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Cahn-Hilliard和粘性Cahn-Hilliard方程解的最大值估计

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摘 要

本文研究了具有一般非线性条件的**Cahn-Hilliard**方程和粘性**Cahn-Hilliard**方程解的最大值估计。首先，我们用能量估计的方法得到解的L^q 范数有界。然后，利用**Nirenberg-Gagliado**不等式得到解的本性上确界有界。

关键词

Cahn-Hilliard方程，粘性**Cahn-Hilliard**方程，最大值估计，**Nirenberg-Gagliado**不等式

The Maximum Estimate of Solution to the Cahn-Hilliard Equation and Viscous Cahn-Hilliard Equation

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Abstract

In this paper, our aim is to prove the maximum estimates of solutions to the Cahn-Hilliard equation and Viscous Cahn-Hilliard equation with a general nonlinear source term. Firstly, we obtain the boundedness of the L^q norm by using energy estimates. Then, the boundedness of essential supremum is demonstrated by Nirenberg-Gagliado inequality.

Keywords

Cahn-Hilliard Equation, Viscous Cahn-Hilliard Equation, Maximum Estimates, Nirenberg-Gagliado Inequality

1. 引言

Cahn-Hillard于1958年[1]提出的以下方程：

$$u_t + \alpha\Delta^2 u - k\Delta f(u) = 0, \alpha, k > 0 \tag{1.1}$$

在材料科学中扮演重要的角色，因为它描述了二元合金中相分离过程中两相系统的重要定性特征。Cahn-Hilliard方程除了应用于二元合金之外，后来也应用于其他科学领域，如肿瘤生长[2, 3]、细菌薄膜[4]。它也有许多扩展和推广(见例[5, 6])，其中A. Novick-Cohen在[7]中提出了用于分析一阶相变动力学的粘性Cahn-Hilliard方程

$$(1-\nu)u_t = \Delta(-\Delta u + f(u) - \nu u_t), x \in \Omega \tag{1.2}$$

其中$\nu \in [0,1], \Omega \subset \mathbb{R}^n$是有界光滑区域，包括作为极限情况的Cahn-Hilliard方程(当$\nu = 0$)和半线性热方程(当$\nu = 1$)。

近来，不少学者通过对标准粘性Cahn-Hilliard方程非线性项做不同假设，得到了结果。例如,刘长春等人在文献[8] 中考虑$f(u) = -u + \gamma_1 u^2 + \gamma_2 u^3$ 的情况，得到经典解的全局存在性和爆破

性；并且指出γ_2 的符号对于解的全局存在性至关重要。特别地，当$\gamma_2 < 0$ 时，Ke和Yin在[9]得到多维(n⩽5)粘性Cahn-Hilliard 方程经典解的存在性。Huang 等人在[10]考虑$\gamma_2 < 0$的情况，证明了初值足够小时解的全局存在性和初值足够大时解的爆破结果。除此之外，刘长春等人在[11]中得到初值足够小时解的全局存在性和对任意非平凡初值，解爆破的一些结果。

本文，我们考虑如下
粘性Cahn-Hilliard方程的初边值问题

$$\begin{cases} (1-\nu)u_t = \Delta(-\Delta u + f(u) - \nu u_t), x \in \Omega \subset R^n, t > 0, \\ u(0,x) = u_0(x), x \in \Omega, \\ \dfrac{\partial u}{\partial N} = \dfrac{\partial(\Delta u)}{\partial N} = 0, x \in \partial\Omega, t > 0. \end{cases} \tag{1.3}$$

以及推广的Cahn-Hilliard方程的初边值问题

$$\begin{cases} u_t = (-\Delta^2 + \varepsilon\Delta)u - (-\Delta + \varepsilon I)f(u), \\ u(0,x) = u_0(x), x \in \Omega, \\ \dfrac{\partial u}{\partial N} = \dfrac{\partial(\Delta u)}{\partial N} = 0, x \in \partial\Omega, t > 0. \end{cases} \tag{1.4}$$

解的最大值估计。本文主要利用文献[12]的方法和技巧进行证明。首先，我们给出主要结果定理2.3和定理2.4；然后，我们给出了定理2.3 和定理2.4的证明。

在本文中，所有的c都是大于0的常数，并且各处的取值不尽相同。文章中所有的范数都是标准的Sobolev空间范数，其中N表示沿$\partial\Omega$的外法向量，$\nu \in [0,1), \varepsilon > 0, Q_T = \Omega \times (0,T), T > 0$。

2. 主要结果

首先，我们对非线性项$f : \Omega \times R \to R$做如下假设：
假设2.1. f是$2r-1$次多项式，F是f的原函数，f的首项系数$a_{2r-1} > 0$，满足

$$\begin{gathered} f(s) = \textstyle\sum_{j=1}^{2r-1} a_j s^j, r \geq 2, \\ F(s) = \textstyle\sum_{j=2}^{2r} b_j s^j. \end{gathered} \tag{2.1}$$

其中，$a_{j-1} = jb_j \geq 0, r \geq 2$
注2.2. 对$\forall u \in R$，由假设2.1 得[13],

$$f'(u) \geq r(2r-1)b_{2r}u^{2r-2} - c, \tag{2.2}$$

$$\frac{1}{2}b_{2r}u^{2r} - c \leq F(u), \tag{2.3}$$

和

$$f(u) \geq rb_{2r}u^{2r-1} - c. \tag{2.4}$$

本文主要结果是：

定理2.3. 如果$u(x,t) \in C^{2,1}(\bar{Q}_T)$是方程(1.3)的解，则存在常数$\bar{C}(u_0)$ 满足

$$\max_{Q_T}|u(x,t)| \leq \bar{C}(u_0). \tag{2.5}$$

定理2.4. 如果$u(x,t) \in C^{2,1}(\bar{Q}_T)$是方程(1.4)的解，则存在常数$\bar{C}(u_0)$ 满足

$$\max_{Q_T}|u(x,t)| \leq \bar{C}(u_0). \tag{2.6}$$

3. 证明

在证明定理2.3和定理2.4之前，我们先给出了几个重要的引理.

引理3.1. [12] *(Nirenberg-Gagliado不等式)*设Ω有界或$\Omega = R^n, 1 \leq s, q \leq \infty, m, j$为整数，且有

$$\frac{j}{m} \leq a \leq 1, \frac{1}{s} = \frac{j}{n} + a(\frac{1}{r} - \frac{m}{n}) + (1-a)\frac{1}{q}. \tag{3.1}$$

那么存在常数$C_1 = C_1(n,j,s,m,a,r,q), C_2 = C_2(n,j,s,m,a,r,q)$，使得

$$||D^j\nu||_{L^s} \leq C_1||D^m\nu||^a_{L^r}||\nu||^{1-a}_{L^r} + C_2||\nu||_{L^q}. \tag{3.2}$$

引理3.2. 如果$u(x,t) \in C^{2,1}(\bar{Q}_T)$是方程(1.3)的解，则存在常数$\bar{C}(u_0)$ 满足

$$\sup_{0\leq t\leq T}||u(x,t)||_q \leq \bar{C}(u_0), q \leq 2. \tag{3.3}$$

证明：将方程(1.3)两边同时乘以u,并在Ω上积分，我们得到

$$\begin{aligned}\int_\Omega (1-\nu)u_t u\,dx &= \int_\Omega \Delta(-\Delta u + f(u) - \nu u_t)u\,dx,\\ &= \int_\Omega -|\Delta u|^2 dx - \int_\Omega f'(u)|\nabla u|^2\,dx - \int_\Omega u_t\Delta u dx.\end{aligned} \tag{3.4}$$

利用条件$u(x,t) \in C^{2,1}(\bar{Q}_T)$，我们得到

$$\int_\Omega -|\Delta u|^2 dx \leq c. \tag{3.5}$$

$$-\int_\Omega u_t\Delta u dx \leq c. \tag{3.6}$$

由方程(2.2)以及条件$u(x,t) \in C^{2,1}(\bar{Q}_T)$，我们得到

$$\begin{aligned} -\int_\Omega f'(u)|\nabla u|^2 dx &\leq \int_\Omega (c - r(2r-1)b_{2r}u^{2r-2})|\nabla u|^2 dx, \\ &\leq \int_\Omega c|\nabla u|^2 dx, \\ &\leq c. \end{aligned} \tag{3.7}$$

$$\begin{aligned} \int_\Omega (1-\nu)u_t u\, dx &= \frac{1-\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2 dx, \\ &\leq c. \end{aligned} \tag{3.8}$$

则由$Gronwall's$引理，得到

$$||u||_{L^2} \leq c.$$

得证。

引理3.3. 如果$u(x,t) \in C^{2,1}(\bar{Q}_T)$是方程(1.4)的解，则存在常数$\bar{C}(u_0)$ 满足

$$\sup_{0 \leq t \leq T} ||u(x,t)||_q \leq \bar{C}(u_0), q \leq 2r. \tag{3.9}$$

证明：将方程(1.4)两边同时乘以$-\Delta u + f(u)$，并在Ω上积分，我们得到

$$\int_\Omega (-\Delta u + f(u))u_t dx = -\int_\Omega [(\nabla(\Delta u - f(u)))^2 + \varepsilon(-\Delta u + f(u))^2] dx \tag{3.10}$$

因为

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\int_\Omega F(u) + \frac{1}{2}|\nabla u|^2 dx) &= \int_\Omega [f(u)u_t + \nabla u \nabla u_t] dx, \\ &= \int_\Omega (-\Delta u + f(u))u_t dx, \\ &\leq 0. \end{aligned} \tag{3.11}$$

令

$$E(t) = E(u(x,t)) = \int_\Omega F(u) + \frac{1}{2}|\nabla u|^2 dx. \tag{3.12}$$

由(3.11)，我们得到，

$$E(u(t)) \leq E(u(0)),$$

即

$$\int_\Omega F(u) + \frac{1}{2}|\nabla u|^2 dx \leq E(u(0)).$$

由方程(2.3)，我们得到

$$\int_{\Omega} \frac{1}{2} b_{2r} u^{2r} dx \leq E(u_0) + c.$$

那么

$$\sup_{0 \leq t \leq T} ||u||_q \leq c, q \leq 2r. \tag{3.13}$$

3.1. 定理2.3的证明

证明.$\forall p \geq 3$,将方程(1.3)中两边同时乘以$u|u|^{p-1}$,并且在Ω上积分，我们得到

$$\begin{aligned}\int_{\Omega} (1-\nu) u_t u|u|^{p-1} dx &= \int_{\Omega} u|u|^{p-1} \Delta(-\Delta u + f(u) - \nu u_t) u\, dx,\\ &= \int_{\Omega} \Delta u \Delta(u|u|^{p-1})\, dx - \int_{\Omega} \nabla f(u) \nabla(u|u|^{p-1})\, dx - \int_{\Omega} \nu \Delta u_t u|u|^{p-1} dx,\\ &= -\int_{\Omega} \Delta|u|\Delta|u|^p\, dx - p\int_{\Omega} f'(u)|\nabla u|^2|u|^{p-1}\, dx - \int_{\Omega} \nu u_t \Delta u|u|^{p-1} dx.\end{aligned} \tag{3.14}$$

利用条件$u(x,t) \in C^{2,1}(\bar{Q}_T)$，我们得到

$$-\int_{\Omega} \nu u_t \Delta u|u|^{p-1} dx \leq c$$

(i)当$\nu = 0$时，方程(1.3)变为标准的Cahn-Hillard方程
由条件$u(x,t) \in C^{2,1}(\bar{Q}_T)$，则

$$\begin{aligned}-\int_{\Omega} \Delta|u|\Delta|u|^p\, dx &\leq c\int_{\Omega} \Delta|u|^p\, dx,\\ &= c(\int_{\Omega} p(p-1)|u|^{p-2}|\nabla u|^2 + p|u|^{p-1}\Delta|u|\, dx),\\ &\leq cp^2 \int_{\Omega} h(u)\, dx.\end{aligned} \tag{3.15}$$

其中$h(u) = \max\{|u|^{p-2}, |u|^{p-1}\}$.
由$Young's\, inequality$，我们得到

$$\begin{aligned}\int_{\Omega} p^2|u|^{p-2} dx &\leq c + c\int_{\Omega} \frac{p-2}{p+1}(p^2|u|^{p-2})^{\frac{p+1}{p-2}}\, dx,\\ &\leq c + c\int_{\Omega} p^{\frac{2(P+1)}{P-2}}|u|^{p+1}\, dx,\\ &\leq c + c\int_{\Omega} p^8|u|^{p+1}\, dx.\end{aligned} \tag{3.16}$$

$$\int_{\Omega} p^2|u|^{p-1}\,dx \le c + c\int_{\Omega}\frac{p-1}{p+1}(p^2|u|^{p-1})^{\frac{p+1}{p-1}}\,dx,$$
$$\le c + c\int_{\Omega} p^{\frac{2(P+1)}{P-1}}|u|^{p+1}\,dx, \tag{3.17}$$
$$\le c + c\int_{\Omega} p^8|u|^{p+1}\,dx.$$

结合条件$f'(u) \ge r(2r-1)b_{2r}u^{2r-2} - c$及$u|u|^{p-1}u_t = \frac{1}{p+1}\frac{\mathrm{d}}{\mathrm{d}t}|u|^{p+1}$，得到

$$\frac{1}{p+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}|u|^p dx + cp\int_{\Omega}|\nabla u|^2|u|^{2r+p-3}\,dx \le c(1+p^8)\int_{\Omega}|u|^{p+1}\,dx. \tag{3.18}$$

因为

$$|\nabla u|^2|u|^{2r+p-3} = \frac{4}{(2r+p-1)^2}|\nabla|u|^{\frac{(2r+p-1)}{2}}|^2. \tag{3.19}$$

综合方程(3.18), (3.19)，得到

$$\frac{1}{p+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}|u|^p dx + \frac{4cp}{(2r+p-1)^2}\int_{\Omega}|\nabla|u|^{\frac{(2r+p-1)}{2}}|^2\,dx \le c(1+p^8)\int_{\Omega}|u|^{p+1}\,dx. \tag{3.20}$$

当$|u(x,t)| > 1$时，因为$2r+p-3 > p-1$，则$|u|^{2r+p-3} > |u|^{p-1}$，故$|\nabla u|^2|u|^{2r+p-3} \ge |\nabla u|^2|u|^{p-1}$，即，

$$\frac{4}{(p+1)^2}|\nabla|u|^{\frac{(p+1)}{2}}|^2 \le \frac{4}{(2r+p-1)^2}|\nabla|u|^{\frac{(2r+p-1)}{2}}|^2. \tag{3.21}$$

由方程(3.20)和(3.21)，得到

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}|u|^{p+1}dx + \frac{4cp}{(p+1)}\int_{\Omega}|\nabla|u|^{\frac{(p+1)}{2}}|^2\,dx \le C(1+p)^9\int_{\Omega}|u|^{p+1}\,dx. \tag{3.22}$$

在(3.1)中，令$s=2, j=0, r=2, m=1$，得到

$$||\nu||_{L^2}^2 \le C_1||D\nu||_{L^2}^{2a}||\nu||_{L^q}^{1-a} + C_2||\nu||_{L^q}^2. \tag{3.23}$$

设$\nu = |u|^{\frac{\mu_k+1}{2}}, \mu_k = 2^k, q = \frac{2(\mu_{k-1}+1)}{\mu_k+1}$，由(3.2)得到

$$a = \frac{n(2-q)}{n(2-q)+2q} = \frac{n}{n+2+2^{2-k}} \tag{3.24}$$

在方程(3.23)中利用$Young's$不等式

$$\int_{\Omega}|u|^{\mu_k+1}dx \le \epsilon\int_{\Omega}|\nabla|u|^{\frac{\mu_k+1}{2}}|^2\,dx + c\epsilon^{-\frac{a}{1-a}}(\int_{\Omega}|u|^{\mu_{k-1}+1}\,dx)^{\frac{\mu_k+1}{\mu_{k-1}+1}}. \tag{3.25}$$

在方程(3.22)中令$p=\mu_k$ 及利用方程(3.25),我们得到

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^{\mu_k+1}dx+\frac{4c_1\mu_k}{(\mu_k+1)}\int_\Omega |\nabla |u|^{\frac{(\mu_k+1)}{2}}|^2\,dx,\\&\le C(1+\mu_k)^9(\epsilon\int_\Omega |\nabla |u|^{\frac{\mu_k+1}{2}}|^2\,dx+c\epsilon^{-\frac{a}{1-a}}(\int_\Omega |u|^{\mu_{k-1}+1}\,dx)^{\frac{\mu_k+1}{\mu_{k-1}+1}}).\end{aligned}\tag{3.26}$$

令$\epsilon=\frac{1}{C(\mu_k+1)^9}\cdot\frac{c_1\mu_k}{\mu_k+1}$，我们得到

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^{\mu_k+1}dx+C_1(k)\int_\Omega |\nabla |u|^{\frac{(\mu_k+1)}{2}}|^2\,dx\le C_2(k)(\int_\Omega |u|^{\mu_{k-1}+1}\,dx)^{\frac{\mu_k+1}{\mu_{k-1}+1}}.\tag{3.27}$$

其中$C_1(k)=\frac{c_1\mu_k}{\mu_k+1}, C_2(k)=C^{\frac{1}{1-a}}.c.(\frac{c_1\mu_k}{\mu_k+1})^{-\frac{1}{1-a}}.(1+\mu_k)^{\frac{9}{1-a}}$
在方程(3.25)中令$\epsilon=1$及由方程(3.27)，我们得到

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^{\mu_k+1}dx+C_1(k)\int_\Omega |u|^{\mu_k+1}\,dx\le C_4(k)(\int_\Omega |u|^{\mu_{k-1}+1}\,dx)^{\frac{\mu_k+1}{\mu_{k-1}+1}}.\tag{3.28}$$

其中$C_4(k)=C_2(k)+c.$
利用$Gronwall$不等式，我们得到

$$\begin{aligned}&\int_\Omega |u|^{\mu_k+1}\,dx\int_\Omega |u_0|^{\mu_k+1}\,dx+\frac{C_4(k)}{C_1(k)}(\sup_{t\ge 0}\int_\Omega |u|^{\mu_{k-1}+1}\,dx)^{\frac{\mu_k+1}{\mu_{k-1}+1}},\\&\le \delta(k)\max\{M_0^{\mu_k+1}|\Omega|,(\sup_{t\ge 0}\int_\Omega |u|^{\mu_{k-1}+1}\,dx)^{\frac{\mu_k+1}{\mu_{k-1}+1}}\}.\end{aligned}\tag{3.29}$$

其中，$\delta(k)=c(1+\mu_k)^\alpha,\alpha=\frac{2}{1-a},M_0=\sup_{x\in\Omega}|u_0|.$

由$H\ddot{o}lder$不等式，我们得到

$$\begin{aligned}\int_\Omega |u|^{\mu_k+1}dx&\le \delta(k)\max\{M_0^{\mu_k+1}|\Omega|,(\sup_{t\ge 0}\int_\Omega |u|^{\mu_{k-1}+1}dx)^{\frac{\mu_k+1}{\mu_{k-1}+1}}\},\\&\le \delta(k)|\Omega|\max\{M_0^{\mu_k+1},(\sup_{t\ge 0}\int_\Omega |u|^2dx)^{\frac{\mu_k+1}{2}}\},\\&\le \prod_{i=0}^{k}(|\Omega|\delta(k-i))^{\frac{\mu_k+1}{\mu_{k-i}+1}}\max\{M_0^{\mu_k+1},(\sup_{t\ge 0}\int_\Omega |u|^2dx)^{\frac{\mu_k+1}{2}}\}.\end{aligned}\tag{3.30}$$

因为$\frac{\mu_k+1}{\mu_{k-i}+1}<2^i$，我们得到

$$\begin{aligned}&\delta(k)\delta(k-1)^{\frac{\mu_k+1}{\mu_{k-1}+1}}\delta(k-2)^{\frac{\mu_k+1}{\mu_{k-2}+1}}...\delta(0)^{\frac{\mu_k+1}{2}},\\&\le c^{2^{k+1}-1}(2^\alpha)^{-k+2^{k+2}-2}.\end{aligned}\tag{3.31}$$

和

$$|\Omega|\cdot|\Omega|^{\frac{\mu_k+1}{\mu_{k-1}+1}}...|\Omega|^{\frac{\mu_k+1}{2}}\le|\Omega|^{2^{k+1}+1}.\tag{3.32}$$

因此结合方程(3.30)，(3.31),(3.32)以及引理3.2，我们得到

$$(\int_\Omega |u|^{2^k+1}dx)^{\frac{1}{2^k+1}} \le C|\Omega|2^{4\alpha}\max\{M_0,(\sup_{t\ge0}(\int_\Omega |u|^2dx)^{\frac{1}{2}}\} \le \bar{C}(u_0). \tag{3.33}$$

因为k是任意的，在方程(3.33)中，令$k\to\infty$，我们得到

$$\|u\|_\infty \le \bar{C}(u_0).$$

因此，

$$\sup_{0\le t\le T}\|u\|_\infty \le \bar{C}(u_0). \tag{3.34}$$

因为$u\in C^{2,1}(\bar{Q}_T)$，那么

$$\max_{Q_T}|u(x,t)| \le \bar{C}(u_0)$$

.

当$u(x)\le 1$**时**，由于$2r+p-2>p$，那么$|u_t||u|^{2r+p-2}\le|u_t||u|^p$。
由条件$u\in C^{2,1}(\bar{Q}_T)$,$|u_t||u|^{2r+p-2}\le|u_t||u|^p$以及$L^{2r+P-1}(\Omega)\subset L^p(\Omega)$，我们得到

$$\begin{aligned}\frac{1}{2r+p-1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^{2r+p-1}dx &\le \frac{1}{2r+p-1}\int_\Omega |\frac{\mathrm{d}}{\mathrm{d}t}|u|^{2r+p-1}|dx,\\ &\le \frac{1}{p+1}\int_\Omega |\frac{\mathrm{d}}{\mathrm{d}t}|u|^{p+1}|dx,\\ &= \int_\Omega |u_t||u|^p\,dx,\\ &\le c\int_\Omega |u|^p\,dx,\\ &\le c\int_\Omega |u|^{2r+P-1}\,dx.\end{aligned} \tag{3.35}$$

由条件$u\in C^{2,1}(\bar{Q}_T)$以及$Young's$不等式，我们得到

$$\begin{aligned}\int_\Omega p|\nabla|^2|u|^{2r+p-3}dx &\le c\int_\Omega p|u|^{2r+p-3}\,dx,\\ &\le c\int_\Omega (p|u|^{2r+p-3})^{\frac{2r+p-1}{2r+p-3}}\,dx + c,\\ &\le c\int_\Omega p^2|u|^{2r+p-1}\,dx.\end{aligned} \tag{3.36}$$

结合方程(3.19),(3.35),(3.36)，我们得到

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^{2r+p-1}dx + \frac{4cp}{(2r+p-1)}\int_\Omega |\nabla|u|^{\frac{(2r+p-1)}{2}}|^2 \le c(2r+p)^3\int_\Omega |u|^{2r+p-1}dx. \tag{3.37}$$

重复方程(3.23) − (3.33)的步骤，当$|u(x,t)| \le 1$时，也有

$$max_{Q_T}|u(x,t)| \le \bar{C}(u_0).$$

(ii)当$\nu \in (0,1)$时，和$\nu = 0$时的处理方法一样，也有

$$max_{Q_T}|u(x,t)| \le \bar{C}(u_0).$$

综上所述，得证。

3.2. 定理2.4的证明

证明：我们在方程(1.4)中两边同时乘以$u|u|^{p-1}$,$(p \ge 3)$，并且在Ω上积分，我们得到

$$\begin{aligned}
&\int_\Omega u|u|^{p-1}u_t dx = \int_\Omega u|u|^{p-1}\Delta(\Delta u + f(u)) + \varepsilon(\Delta u - f(u))u|u|^{p-1}dx,\\
&= \int_\Omega \Delta u\Delta(u|u|^{p-1})\,dx - \int_\Omega \nabla f(u)\nabla(u|u|^{p-1})\,dx - \varepsilon\int_\Omega \nabla u.\nabla(u|u|^{p-1})dx - \varepsilon\int_\Omega f(u)u|u|^{p-1}dx,\\
&= -\int_\Omega \Delta|u|\Delta|u|^p\,dx - p\int_\Omega f'(u)|\nabla u|^2|u|^{p-1}\,dx - \varepsilon p\int_\Omega |\nabla u|^2|u|^{p-1}dx - \varepsilon\int_\Omega f(u)u|u|^{p-1}dx,\\
&\le -\int_\Omega \Delta|u|\Delta|u|^p\,dx - p\int_\Omega f'(u)|\nabla u|^2|u|^{p-1}\,dx - \varepsilon\int_\Omega f(u)u|u|^{p-1}dx.
\end{aligned} \tag{3.38}$$

由方程(2.4)以及$L^{p+1}(\Omega) \subset L^p(\Omega)$，我们得到

$$\begin{aligned}
-\varepsilon\int_\Omega f(u)u|u|^{p-1}dx \;&\le -\int_\Omega \{r(2r-1)b_{2r}u^{2r-1}u|u|^{p-1} - cu|u|^{p-1}\}dx,\\
&= \int_\Omega cu|u|^{p-1}dx - c\int_\Omega u^{2r}|u|^{p-1}dx,\\
&\le \int_\Omega c|u|^p dx,\\
&\le c(p+1)\int_\Omega |u|^{p+1}dx.
\end{aligned} \tag{3.39}$$

接来下，重复方程(3.15) − (3.33)的步骤，即可以得到本定理的结论，故不再给出详细证明。得证。

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