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一类带非线性边界条件的二阶差分方程正解的多解性

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摘 要

带非线性边界条件的差分方程边值问题在采用聚氨酯水泥钢丝绳加固桥梁以及求解环形域上椭圆型方程组正径向解等方面有着重要的应用。本文运用不动点指数定理和上下解方法，当非线性项为正函数且在无穷远处超线性增长时，对充分小的参数，建立了上述问题正解的存在性及多解性的结果，这为微分方程边值问题的数值解提供了理论方法。最后通过一个例子说明定理结论的有效性。

关键词

多解性，正解，上下解方法，拓扑度理论

Multiplicity of Positive Solutions for a Class of Second-Order Difference Equation with Nonlinear Boundary Conditions

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Abstract

Boundary value problems of difference equations with nonlinear boundary conditions have many important applications such as in strengthening Bridges with polyurethane cement wire ropes and solving positive radial solutions of elliptic equations in annular domain. In this paper, by using the fixed point index theorem and the method of upper and lower solutions, we obtain the existence and multiplicity of positive solutions to the above problems for sufficiently small parameters when the nonlinear term is a positive function and superlinear growth at infinity. The results provide a theoretical method for numerical solution of boundary value problems of differential equations. Finally, we give an example to illustrate the validity of the main results.

Keywords

Multiplicity, Positive Solution, Upper and Lower Solutions, Topological Degree Theory

1. 引言

带非线性边界条件的微分方程/差分方程边值问题是一类重要的非线性问题,受到了许多学者的研究 [1–11].1998年,DUNNINGER D R和王海燕 [1]运用上下解方法和拓扑度理论研究了带非线性边界条件的二阶微分方程边值问题

$$\begin{cases} u''(x) + \lambda h(x) f(u(x)) = 0, \quad 0 < x < 1, \\ \alpha u(0) - \beta u'(0) = \lambda g_1(u(0)), \\ \gamma u(1) + \delta u'(1) = \lambda g_2(u(1)) \end{cases}$$

正解的多解性,其中$\lambda > 0$, $f : [0, \infty) \to (0, \infty)$连续且$f_\infty = \lim\limits_{u\to\infty} f(u)/u = \infty$, $g_1, g_2 : [0, \infty) \to (0, \infty)$ 连续.此后,这类问题正解的存在性和多解性研究获得了丰硕的成果,参见Umezu, Kenichiro [2], Abdou K.Drame,David G.Costa [5], Shivaji等 [6, 7],马如云等 [9]的工作及其参考文献.特别

地,当$h(t) \equiv 1$时,Abdou K.Drame,David G.Costa [5]运用时间映象方法获得了问题

$$\begin{cases} -u'' = \lambda f(u), \\ -u'(0) + u(0) = g(u(0)), \\ u'(1) + u(1) = g(u(1)) \end{cases}$$

正解的存在性,其中$\lambda > 0$为正参数, $f : [0, \infty) \to \mathbb{R}$是一个局部Lipschitz连续(非线性)函数,且$f(0) < 0$, $g : [0, \infty) \to \mathbb{R}$是一个连续的非线性函数.

近年来,随着科学技术的发展,差分方程边值问题的应用领域越来越广泛,比如计算机科学,经济学,神经网络,生态学及控制论等学科过程中出现了大量的差分方程 [12, 13],这就需要对差分方程进行更深入和广泛的研究.特别地,差分方程边值问题正解的存在性和多解性引起了很多学者的兴趣 [12, 13]. 2013年,李艳,唐淑瑰 [10]运用Guo-Krasnoselskii不动点理论研究了二阶带参数非线性差分方程边值问题

$$\begin{cases} \Delta^2 u(k+1) + \lambda a(k) f(u(k)) = 0, \quad k \in [1, T]_{\mathbb{Z}}, \\ \alpha \Delta u(0) = 0, u(T+1) = \beta u(k) \end{cases}$$

正解的存在性.2014年,路艳琼,马如云 [11]运用分歧理论研究了带非线性边界条件的二阶差分方程

$$\begin{cases} -\Delta[p(k-1)\Delta y(k-1)] + q(k)y(k) = \lambda a(k) f(y(k)), \quad k \in I, \\ -\Delta y(0) + \alpha g(y(0)) = 0, \Delta y(N) + \beta g(y(N+1)) = 0 \end{cases}$$

正解的全局结构,其中$\alpha, \beta \geq 0$为常数,$I := \{1, \cdots\cdots, N\}$,$p : \{0, 1, \cdots\cdots, N\} \to (0, \infty)$, $q, a : I \to [0, \infty)$, 当$k \in I$ 时,$a(k) > 0$.

受上述文献 [1, 10, 11]启发,本文讨论带非线性边界条件的二阶差分方程边值问题

$$\begin{cases} \Delta^2 u(t-1) + \lambda h(t) f(u(t)) = 0, \qquad t \in [1, T]_{\mathbb{Z}}, \\ \alpha u(0) - \beta \Delta u(0) = \lambda g_1(u(0)), \\ \gamma u(T+1) + \delta \Delta u(T) = \lambda g_2(u(T)) \end{cases} \tag{1}$$

正解的存在性及多解性,其中$\alpha, \beta, \gamma, \delta \geq 0$, $\varrho := \alpha\delta + \alpha\gamma(T+1) + \gamma\beta > 0$为参数, $\lambda > 0$, $[1, T]_{\mathbb{Z}} = \{1, 2, \cdots, T\}$, $h : [1, T]_{\mathbb{Z}} \to [0, \infty)$ 且$h \not\equiv 0$;并获得了问题(1)正解的存在性和多解性结果.

定理1.1 假设下列条件

(A1) $f : [0, \infty) \to (0, \infty)$连续;

(A2) $g_1, g_2 : [0, \infty) \to (0, \infty)$连续;

(A3) $f_\infty = \lim\limits_{u \to \infty} f(u)/u = \infty$

成立,则存在一个$\lambda^* > 0$,使得当$0 < \lambda < \lambda^*$时,问题(1)至少有两个正解;当$\lambda = \lambda^*$时,问题(1)至少有一个正解;当$\lambda > \lambda^*$时,问题(1)无正解.

定理1.2 假设$f \equiv 0$且(A2)和

(A4) $(g_1)_\infty = \lim\limits_{u \to \infty} g_1(u)/u = \infty$, $(g_2)_\infty = \lim\limits_{u \to \infty} g_2(u)/u = \infty$

成立,则存在一个$\lambda^* > 0$,使得当$0 < \lambda < \lambda^*$时,问题(1)至少存在两个正解;当$\lambda = \lambda^*$时,问题(1)至少存在一个正解;当$\lambda > \lambda^*$时,问题(1) 无正解.

2. 预备知识

由差分方程的理论知识 [12],易知问题(1)的解等价于问题

$$\begin{cases} \Delta^2 u(t-1) + \lambda h(t) f(u(t)) = 0, \quad t \in [1,T]_{\mathbb{Z}}, \\ \alpha u(0) - \beta \Delta u(0) = 0, \\ \gamma u(T+1) + \delta \Delta u(T) = 0 \end{cases} \tag{2}$$

的解和问题

$$\begin{cases} \Delta^2 u(t-1) = 0, & t \in [1,T]_{\mathbb{Z}}, \\ \alpha u(0) - \beta \Delta u(0) = \lambda g_1(u(0)), \\ \gamma u(T+1) + \delta \Delta u(T) = \lambda g_2(u(T)) \end{cases} \tag{3}$$

的解之和.通过计算可得问题(2)的解为

$$u_1(t) = \lambda \sum_{s=1}^{T} G(t,s) h(s) f(u(s)),$$

问题(3)的解为

$$u_2(t) = \frac{\alpha g_2(u(T)) - \gamma g_1(u(0))}{\varrho} t + \frac{(\delta + \gamma(T+1)) g_1(u(0)) + \beta g_2(u(T))}{\varrho}.$$

因此不难验证问题(1)等价于和分方程

$$u(t) = \lambda(P(u(0), u(T))t + Q(u(0), u(T))) + \lambda \sum_{s=1}^{T} G(t,s) h(s) f(u(s)), \tag{4}$$

其中

$$P(s,t) = \frac{\alpha g_2(t) - \gamma g_1(s)}{\varrho}, \qquad Q(s,t) = \frac{(\delta + \gamma(T+1)) g_1(s) + \beta g_2(t)}{\varrho} \tag{5}$$

且格林函数

$$G(t,s) = \frac{1}{\varrho} \begin{cases} (\delta + \gamma(T+1) - \gamma s)(\beta + \alpha t), & 0 \le t \le s \le T+1, \\ (\delta + \gamma(T+1) - \gamma t)(\beta + \alpha s), & 0 \le s \le t \le T+1. \end{cases} \tag{6}$$

引理2.1 设 $G(t,s)$ 是(6)中定义的格林函数,则 $G(t,s)$ 满足如下性质:

(i) $G(t,s) \ge 0, \quad t,s \in [0,T+1]_{\mathbb{Z}}$;

(ii) $G(t,s) \ge \frac{1}{T+1} G(\tau,s), \quad t \in [1,T]_{\mathbb{Z}}, \ s,\tau \in [0,T+1]_{\mathbb{Z}}$.

证明 由$G(t,s)$的定义知

$$G(t,s) \geq \frac{1}{\varrho}\delta\beta \geq 0 \qquad t,s \in [0,T+1]_{\mathbb{Z}}.$$

特别地,当$t \in [1,T]_{\mathbb{Z}}$时,有

$$G(t,s) \geq \frac{1}{\varrho}\begin{cases} (\delta+\gamma)\beta, & s=0, \\ (\delta+\gamma)(\beta+\alpha), & s\in[1,T]_{\mathbb{Z}}, \\ (\beta+\alpha)\delta, & s=T+1. \end{cases}$$

另一方面,对于任意的$s,\tau \in [0,T+1]_{\mathbb{Z}}$,有

$$G(\tau,s) \leq \frac{1}{\varrho}(\delta+\gamma(T+1)-\gamma s)(\beta+\alpha s)$$

因此$G(t,s) \geq \frac{1}{T+1}G(\tau,s)$成立.

引理2.2 函数$P(x,y)t+Q(x,y)$满足如下性质:
(i) $P(x,y)t+Q(x,y) \geq \min\{Q(x,y), P(x,y)(T+1)+Q(x,y)\} \geq 0, \quad t\in[0,T+1]_{\mathbb{Z}}, \quad x,y>0,$
(ii) $P(x,y)t+Q(x,y) \leq \max\{Q(x,y), P(x,y)(T+1)+Q(x,y)\}, \quad t\in[0,T+1]_{\mathbb{Z}}, \quad x,y>0,$
(iii) $P(x,y)t+Q(x,y) \geq \frac{1}{T+1}(P(x,y)\tau+Q(x,y)), \quad t\in[1,T]_{\mathbb{Z}}, \quad \tau\in[0,T+1]_{\mathbb{Z}}, \quad x,y>0.$

证明 显然,对每个$x,y \geq 0$且g_1,g_2是非负函数的条件下,对$P(x,y)t+Q(x,y)$进行分析,对于任意的$t\in[0,T+1]_{\mathbb{Z}}, \tau\in[0,T+1]_{\mathbb{Z}}$有

(1) 当$P(x,y)\geq 0$时,$P(x,y)t+Q(x,y) \geq Q(x,y) \geq 0$;

(2) 当$P(x,y) < 0$时,$P(x,y)t+Q(x,y) \geq P(x,y)(T+1)+Q(x,y) \geq 0$,

因此性质(i)成立.同理可证性质(ii) 成立.

下证性质(iii)成立.对于任意的$t\in[1,T]_{\mathbb{Z}}, \tau\in[0,T+1]_{\mathbb{Z}}$有

(1) 当$P(x,y)\geq 0$时,

$$\begin{aligned} P(x,y)t+Q(x,y) &\geq P(x,y)+Q(x,y) = \frac{1}{T+1}(P(x,y)(T+1)+Q(x,y)(T+1)) \\ &\geq \frac{1}{T+1}(P(x,y)(T+1)+Q(x,y)) \geq \frac{1}{T+1}(P(x,y)\tau+Q(x,y)); \end{aligned}$$

(2) 当$P(x,y)<0$时,

$$\begin{aligned} P(x,y)t+Q(x,y) &\geq P(x,y)T+Q(x,y) \geq P(x,y)(T+1)+Q(x,y) \\ &= \frac{1}{T+1}(P(x,y)+Q(x,y)(T+1)) = \frac{1}{T+1}(P(x,y)+Q(x,y)T+Q(x,y)) \\ &\geq \frac{1}{T+1}Q(x,y) \geq \frac{1}{T+1}(P(x,y)\tau+Q(x,y)), \end{aligned}$$

因此性质(iii)成立.

引理2.3 ([14]) 设E是Banach空间,$K\subset E$是E中的一个锥.对任意的$r>0$,记$K_r=\{u\in K|\ \|$

$u \parallel\leq r\}$.假设算子$T:\bar{K}_r\to K$全连续,当$u\in\partial K_r$时,使得$Tu\neq u$.

(i) 若$u\in\partial K_r$时,满足$\parallel Tu\parallel\geq\parallel u\parallel$,则$i(T,K_r,K)=0$.

(ii) 若$u\in\partial K_r$时,满足$\parallel Tu\parallel\leq\parallel u\parallel$,则$i(T,K_r,K)=1$.

令$E=\{u|u:[0,T+1]_{\mathbb{Z}}\to\mathbb{R}\}$,则$E$按范数$\parallel u\parallel=\max\limits_{t\in[0,T+1]_{\mathbb{Z}}}|u(t)|$构成Banach空间.

定义集合$K_0=\{u\in E|u\geq 0\}$,则K_0为E中的非负锥.定义E上的锥K如下:

$$K=\{u\in E|u\geq 0,\min_{t\in[1,T]_{\mathbb{Z}}}u(t)\geq\frac{1}{T+1}\parallel u\parallel\}. \tag{7}$$

易见$K\subset K_0$,且(4)可写作不动点方程$Tu=u$, 其中算子$T:E\to E$定义如下:

$$Tu(t)=\lambda(P(u(0),u(T))t+Q(u(0),u(T)))+\lambda\sum_{s=1}^{T}G(t,s)h(s)f(u(s)). \tag{8}$$

引理2.4 $T(K_0)\subset K$,且算子$T:K_0\to K$全连续.

证明 对于任意的$u\in K_0$,当$t\in[1,T]_{\mathbb{Z}}$时,由 引理2.1(ii)和引理2.2(iii)可得

$$\begin{aligned}Tu(t)&\geq\lambda\frac{1}{T+1}\{P(u(0),u(T))\tau+Q(u(0),u(T))+\sum_{s=1}^{T}G(\tau,s)h(s)f(u(s))\}\\&\geq\frac{1}{T+1}Tu(\tau),\quad\tau\in[0,T+1]_{\mathbb{Z}}.\end{aligned}$$

因此

$$\min_{t\in[1,T]_{\mathbb{Z}}}Tu(t)\geq\frac{1}{T+1}\parallel Tu\parallel$$

即$Tu\in K$,故$T(K_0)\subset K$.又因为E为有限维空间,所以由f的连续性,易证$T:K_0\to K$全连续.

3. 正解的存在性与不存在性

定理3.1 假设$(A1)-(A3)$.则当$\lambda>0$充分小时,问题(1)至少存在一个正解;当$\lambda>0$充分大时,问题(1)无正解.

证明 令

$$M=\max_{(t,s)\in[0,T+1]_{\mathbb{Z}}\times[0,T+1]_{\mathbb{Z}}}G(t,s),\quad m=\min_{(t,s)\in[1,T]_{\mathbb{Z}}\times[1,T]_{\mathbb{Z}}}G(t,s),$$

则由引理2.1知,$M,m>0$,对任意的$q>0$,记

$$I(q)=M\max_{u\in K,\|u\|=q}\sum_{s=1}^{T}h(s)f(u(s))>0.$$

对任意的$r_1>0$,记$K_{r_1}=\{u\in K|\parallel u\parallel<r_1\}$.对任意的$u\in\partial K_{r_1}$,存在充分小的$\sigma>0$满足

$$\sigma\leq\frac{r_1}{2I(r_1)},\quad\sigma\max(Q(u(0),u(T)),P(u(0),u(T))+Q(u(0),u(T)))\leq r_1/2.$$

当$\lambda \le \sigma$时,对任意的$t \in [0, T+1]_{\mathbb{Z}}$,由引理2.2(ii)知

$$Tu(t) \le \frac{r_1}{2} + \sigma M \sum_{s=1}^{T} h(s) f(u(s)) \le \frac{r_1}{2} + \sigma I(r_1) \le r_1,$$

即$\| Tu \| \le r_1 = \| u \|, u \in \partial K_{r_1}$.由引理2.3知

$$\deg(T, K_{r_1}, K) = 1.$$

对给定的$\lambda \le \sigma$,因$f_\infty = \infty$,故存在常数$p > 0$,使得对任意的$u \ge p$,有$f(u) \ge \eta u$,其中$\eta > 0$充分大且满足

$$\frac{\lambda m \eta}{T+1} \sum_{s=1}^{T} h(s) \ge 1.$$

取$r_2 \ge \max\{(T+1)p, r_1+1\}$,记$K_{r_2} = \{u \in K | \| u \| < r_2\}$.当$u \in \partial K_{r_2}$,时,$\min\limits_{t \in [1,T]_{\mathbb{Z}}} u(t) \ge \frac{1}{T+1} \| u \| \ge p$. 因此,对任意的$t \in [1, T]_{\mathbb{Z}}$,有

$$\begin{aligned} Tu(t) &\ge \lambda m \sum_{s=1}^{T} h(s) f(u(s)) \ge \lambda m \eta \sum_{s=1}^{T} h(s) f(u(s)) \\ &\ge \frac{\lambda m \eta}{T+1} \| u \| \sum_{s=1}^{T} h(s) \ge \| u \|, \end{aligned}$$

即$\| Tu \| \ge \| u \|, u \in \partial K_{r_2}$,由引理 2.3 知

$$\deg(T, K_{r_2}, K) = 0.$$

由不动点指数的可加性得,$\deg(T, K_{r_2} \backslash \bar{K}_{r_1}, K) = -1$,因此,算子$T$在$K_{r_2} \backslash \bar{K}_{r_1}$至少存在一个不动点,即问题(1)至少存在一个正解.

由$(A1), (A3)$可知,存在一个常数$c > 0$,使得对任意的$u \ge 0$,有$f(u) \ge cu$.假设问题(4)存在正解$u \in E$,由引理2.4知$u \in K$. 取充分大的$\lambda > 0$,使得

$$\frac{\lambda mc}{T+1} \sum_{s=1}^{T} h(s) > 1.$$

则对任意的$t \in [1, T]_{\mathbb{Z}}$,有

$$u(t) \ge \lambda mc \sum_{s=1}^{T} h(s) u(s) \ge \frac{\lambda mc}{T+1} \| u \| \sum_{s=1}^{T} h(s) > \| u \|,$$

这与假设矛盾.因此,当λ充分大时,问题(4)无正解.

4. 多个正解的存在性

为了获得问题(1)多个正解的存在性,我们引入问题(1)的上下解方法.

定义4.1 若$\bar{u}\in E$满足

$$\begin{cases}\Delta^2\bar{u}(t-1)+\lambda h(t)f(\bar{u}(t))\le 0, & t\in[1,T]_{\mathbb{Z}},\\ \alpha\bar{u}(0)-\beta\Delta\bar{u}(0)\ge\lambda g_1(\bar{u}(0)),\\ \gamma\bar{u}(T+1)+\delta\Delta\bar{u}(T)\ge\lambda g_2(\bar{u}(T)),\end{cases}$$

则称$\bar{u}$为问题(1)的上解;若$\underline{u}\in E$满足

$$\begin{cases}\Delta^2\underline{u}(t-1)+\lambda h(t)f(\underline{u}(t))\ge 0, & t\in[1,T]_{\mathbb{Z}},\\ \alpha\underline{u}(0)-\beta\Delta\underline{u}(0)\le\lambda g_1(\underline{u}(0)),\\ \gamma\underline{u}(T+1)+\delta\Delta\underline{u}(T)\le\lambda g_2(\underline{u}(T)),\end{cases}$$

则称$\underline{u}$为问题(1)的下解.

下面给出问题(1)解的存在性结果.

引理4.1 设$\underline{u},\bar{u}\in E$分别为问题(1)的下解和上解,且满足$\underline{u}\le\bar{u}$,则问题(1)存在一个解$u$满足$\underline{u}(t)\le u(t)\le\bar{u}(t), t\in[0,T+1]_{\mathbb{Z}}$.

证明 考察辅助问题

$$\begin{cases}\Delta^2u(t-1)+\lambda h(t)f^*(u(t))=0, & t\in[1,T]_{\mathbb{Z}},\\ \alpha u(0)-\beta\Delta u(0)=\lambda g_1^*(u(0)),\\ \gamma u(T+1)+\delta\Delta u(T)=\lambda g_2^*(u(T)),\end{cases}\tag{9}$$

其中

$$f^*(u(t))=\begin{cases}f(\bar{u}(t)), & u(t)\ge\bar{u}(t),\\ f(u(t)), & \underline{u}(t)\le u(t)\le\bar{u}(t),\\ f(\underline{u}(t)), & u(t)\le\underline{u}(t),\end{cases}$$

$$g_1^*(u(t))=\begin{cases}g_1(\bar{u}(t)), & u(t)\ge\bar{u}(t),\\ g_1(u(t)), & \underline{u}(t)\le u(t)\le\bar{u}(t),\\ g_1(\underline{u}(t)), & u(t)\le\underline{u}(t),\end{cases}$$

$$g_2^*(u(t))=\begin{cases}g_2(\bar{u}(t)), & u(t)\ge\bar{u}(t),\\ g_2(u(t)), & \underline{u}(t)\le u(t)\le\bar{u}(t),\\ g_2(\underline{u}(t)), & u(t)\le\underline{u}(t).\end{cases}$$

易见问题(9)等价于和分方程

$$u(t)=\lambda(P^*(u(0),u(T))t+Q^*(u(0),u(T)))+\lambda\sum_{s=1}^{T}G(t,s)h(s)f^*(u(s)),$$

其中

$$P^*(s,t)=\frac{\alpha g_2^*(t)-\gamma g_1^*(s)}{\varrho},$$

$$Q^*(s,t)=\frac{(\delta+\gamma(T+1))g_1^*(s)+\beta g_2^*(t)}{\varrho},$$

$G(t,s)$是相对应的格林函数.令

$$T^*u(t) = \lambda(P^*(u(0),u(T))t + Q^*(u(0),u(T))) + \lambda\sum_{s=1}^{T} G(t,s)h(s)f^*(u(s)),$$

由引理2.4易证算子$T^*: E \to E$全连续.由于f^*, g_1^*, g_2^* 是有界的,因此T^*有界.根据 Schauder 不动点定理知,T^*有一个不动点u,即u为问题(9)的一个解.
下面证明问题(9)的解u满足$\underline{u}(t) \le u(t) \le \bar{u}(t)$.
首先证明$u(t) \le \bar{u}(t)$.假设存在$t_0 \in [0,T+1]_{\mathbb{Z}}$,使得$u(t_0) > \bar{u}(t_0)$.则有以下四种情况:

(i) $u(t) > \bar{u}(t)$,$t \in [0,T+1]_{\mathbb{Z}}$.有

$$f^*(u(t)) = f(\bar{u}(t)), \quad g_1^*(u(0)) = g_1(\bar{u}(0)), \quad g_2^*(u(T)) = g_2(\bar{u}(T)),$$

因此

$$\Delta^2(\bar{u}-u)(t-1) \le 0, \quad \alpha(\bar{u}-u)(0) - \beta\Delta(\bar{u}-u)(0) \ge 0, \quad \gamma(\bar{u}-u)(T+1) + \delta\Delta(\bar{u}-u)(T) \ge 0,$$

根据极大值原理 [13]知,对于任意的$t \in [0,T+1]_{\mathbb{Z}}$有$\bar{u}(t) \ge u(t)$,矛盾!

(ii) $u(t) > \bar{u}(t)$,$t \in [a,b]_{\mathbb{Z}}, a, b \in [0,T+1]$,且$u(a-1) \le \bar{u}(a-1), u(b+1) \le \bar{u}(b+1)$.
则

$$\Delta^2(\bar{u}-u)(t-1) \le 0, \quad t \in [a,T+1]_{\mathbb{Z}},$$

$$\alpha(\bar{u}-u)(a-1) - \beta\Delta(\bar{u}-u)(a-1) \ge 0, \quad \gamma(\bar{u}-u)(b+1) + \delta\Delta(\bar{u}-u)(b) \ge 0.$$

再次由极大值原理 [13]知,对于任意的$t \in [a,b]_{\mathbb{Z}}$,$\bar{u}(t) \ge u(t)$,矛盾!

运用相似的方法可处理以下两种情况:

(iii) $u(t) > \bar{u}(t)$,$t \in [0,a-1]_{\mathbb{Z}}, a \in [0,T+1]_{\mathbb{Z}}$,且$u(a) \le \bar{u}(a)$.

(iv) $u(t) > \bar{u}(t)$,$t \in [a+1,T+1]_{\mathbb{Z}}, a \in [0,T+1]_{\mathbb{Z}}$,且$u(a) \le \bar{u}(a)$.

同理可证$\underline{u}(t) \le u(t), t \in [0,T+1]_{\mathbb{Z}}$.从而$\underline{u}(t) \le u(t) \le \bar{u}(t)$.因此$f^* = f, g_1^* = g_1, g_2^* = g_2$,即$u$是问题(1) 的一个解.

由于我们考虑的是问题(1)的正解,规定当$u < 0$时,有$f(u) = f(0), g_1(u) = g_1(0), g_2(u) = g_2(0)$.

引理4.2 设(A1)-(A3)成立,令$I \subset (0,\infty)$为紧子集,若$\lambda \in I$,则存在常数$b_I > 0$,使得问题(1)的所有解u满足$\| u \| \le b_I$.

证明 假设$\{u_n\}_{n=1}^{\infty}$是问题(1)的解,且$\lim\limits_{n\to\infty} | u_n | \ne \infty$,$\lambda_n \in I$.由引理2.4知$u_n \in K$.由$f_\infty = \infty$知,选取$\eta > 0$充分大满足$\frac{\lambda_n \eta}{T}\sum\limits_{s=1}^{T} h(s) \ge 2$. 则存在一个$p > 0$,使得当$u \ge p$时,$f(u) \ge \eta u$.
因$\lim\limits_{n\to\infty} | u_n | = \infty$,故对任意的$p > 0$,存在$N > 0$,使得当$n \ge N$时,有$\min\limits_{t\in[1,T]} u_n(t) \ge \frac{1}{T+1} \| u_n \| \ge p$.因此

$$u_n(t) \ge \lambda_n \sum_{s=1}^{T} h(s)f(u_n(s)) \ge \frac{\lambda_n \eta}{T+1} \| u_n \| \sum_{s=1}^{T} h(s) \ge 2 \| u_n \|,$$

矛盾!因此假设错误,原命题正确.

令$\Gamma = \{\lambda > 0 \,|$问题(1)存在一个正解$\}$,且令$\lambda^* = \sup \Gamma$.由定理3.1知$\Gamma \neq \emptyset$,且$0 < \lambda^* < \infty$.下证$\lambda^* \in \Gamma$. 设$\lambda_n \in \Gamma$且$\lambda_n \to \lambda^*$.因为$\lambda_n$是有界的,所以由引理4.2知$\lambda_n$相对应的解$u_n$是有界的.由和分算子$\Gamma$的全连续性知$\lambda^* \in \Gamma$.

令λ^*是问题(1)的一个正解u^*所对应的参数,且定义

$$\tilde{f}(u(t)) = \begin{cases} f(u^*(t)+\varepsilon), & u(t) \geq u^*(t)+\varepsilon, \\ f(u(t)), & -\varepsilon \leq u(t) \leq u^*(t)+\varepsilon, \\ f(-\varepsilon), & u(t) \leq -\varepsilon, \end{cases}$$

$$\tilde{g}_1(u(t)) = \begin{cases} g_1(u^*(t)+\varepsilon), & u(t) \geq u^*(t)+\varepsilon, \\ g_1(u(t)), & -\varepsilon \leq u(t) \leq u^*(t)+\varepsilon, \\ g_1(-\varepsilon), & u(t) \leq -\varepsilon, \end{cases}$$

$$\tilde{g}_2(u(t)) = \begin{cases} g_2(u^*(t)+\varepsilon), & u(t) \geq u^*(t)+\varepsilon, \\ g_2(u(t)), & -\varepsilon \leq u(t) \leq u^*(t)+\varepsilon, \\ g_2(-\varepsilon), & u(t) \leq -\varepsilon. \end{cases}$$

令

$$\tilde{T}_\lambda(u(t)) = \lambda(\tilde{P}(u(0),u(T))t + \tilde{Q}(u(0),u(T))) + \lambda \sum_{s=1}^{T} G(t,s)h(s)\tilde{f}(u(s)),$$

其中

$$\tilde{P}(s,t) = \frac{\alpha \tilde{g}_2(t) - \gamma \tilde{g}_1(s)}{\varrho}, \qquad \tilde{Q}(s,t) = \frac{(\delta+\gamma(T+1))\tilde{g}_1(s) + \beta \tilde{g}_2(t)}{\varrho}.$$

考察

$$\Omega = \{u \in E | -\varepsilon < u(t) < u^*(t)+\varepsilon\}.$$

引理4.3 设$u \in E$,存在一个充分小的$\varepsilon > 0$,当$0 < \lambda < \lambda^*$时满足$\tilde{T}_\lambda u = u$,则$u \in \bar{\Omega}$.

证明 显然$u \geq 0$.为了证明$u \leq u^* + \varepsilon$,首先说明$u^* + \varepsilon$是问题(1)的一个上解.因为$u^* \geq 0$,所以存在一个常数$c > 0$,使得当$t \in [1,T]_{\mathbb{Z}}$时,有$f(u^*(t)) > c$.通过一致连续性,存在一个ε_0, 使得当$0 \leq \varepsilon \leq \varepsilon_0$时,有

$$|\ f(u^*(t)+\varepsilon) - f(u^*(t))\ | < c(\lambda^* - \lambda)/\lambda.$$

则

$$\begin{aligned} \Delta^2(u^*+\varepsilon)(t-1) &= \Delta^2 u^*(t-1) = -\lambda^* h(t) f(u^*(t)) \\ &= -\lambda h(t) f(u^*(t)+\varepsilon) + \lambda[h(t)f(u^*(t)+\varepsilon) - h(t)f(u^*(t))] + (\lambda - \lambda^*)h(t)f(u^*(t)) \\ &< -\lambda h(t) f(u^*(t)+\varepsilon) + ch(t)(\lambda^* - \lambda) + ch(t)(\lambda - \lambda^*) = -\lambda h(t) f(u^*(t)+\varepsilon). \end{aligned}$$

因此

$$\Delta^2(u^*+\varepsilon)(t-1) + \lambda h(t) f(u^*(t)+\varepsilon) \leq 0.$$

同理,当$\varepsilon > 0$充分小时,由g_1, g_2的一致连续性可得

$$\alpha(u^*(0)+\varepsilon) - \beta\Delta(u^*(0)+\varepsilon) \geq \lambda g_1(u^*(0)+\varepsilon), \quad \gamma(u^*(T+1)+\varepsilon) + \delta\Delta(u^*(T)+\varepsilon) \geq \lambda g_2(u^*(T)+\varepsilon).$$

故$u^*+\varepsilon$是问题(1)的一个上解.根据引理4.1知$u\le(u^*+\varepsilon)$.

定理1.1的证明 令$0<\lambda<\lambda^*$,因为u^*和0分别是问题(1)的上解和下解,所以由引理4.1知,问题(1)存在一个正解u_λ满足$0\le u_\lambda\le u^*$, 因此当$0<\lambda\le\lambda^*$时,问题(1)存在一个正解;当$\lambda>\lambda^*$时,问题(1)无正解.进一步,由引理4.3知,当$0<\lambda<\lambda^*$ 时,$u_\lambda\in\Omega$.

记$B(u_\lambda,R)$为E中以u_λ为心,R为半径的球.对充分大的R,由$\tilde{T}_\lambda$在λ的紧区间上有界知,

$$\deg(I-\tilde{T}_\lambda,B(u_\lambda,R),0)=1.$$

若存在$u\in\partial\Omega$,使得$u=\tilde{T}_\lambda u$,则$f=\tilde{f},g=\tilde{g}$,即u为问题(1)的第二个解.假设对所有的$u\in\partial\Omega$,有$u\neq\tilde{T}_\lambda u$,故$\deg(I-\tilde{T}_\lambda,\Omega,0)$有定义.由引理4.3知,$\tilde{T}_\lambda$在$B(u_\lambda,R)\setminus\Omega$上没有不动点,由拓扑度的切除性知

$$\deg(I-T_\lambda,\Omega,0)=\deg(I-\tilde{T}_\lambda,\Omega,0)=1.$$

另一方面,由引理4.2知,对任意给定的$\lambda\in I$,问题(1)的可能解有界,因此

$$\deg(I-T_\lambda,B(0,M),0)=c,$$

其中c是常数,$M>0$为充分大的常数.因为当$\lambda>\lambda^*$时,问题(1)无正解,所以$c=0$.由拓扑度的切除性可知,

$$\deg(I-T_\lambda,B(0,M)\setminus\Omega,0)=-1,$$

因此当$0<\lambda<\lambda^*$时,问题(1)存在第二个正解.

定理1.2的证明 假设问题(1)中$f\equiv 0$.在这种情况下,问题(1)等价于算子方程$u=Tu$, 其中

$$Tu(t)=\lambda(\frac{\alpha g_2(u(T))-\gamma g_1(u(0))}{\varrho}t+\frac{(\delta+\gamma(T+1))g_1(u(0))+\beta g_2(u(T))}{\varrho}).$$

若用以下锥代替在(7)中所定义的锥

$$K_1=\{u\in E|u=at+b,a,b\in\mathbb{R},t\in[0,T+1]_{\mathbb{Z}}\},$$

则$T(K_0)\subset K_1$全连续.进一步,若$u\in K_1$,记$\| u \|=\max\{u(0),u(T+1)\}$.

利用条件(A4)中的g_1和g_2的相应估计替换先前条件(A3)的f估计,下面的证明与定理1.1类似,此处略去.

例4.1 考察二阶离散边值问题

$$\begin{cases}\Delta^2u(t-1)+\lambda t(u^2(t)+1)=0, \quad t\in[1,8]_{\mathbb{Z}},\\ \alpha u(0)-\beta\Delta u(0)=\lambda e^{u(0)},\\ \gamma u(9)+\delta\Delta u(8)=\lambda u^3(8),\end{cases}\tag{10}$$

其中$h(t)=t\in[1,8]_{\mathbb{Z}}$且$h(t)\not\equiv 0,f(u)=u^2+1$.容易验证

$$\lim_{u\to+\infty}\frac{f(u)}{u}=\lim_{u\to+\infty}\frac{u^2+1}{u}=+\infty$$

成立,即条件(A1)-(A3)成立.则由定理1.1知对充分小的$\lambda > 0$,存在一个$\lambda^* > 0$,使得当$0 < \lambda < \lambda^*$时,问题(10)至少有两个正解;当$\lambda = \lambda^*$时,问题(10)至少有一个正解;当$\lambda > \lambda^*$时,问题(10)无正解.

若取$f(u) \equiv 0$且不难验证

$$\lim_{u\to+\infty}\frac{g_1(u)}{u} = \lim_{u\to+\infty}\frac{e^u}{u} = +\infty, \quad \lim_{u\to+\infty}\frac{g_2(u)}{u} = \lim_{u\to+\infty}\frac{u^3}{u} = +\infty$$

成立,即条件(A2)和(A4)成立.因此由定理1.2知对充分小的$\lambda > 0$,存在一个$\lambda^* > 0$,使得当$0 < \lambda < \lambda^*$时,问题(10)至少有两个正解;当$\lambda = \lambda^*$时,问题(10)至少有一个正解;当$\lambda > \lambda^*$时,问题(10)无正解.

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参考文献

[1] Dunninger, D.R. and Wang, H.Y. (1998) Multiplicity of Positive Solutions for a Nonlinear Differential Equation with Nonlinear Boundary Conditions. *Annales Polonici Mathematici*, **69**, 155-165. https://doi.org/10.4064/ap-69-2-155-165

[2] Umezu, K. (2000) Global Positive Solution Branches of Positone Problems with Nonlinear Boundary Conditions. *Differential Integral Equations*, **13**, 669-686.

[3] Hu, L. and Wang, L.L. (2007) Multiple Positive Solutions of Boundary Value Problem for Systems of Nonlinear Second-Order Differential Equations. *Journal of Mathematical Analysis and Applications*, **335**, 1052-1060. https://doi.org/10.1016/j.jmaa.2006.11.031

[4] Zheng, D.M. and Lu, S.P. (2011) Positive Solutions of Boundary Value Problems for Systems of Nonlinear Second-Order Differential Equations. *Chinese Quarterly Journal of Mathematics*, **26**,: 179-184.

[5] Abdou, K.D. and David, G.C. (2012) On Positive Solutions of One-Dimensional Semipositone Equations with Nonlinear Boundary Conditions. *Applied Mathematics Letters*, **25**, 2411-2416. https://doi.org/10.1016/j.aml.2012.07.015

[6] Mallick, M., Sankar, L., Shivaji, R. and Sundar, S. (2018) Infinite Semipositone Problems with a Falling Zero and Nonlinear Boundary Conditions. *Electronic Journal of Differential Equations*, **193**, 1-13.

[7] Goddard, J., Morris, Q., Shivaji, R. and Son, B. (2018) Bifurcation Curves for Singular and Nonsingular Problems with Nonlinear Boundary Conditions. *Electronic Journal of Differential Equations*, **2018**, 1-12.

[8] Su, X.X. (2019) Existence of Positive Solutions for a Class of Singular Second-Order Ordinary Differential Equations with Nonlinear Boundary Condition. *Journal of Sichuan University (Natural Science Edition)*, **56**, 1019-1025.

[9] Ma, R.Y. and Wang, S.Y. (2020) Positive Solutions for Some Semi-Positone Problems with Nonlinear Boundary Conditions via Bifurcation Theory. *Mediterranean Journal of Mathematics*, **17**, Article No. 12. https://doi.org/10.1007/s00009-019-1443-6

[10] 李艳, 唐淑瑰. 二阶非线性差分方程边值问题正解的存在性[J]. 生物数学学报, 2013, 28(2): 279-301.

[11] Lu, Y.Q. and Ma, R.Y. (2014) Global Structure of Positive Solutions for Second-Order Difference Equation with Nonlinear Boundary Value Condition. *Advances in Difference Equations*, **2014**, Article No. 188. https://doi.org/10.1186/1687-1847-2014-188

[12] 马如云, 高承华, 马慧莉, 路艳琼. 差分方程理论及其应用[M]. 北京: 科学出版社, 2018.

[13] Kelley, W.G. and Peterson, A.C. (2001) Difference Equations. Academic Press, San Diego, CA.

[14] Guo, D.J. and Lakshmikantham, V. (1988) Nonlinear Problems in Abstract Cones. Academic Press, New York, San Diego.