}_{\xi^+}^{\ c}D_t^{\beta-1}x_1(1)\right)\\&\le 0,\end{aligned}$$

$$\begin{aligned}\Delta_2^{x_2}-\Delta_2^{x_1}&=-\frac{1}{m_1+n_1}\Big(n_1\left(I(\xi,x_2(\xi))-I(\xi,x_1(\xi))\right)\Big)\\&\quad+n_1\left(1-\xi+\frac{n_2(1-\xi)^{2-\beta}}{\Delta_1\Gamma(3-\beta)}\right)\left(Q(\xi,x_2(\xi))-Q(\xi,x_1(\xi))\right)\\&\quad-\frac{n_1}{\Delta_1}\left(\gamma_1(x_2)-\gamma_1(x_1)\right)-\left(\gamma_0(x_2)-\gamma_0(x_1)\right)\ge 0,\end{aligned}$$

$$\begin{aligned}&Tx_2(t)-Tx_1(t)\\&=\int_0^\xi G_1(t,s)\left(f_1(s,x_2(s),x_2(s+\tau_1))-f_1(s,x_1(s),x_1(s+\tau_1))\right)\mathrm{d}s+\Delta_2^{x_2}-\Delta_2^{x_1}\\&\quad+\int_\xi^1 g_2(t,s)\left(f_2(t,x_2(t),x_2(t-\tau_2))-f_2(t,x_1(t),x_1(t-\tau_2))\right)\mathrm{d}s\\&\quad+\frac{t}{\Delta_1}\left(\frac{\left(Q(\xi,x_2(\xi))-Q(\xi,x_1(\xi))\right)n_2(1-\xi)^{2-\beta}}{\Gamma(3-\beta)}-\gamma_1(x_2)-\gamma_1(x_1)\right)\\&>\int_0^\xi G_1(t,s)\left(f_1(s,x_2(s),x_2(s+\tau_1))-f_1(s,x_1(s),x_1(s+\tau_1))\right)\mathrm{d}s\\&>0.\end{aligned}$$

同理当 $[a,b]\subset(\xi,1]$ 时，

$$\left(f_2(t,x_2(t),x_2(t-\tau_2))-f_2(t,x_1(t),x_1(t-\tau_2))\right)>0,$$

$$\begin{aligned}\Delta_3^{x_2}-\Delta_3^{x_1}&=-\frac{1}{m_1+n_1}\Big(-m_1\left(I(\xi,x_2(\xi))-I(\xi,x_1(\xi))\right)\Big)\\&\quad+\left(m_1\xi+n_1+\frac{n_1n_2}{\Delta_1\Gamma(3-\beta)}\right)\left(Q(\xi,x_2(\xi))-Q(\xi,x_1(\xi))\right)\\&\quad-\frac{n_1}{\Delta_1}\left(\gamma_1(x_2)-\gamma_1(x_1)-\left(\gamma_0(x_2)-\gamma_0(x_1)\right)\right)\\&\ge 0,\end{aligned}$$

$$\begin{aligned}
&Tx_2(t)-Tx_1(t)\\
&=\int_\xi^1 G_2(t,s)\Big(f_2\big(s,x_2(s),x_2(s-\tau_2)\big)-f_2\big(s,x_1(s),x_1(s-\tau_2)\big)\Big)\mathrm{d}s+\Delta_3^{x_2}-\Delta_3^{x_1}\\
&\quad+\int_0^\xi g_1(t,s)\Big(f_1\big(t,x_2(t),x_2(t+\tau_1)\big)-f_1\big(t,x_1(t),x_1(t+\tau_1)\big)\Big)\mathrm{d}s\\
&\quad+\frac{t}{\Delta_1}\left(\frac{\big(Q(\xi,x_2(\xi))-Q(\xi,x_1(\xi))\big)m_2}{\Gamma(3-\beta)}\xi^{2-\alpha}-\gamma_1(x_2)-\gamma_1(x_1)\right)\\
&>\int_\xi^1 G_2(t,s)\Big(f_2\big(s,x_2(s),x_2(s-\tau_2)\big)-f_2\big(s,x_1(s),x_1(s-\tau_2)\big)\Big)\mathrm{d}s\\
&>0.
\end{aligned}$$

综上所述，对任意 $t\in[0,1]$ 有 $(Tx_2)(t)>(Tx_1)(t)$，则 T 为强增算子。

定义 2：令 $\alpha,\beta\in E$。称 α 为边值问题(1)的一个下解，若 α 满足

$$\begin{cases}
{}_t^cD_{\xi^-}^{\alpha}\alpha(t)\le f_1\big(t,\alpha(t),\alpha(t+\tau_1)\big), & t\in[0,\xi],\\
{}_{\xi^+}^{\ c}D_t^{\beta}\alpha(t)\le f_2\big(t,\alpha(t),\alpha(t-\tau_2)\big), & t\in(\xi,1],\\
\Delta\alpha(\xi)\le I\big(\xi,u(\xi)\big),\ \Delta\alpha'(\xi)\le Q\big(\xi,\alpha(\xi)\big), & \\
h_0\big(\alpha(0),\alpha(1)\big)\le 0,\ h_1\Big({}_t^cD_{\xi^-}^{\alpha-1}\alpha(0),{}_{\xi^+}^{\ c}D_t^{\beta-1}\alpha(1)\Big)\ge 0. &
\end{cases}$$

称 β 为边值问题(1)的一个下解，若 β 满足

$$\begin{cases}
{}_t^cD_{\xi^-}^{\alpha}\beta(t)\ge f_1\big(t,\beta(t),\beta(t+\tau_1)\big), & t\in[0,\xi],\\
{}_{\xi^+}^{\ c}D_t^{\beta}\beta(t)\ge f_2\big(t,\beta(t),\beta(t-\tau_2)\big), & t\in(\xi,1],\\
\Delta\beta(\xi)\ge I\big(\xi,\beta(\xi)\big),\ \ \Delta\beta'(\xi)\ge Q\big(\xi,\beta(\xi)\big), & \\
h_0\big(\beta(0),\beta(1)\big)\ge 0,\ h_1\Big({}_t^cD_{\xi^-}^{\alpha-1}\beta(0),{}_{\xi^+}^{\ c}D_t^{\beta-1}\beta(1)\Big)\le 0. &
\end{cases}$$

4. 主要结论

定理 1：假设(H1)、(H2)、(H3)成立，且边值问题(1)存在两个下解 α_1,α_2 和两个上解 β_1,β_2，且 α_2,β_1 不是边值问题(1)的解，

$$\alpha_1\prec\beta_1\prec\alpha_2\prec\beta_2.$$

则边值问题(1)至少存在三个不同的解 u_1，u_2，u_3 满足：

$$\alpha_1\preceq u_1\prec\prec\beta_1,\alpha_2\prec\prec u_2\preceq\beta_2,\alpha_2\npreceq u_3\npreceq\beta_1.$$

证明：令算子 T 在 $[\alpha_1,\beta_1]$ 上，$T|_{[\alpha_1,\beta_1]}$ 也记作 T。由引理 5 和引理 6 可得，$T:[\alpha_1,\beta_1]\to P$ 是一个全连续强增算子。

通过定义算子 T 可得，

$$\begin{cases} {}_{t}^{c}D_{\xi^-}^{\alpha}(T\alpha_1)(t) = f_1(t,\alpha_1(t),\alpha_1(t+\tau_1)), \quad t\in(0,\xi), \\ {}_{\xi^+}^{\ c}D_{t}^{\beta}(T\alpha_1)(t) = f_2(t,\alpha_1(t),\alpha_1(t-\tau_2)), \quad t\in(\xi,1), \\ \Delta(T\alpha_1)(\xi) = I(\xi,\alpha_1(\xi)), \quad \Delta(T\alpha_1)'(\xi) = Q(\xi,\alpha_1(\xi)), \\ m_1(T\alpha_1)(0)+n_1(T\alpha_1)(1) = h_0(\alpha_1(0),\alpha_1(1))+m_1\alpha_1(0)+n_1\alpha_1(1), \\ m_2\,{}_{t}^{c}D_{\xi^-}^{\alpha-1}(T\alpha_1)(0)+n_2\,{}_{\xi^+}^{\ c}D_{t}^{\beta-1}(T\alpha_1)(1) \\ = h_1\left({}_{t}^{c}D_{\xi^-}^{\alpha-1}\alpha_1(0),\ {}_{\xi^+}^{\ c}D_{t}^{\beta-1}\alpha_1(1)\right)+m_2\,{}_{t}^{c}D_{\xi^-}^{\alpha-1}\alpha_1(0)+n_2\,{}_{\xi^+}^{\ c}D_{t}^{\beta-1}\alpha_1(1). \end{cases}$$

令 $\alpha(t)=(T\alpha_1)(t)-\alpha_1(t)$。由于 α_1 是边值问题(1)的一个下解，则

$${}_{t}^{c}D_{\xi^-}^{\alpha}\alpha(t) = {}_{t}^{c}D_{\xi^-}^{\alpha}(T\alpha_1)(t)-{}_{t}^{c}D_{\xi^-}^{\alpha}\alpha_1(t) = f_1(t,\alpha_1(t),\alpha_1(t+\tau_1))-{}_{t}^{c}D_{\xi^-}^{\alpha}\alpha_1(t)\ge 0,\ \ t\in(0,\xi),$$

$${}_{\xi^+}^{\ c}D_{t}^{\beta}\alpha(t) = {}_{\xi^+}^{\ c}D_{t}^{\beta}(T\alpha_1)(t)-{}_{\xi^+}^{\ c}D_{t}^{\beta}\alpha_1(t) = f_2(t,\alpha_1(t),\alpha_1(t-\tau_2))-{}_{\xi^+}^{\ c}D_{t}^{\beta}\alpha_1(t)\ge 0,\ \ t\in(\xi,1),$$

且

$$\begin{aligned} m_1\alpha(0)+n_1\alpha(1) &= m_1((T\alpha_1)(0)-\alpha_1(0))+n_1((T\alpha_1)(1)-\alpha_1(1)) \\ &= (m_1(T\alpha_1)(0)+n_1(T\alpha_1)(1))-(m_1\alpha_1(0)+n_1\alpha_1(1)) \\ &= h_0(\alpha_1(0),\alpha_1(1))+m_1\alpha_1(0)+n_1\alpha_1(1)-(m_1\alpha_1(0)+n_1\alpha_1(1)) \\ &= h_0(\alpha_1(0),\alpha_1(1)) \\ &\le 0. \end{aligned}$$

$$\begin{aligned} & m_2\,{}_{t}^{c}D_{\xi^-}^{\alpha-1}\alpha(0)+n_2\,{}_{\xi^+}^{\ c}D_{t}^{\beta-1}\alpha(1) \\ & = m_2\,{}_{t}^{c}D_{\xi^-}^{\alpha-1}((T\alpha_1)(0)-\alpha_1(0))+n_2\,{}_{\xi^+}^{\ c}D_{t}^{\beta-1}((T\alpha_1)(1)-\alpha_1(1)) \\ & = m_2\,{}_{t}^{c}D_{\xi^-}^{\alpha-1}(T\alpha_1)(0)+n_2\,{}_{\xi^+}^{\ c}D_{t}^{\beta-1}(T\alpha_1)(1)-\left(m_2\,{}_{t}^{c}D_{\xi^-}^{\alpha-1}\alpha_1(0)+n_2\,{}_{\xi^+}^{\ c}D_{t}^{\beta-1}\alpha_1(1)\right) \\ & = h_1\left({}_{t}^{c}D_{\xi^-}^{\alpha-1}\alpha_1(0),\ {}_{\xi^+}^{\ c}D_{t}^{\beta-1}\alpha_1(1)\right)+m_2\,{}_{t}^{c}D_{\xi^-}^{\alpha-1}\alpha_1(0)+n_2\,{}_{\xi^+}^{\ c}D_{t}^{\beta-1}\alpha_1(1) \\ & \quad -\left(m_2\,{}_{t}^{c}D_{\xi^-}^{\alpha-1}\alpha_1(0)+n_2\,{}_{\xi^+}^{\ c}D_{t}^{\beta-1}\alpha_1(1)\right) = h_1\left({}_{t}^{c}D_{\xi^-}^{\alpha-1}\alpha_1(0),\ {}_{\xi^+}^{\ c}D_{t}^{\beta-1}\alpha_1(1)\right)\ge 0. \end{aligned}$$

$$\Delta\alpha(\xi) = \Delta(T\alpha_1)(\xi)-\Delta\alpha_1(\xi) = I(\xi,\alpha_1(\xi))-\Delta\alpha_1(\xi)\ge 0,$$

$$\Delta\alpha'(\xi) = \Delta(T\alpha_1)'(\xi)-\Delta\alpha_1'(\xi) = Q(\xi,\alpha_1(\xi))-\Delta\alpha_1'(\xi)\ge 0.$$

由引理 4 可知，$\alpha(t)\ge 0$。

因此，$\alpha_1 \preceq T\alpha_1$。同理可得，$\alpha_2 \preceq T\alpha_2$。

由于 α_2 是边值问题(1)的一个下解但不是边值问题(1)的解，则 $(T\alpha_2)\ne\alpha_2$。因此，

$$\alpha_2 \prec T\alpha_2.$$

类似可得，

$$T\beta_1 \prec \beta_1, T\beta_2 \preceq \beta_2.$$

由引理 1 可知，算子 T 至少有三个不动点 $x_1, x_2, x_3 \in [\alpha_1, \beta_2]$ 使得

$$\alpha_1 \preceq x_1 \prec\prec \beta_1, \alpha_2 \prec\prec x_2 \preceq \beta_2, \alpha_2 \npreceq x_2 \npreceq \beta_2.$$

因此，边值问题(1)至少有三个不同解。

证毕。

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