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BiCR算法求解Sylvester矩阵方程组的Perhermitian解

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摘 要

对于给定的矩阵 $X \in \mathbb{C}^{n\times n}$，如果 $SXS = S^H$，其中S是给定的反射矩阵，即 $S^H = S$，$S^2 = I$，则称矩阵X为perhermitian矩阵。本文提出一种用于求解Sylvester矩阵方程组的perhermitian解的双共轭残差(BiCR)算法，并且证明了该算法的收敛性。通过选择任意初始perhermitian矩阵，可以在有限步求解出Sylvester矩阵方程组的唯一最小范数perhermitian解。最后，我们给出了一些数值算例来验证该算法的有效性和可行性。

关键词

Sylvester矩阵方程组，BiCR算法，Perhermitian解，最小范数Perhermitian解

The BiCR Algorithm for Solving the Perhermitian Solutions of Sylvester Matrix Equations

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Abstract

For a given matrix $X \in \mathbb{C}^{n\times n}$, matrix X is said to be perhermitian if $SXS = S^H$, where S is a given

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reflection matrix, i.e., $S^H = S$, $S^2 = I$. In this paper, we propose the Bi-Conjugate Residual (BiCR) algorithm for solving the perhermitian solutions of Sylvester matrix equations and prove the convergence of the algorithm. By choosing any initial perhermitian matrices, the unique minimum-norm perhermitian solutions of the Sylvester matrix equations can be solved in finite steps. Finally, we give some numerical examples to verify the validity and feasibility of the algorithm.

Keywords

Sylvester Matrix Equations, BiCR Algorithm, Perhermitian Solution, Minimum-Norm Perhermitian Solution

1. 引言

矩阵方程在控制理论[1]、系统理论[2]和应用数学[3]等许多领域都有重要的作用，比如 Sylvester 矩阵方程可以用来恢复缺失或者损坏的信号或图像[4]，也可以用来研究线性时不变系统的稳定性[5]。另外，五点差分格式求解椭圆方程时会遇到 Sylvester 矩阵方程[6]，在常微分方程定性理论研究及数值求解的隐式 Runge-Kwutta 方法与块方法也会涉及到 Sylvester 矩阵方程[6]。对于它的一般解，基于 Arnoldi 过程，Hu 等人[7]给出了求解 $AX - XB = C$ 的 Galerkin 算法和最小残差法(MINRES)；基于块 Arnoldi 过程，El Guennouni A 等人[8]给出了求解 $AX - XB = C$ 的块 GMRES 方法(BGMRES)；为了提高收敛速度和缩短运算时间，基于全局 Arnoldi 过程，Fatemeh Panjeh Ali Beika 等人[9]提出了求解 Sylvester 矩阵方程组的全局完全正交化方法(Gl-FOM)和全局极小残量法(Gl-GMRES)。由于 Krylov 子空间方法的收敛速度依赖于方程系数矩阵谱的性质，选取合适的预条件矩阵，往往可以提高收敛速度，为了进一步提高算法的收敛速度，Bouhamidi A 等人[10]给出了预条件块 Arnoldi 方法求解 Sylvester 矩阵方程，Kaabi [11]等人提出了预条件 Galerkin 算法(PGal)和预条件最小残差法(PMR)求解 Sylvester 矩阵方程。基于预条件全局 Arnoldi 过程，徐冬梅等人[12]给出了求解 Sylvester 矩阵方程组的预条件全局正交化方法(PG-FOM)和预条件全局极小残量法(PG-GMRES)。对于约束解，Dehghan 和 Hajarian [13] [14]提出了求解广义 Sylvester 矩阵方程组的自反解和广义双对称解的 CGNE 方法。Dehghan 和 Hajarian [15] [16]也通过推广 CGNE 方法得到了一般矩阵方程组的解和广义双对称解。但是，一般情况下，CGNE 方法收敛速度较慢。最近，吕长青等人[17]和 Masoud Hajarian [18]提出了利用 BiCR 算法求解广义 Sylvester 矩阵方程组的中心对称解和反中心对称解，证明了这种 BiCR 算法可以在有限步内找到广义 Sylvester 矩阵方程组的中心对称解，并且选取特定的初始矩阵，可以得到最小范数解；闫同新等人[19]提出了利用 BiCR 算法求解广义 Sylvester 矩阵方程组的自反解和反自反解以及最小 Frobenius 范数对称解；Masoud Hajarian [18]提出了利用 BiCR 算法求解广义 Sylvester 矩阵方程组的对称解，收敛性分析表明，BiCR 算法在不存在舍入误差的情况下，可以在有限次迭代内计算出广义 Sylvester 矩阵方程组的最小 Frobenius 范数对称解对。然而，Sylvester 矩阵方程组的 perhermitian 解目前还没有被研究过。因此，本文将给出求解 Sylvester 矩阵方程组的 perhermitian 解的 BiCR 算法。

在本文中，考虑如下 Sylvester 矩阵方程组

$$\sum_{j=1}^{q} A_{ij} X_j B_{ij} = C_i,\ i = 1,2,\cdots,p, \tag{1.1}$$

其中，已知系数矩阵 $A_{ij} \in \mathbb{C}^{m\times n}, B_{ij} \in \mathbb{C}^{n\times l}$ 和常数矩阵 $C_i \in \mathbb{C}^{m\times l}$， $X_j \in \mathbb{C}^{n\times n}$ 是未知矩阵。

2. 预备知识

定义 2.1 [20] [21]对给定的反射矩阵 $S \in \mathbb{C}^{n\times n}$，即 $S^H = S$， $S^2 = I_n$。如果矩阵 $X \in \mathbb{C}^{n\times n}$ 满足条件 $SXS = S^H$ （$SXS = -S^H$），那么称其为反射矩阵 S 的 perhermitian 矩阵(skew-perhermitian 矩阵)，记为 $X \in \mathbb{PC}^{n\times n}$ （$X \in \mathbb{SPC}^{n\times n}$）。

定义 2.2 [22] [23]设任意 $X,Y \in \mathbb{C}^{m\times n}$，规定

$$\langle X,Y\rangle = \mathrm{Re}\left[tr\left(X^H Y\right)\right],$$

则称 $\langle X,Y\rangle$ 为矩阵 X,Y 的实内积，记作 $\left(\mathbb{C}^{m\times n},\mathbb{R},\langle\cdot,\cdot\rangle\right)$。

定义 2.3 [22] [23]设矩阵 $X \in \mathbb{C}^{n\times n}$，规定

$$\|X\| = \sqrt{\langle X,X\rangle} = \sqrt{\mathrm{Re}\left[tr\left(X^H X\right)\right]},$$

则称 $\|X\|$ 为矩阵 X 的 Frobenius 范数，简称 F 范数。

定义 2.4 [22] [23]设矩阵 $X,Y \in \mathbb{C}^{n\times n}$，如果 $\langle X,Y\rangle = 0$，那么矩阵 X,Y 是正交的。

引理 2.1 设矩阵 $X,Y \in M_n(\mathbb{C})$，有

$$\mathrm{Re}\left[tr(XY)\right] = \mathrm{Re}\left[tr\left(\overline{XY}\right)\right] = \mathrm{Re}\left[tr(YX)\right] = \mathrm{Re}\left[tr\left(X^{\mathrm{T}}Y^{\mathrm{T}}\right)\right] = \mathrm{Re}\left[tr\left(X^H Y^H\right)\right].$$

引理 2.2 设矩阵 $X \in \mathbb{PC}^{n\times n}, Y \in \mathbb{SPC}^{n\times n}$，则矩阵 X,Y 是正交的。

证明： 因为

$$\begin{aligned}\langle X,Y\rangle &= \mathrm{Re}\left[tr\left(X^H Y\right)\right] = -\mathrm{Re}\left[tr\left(SXS\cdot SY^H S\right)\right] = -\mathrm{Re}\left[tr\left(XY^H\right)\right] \\ &= -\mathrm{Re}\left[tr\left(XY^H\right)^H\right] = -\mathrm{Re}\left[tr\left(YX^H\right)\right] = -\mathrm{Re}\left[tr\left(X^H Y\right)\right] \\ &= -\langle X,Y\rangle,\end{aligned}$$

所以 $\langle X,Y\rangle = 0$，因此矩阵 X,Y 正交。 □

引理 2.3 设矩阵 $X \in \mathbb{C}^{n\times n}$，那么 $X + SX^H S \in \mathbb{PC}^{n\times n}$。

证明： 因为

$$S\left(X + SX^H S\right)S = SXS + X^H = \left(X + SX^H S\right)^H,$$

所以结论成立。 □

根据**引理 2.3**，我们可以很容易地构造一个 perhermitian 矩阵。

引理 2.4 设矩阵 $X \in \mathbb{PC}^{n\times n}$， $Y \in \mathbb{C}^{n\times n}$， $S \in \mathbb{C}^{n\times n}$ 为反射矩阵，有

$$\frac{1}{2}\mathrm{Re}\left(tr\left(X^H\left(Y + SY^H S\right)\right)\right) = \mathrm{Re}\left(tr\left(X^H Y\right)\right).$$

证明：

$$
\begin{aligned}
\frac{1}{2}\operatorname{Re}\left(tr\left(X^H\left(Y+SY^HS\right)\right)\right) &= \frac{1}{2}\left(\operatorname{Re}\left(tr\left(X^HY\right)\right)+\operatorname{Re}\left(tr\left(X^HSY^HS\right)\right)\right) \\
&= \frac{1}{2}\left(\operatorname{Re}\left(tr\left(X^HY\right)\right)+\operatorname{Re}\left(tr\left(SX^HSY^H\right)\right)\right) \\
&= \frac{1}{2}\left(\operatorname{Re}\left(tr\left(X^HY\right)\right)+\operatorname{Re}\left(tr\left(XY^H\right)\right)\right) \\
&= \frac{1}{2}\left(\operatorname{Re}\left(tr\left(X^HY\right)\right)+\operatorname{Re}\left(tr\left(YX^H\right)^H\right)\right) \\
&= \frac{1}{2}\left(\operatorname{Re}\left(tr\left(X^HY\right)\right)+\operatorname{Re}\left(tr\left(YX^H\right)\right)\right) \\
&= \frac{1}{2}\left(\operatorname{Re}\left(tr\left(X^HY\right)\right)+\operatorname{Re}\left(tr\left(X^HY\right)\right)\right) \\
&= \operatorname{Re}\left(tr\left(X^HY\right)\right).
\end{aligned}
$$

证毕。 □

引理 2.5 如果矩阵 $X,Y\in\mathbb{PC}^{n\times n}$， $\alpha,\beta\in\mathbb{R}$，那么 $\alpha X+\beta Y\in\mathbb{PC}^{n\times n}$，换句话说， $\mathbb{PC}^{n\times n}$ 是 $\mathbb{C}^{n\times n}$ 的一个子空间。

证明：因为

$$
S\left(\alpha X+\beta Y\right)S=\alpha SXS+\beta SYS=\alpha X^H+\beta Y^H=\left(\alpha X+\beta Y\right)^H.
$$

证毕。 □

3. 迭代方法

在这个部分，我们提出一种用于求解 Sylvester 矩阵方程组的 perhermitian 解的 BiCR 算法。

3.1. BiCR 算法求解 Sylvester 矩阵方程组的 Perhermitian 解(表 1)

首先，给出该算法的迭代过程。

Table 1. TheBiCR algorithm for solving the perhermitian solutions of the matrix equations (1.1)
表 1. BiCR 算法求解矩阵方程组(1.1)的 perhermitian 解

算法 3.1 ：BiCR 算法求解矩阵方程组(1.1)的 perhermitian 解

1. 设 $A_{ij}\in\mathbb{C}^{m\times n}, B_{ij}\in\mathbb{C}^{n\times l}, C_{ij}\in\mathbb{C}^{m\times l}$，适当维度的自反矩阵 $S\in\mathbb{C}^{n\times n}$， $\tilde{U}_j(0)=W_j(0)=0\in\mathbb{PC}^{n\times n}, j=1,2,\cdots,q$，

$\tilde{V}_i(0)=T_i(0)=0\in\mathbb{C}^{m\times l}, i=1,2,\cdots,p$，

$\sigma(0)=\tau(0)=1$.

任意给定初始值 $X_j(1)\in\mathbb{PC}^{n\times n}$， $Z_j(1)\in\mathbb{PC}^{n\times n}$， $\varepsilon>0$。

计算以下式子：

$R_i(1)=C_i-\sum_{j=1}^{q}A_{ij}X_j(1)B_{ij}, i=1,2,\cdots,p,$

$U_j(1)=Z_j(1),\ V_i(1)=R_i(1),$

$\tilde{U}_j(1)=\dfrac{U_j(1)}{\sum_{j=1}^{q}\left\|U_j(1)\right\|},\ \tilde{V}_i(1)=\dfrac{V_i(1)}{\sum_{i=1}^{p}\left\|V_i(1)\right\|},$

Continued

$T_i(1)=\sum_{j=1}^{q}A_{ij}\tilde{U}_j(1)B_{ij}$, $W_j(1)=\frac{1}{2}\sum_{i=1}^{p}\left(A_{ij}^H\tilde{V}_i(1)B_{ij}^H+S\left(A_{ij}^H\tilde{V}_i(1)B_{ij}^H\right)^H S\right)$,

$\sigma(1)=\sum_{i=1}^{p}\|T_i(1)\|^2$, $\tau(1)=\sum_{j=1}^{q}\|W_j(1)\|^2$, $r(1)=\sqrt{\sum_{i=1}^{p}\|R_i(1)\|^2}$,

$k=1$.

2. 如果 $r(k)<\varepsilon$，则停止；否则，转步骤 3。

3. 计算以下式子：

$$\alpha(k)=\frac{\sum_{i=1}^{p}\mathrm{Re}\left(tr\left(T_i^H(k)R_i(k)\right)\right)}{\sigma(k)},$$

$$X_j(k+1)=X_j(k)-\alpha(k)\tilde{U}_j(k),$$

$$R_i(k+1)=R_i(k)-\alpha(k)T_i(k),$$

$$M_i(k)=\sum_{j=1}^{q}A_{ij}W_j(k)B_{ij},$$

$$U_j(k+1)=W_j(k)-\frac{\sum_{i=1}^{p}\mathrm{Re}\left(tr\left(T_i^H(k)M_i(k)\right)\right)}{\sigma(k)}\tilde{U}_j(k)$$
$$-\frac{\sum_{i=1}^{p}\mathrm{Re}\left(tr\left(T_i^H(k-1)M_i(k)\right)\right)}{\sigma(k-1)}\tilde{U}_j(k-1),$$

$$N_j(k)=\sum_{i=1}^{p}A_{ij}^HT_i(k)B_{ij}^H,$$

$$V_i(k+1)=T_i(k)-\frac{\sum_{i=1}^{p}\mathrm{Re}\left(tr\left(T_i^H(k)M_i(k)\right)\right)}{\tau(k)}\tilde{V}_i(k)$$
$$-\frac{\sum_{j=1}^{q}\mathrm{Re}\left(tr\left(W_j^H(k-1)N_j(k)\right)\right)}{\tau(k-1)}\tilde{V}_i(k-1),$$

$$\tilde{U}_j(k+1)=\frac{U_j(k+1)}{\sum_{j=1}^{q}\|U_j(k+1)\|},\quad \tilde{V}_i(k+1)=\frac{V_i(k+1)}{\sum_{i=1}^{p}\|V_i(k+1)\|},$$

$$T_i(k+1)=\sum_{j=1}^{q}A_{ij}\tilde{U}_j(k+1)B_{ij},$$

$$W_j(k+1)=\frac{1}{2}\sum_{i=1}^{p}\left(A_{ij}^H\tilde{V}_i(k+1)B_{ij}^H+S\left(A_{ij}^H\tilde{V}_i(k+1)B_{ij}^H\right)^H S\right),$$

$$\sigma(k+1)=\sum_{i=1}^{p}\|T_i(k+1)\|^2,$$

$$\tau(k+1)=\sum_{j=1}^{q}\|W_j(k+1)\|^2,$$

$$r(k+1)=\sqrt{\sum_{i=1}^{p}\|R_i(k+1)\|^2}.$$

4. $k=k+1$，返回步骤 2。

通过算法 3.1，我们可以知道 $X_j(k), W_j(k), U_j(k)$ 和 $\tilde{U}_j(k)$ 都是反射矩阵 S 的 perhermitian 矩阵，$X_j(k)\in\mathbb{PC}^{n\times n}$，$W_j(k)\in\mathbb{PC}^{n\times n}$，$U_j(k)\in\mathbb{PC}^{n\times n}$，$\tilde{U}_j(k)\in\mathbb{PC}^{n\times n}$，其中 $j=1,2,\cdots,q$。

3.2. 收敛性分析

接下来，对该算法进行收敛性分析。我们可以通过以下定理来证明所提算法的收敛性。

定理 3.1 如果对于正整数 m，$R_i(k)\neq 0$、$T_i(k)\neq 0$，$k=1,2,\cdots,m$，那么

$$\sum_{i=1}^{p}\mathrm{Re}\left(tr\left(R_i^H(t)T_i(s)\right)\right)=0,\quad s,t=1,2,\cdots,m,\quad s<t \tag{3.1}$$

$$\sum_{j=1}^{q}\mathrm{Re}\left(tr\left(W_j^H(t)W_j(s)\right)\right)=0,\quad s,t=1,2,\cdots,m,\quad s\neq t \tag{3.2}$$

$$\sum_{i=1}^{p}\mathrm{Re}\left(tr\left(T_i^H(t)T_i(s)\right)\right)=0,\quad s,t=1,2,\cdots,m,\quad s\neq t \tag{3.3}$$

证明：我们采用数学归纳法来证明这个定理，可以得到上述(3.1)~(3.3)。

首先，当 $s<n<m$，我们假设有

$$\sum_{i=1}^{p}\mathrm{Re}\left(tr\left(R_i^H(n)T_i(s)\right)\right)=0,\quad s,t=1,2,\cdots,m,\quad s<t$$

$$\sum_{j=1}^{q}\mathrm{Re}\left(tr\left(W_j^H(n)W_j(s)\right)\right)=0,\quad s,t=1,2,\cdots,m,\quad s\neq t$$

$$\sum_{i=1}^{p}\mathrm{Re}\left(tr\left(T_i^H(n)T_i(s)\right)\right)=0,\quad s,t=1,2,\cdots,m,\quad s\neq t$$

根据上述归纳假设，接下来我们来证明 $n+1$ 时(3.1)~(3.3)的情况，可得

$$\begin{aligned}
&\sum_{i=1}^{p}\mathrm{Re}\left(tr\left(R_i^H(n+1)T_i(s)\right)\right)\\
&=\sum_{i=1}^{p}\mathrm{Re}\left(tr\left(\left(R_i(n)-\alpha(n)T_i(n)\right)^H T_i(s)\right)\right)\\
&=\sum_{i=1}^{p}\mathrm{Re}\left(tr\left(R_i^H(n)T_i(s)\right)\right)-\alpha(n)\sum_{i=1}^{p}\mathrm{Re}\left(tr\left(T_i^H(n)T_i(s)\right)\right)\\
&=0
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=1}^{q}\mathrm{Re}\left(tr\left(W_j^H(n+1)W_j(s)\right)\right)\\
&=\sum_{j=1}^{q}\mathrm{Re}\left(tr\left(\frac{1}{2}\sum_{i=1}^{p}\left(A_{ij}^H\tilde{V}_i(n+1)B_{ij}^H+S\left(A_{ij}^H\tilde{V}_i(n+1)B_{ij}^H\right)^H S\right)^H W_j(s)\right)\right)\\
&=\frac{1}{\sum_{i=1}^{p}\left\|V_i(n+1)\right\|}\sum_{j=1}^{q}\mathrm{Re}\left(tr\left(\frac{1}{2}\sum_{i=1}^{p}\left(A_{ij}^H V_i(n+1)B_{ij}^H+S\left(A_{ij}^H V_i(n+1)B_{ij}^H\right)^H S\right)^H W_j(s)\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sum_{i=1}^{p}\left\|V_i(n+1)\right\|}\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(\frac{1}{2}\left(\sum_{i=1}^{p}B_{ij}\left(T_i(n)-\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)M_i(n)\right)\right)}{\tau(n)}\tilde{V}_i(n)\right.\right.\right.\right.\\
&\left.-\frac{\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n-1)N_j(n)\right)\right)}{\tau(n-1)}\tilde{V}_i(n-1)\right)^H A_{ij}\\
&+S\left(A_{ij}^H\left(T_i(n)-\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)M_i(n)\right)\right)}{\tau(n)}\tilde{V}_i(n)\right.\right.\\
&\left.\left.\left.\left.\left.-\frac{\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n-1)N_j(n)\right)\right)}{\tau(n-1)}\tilde{V}_i(n-1)\right)B_{ij}^H\right)S\right)W_j(s)\right)\right)\\
&= \frac{1}{\sum_{i=1}^{p}\left\|V_i(n+1)\right\|}\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(\frac{1}{2}\sum_{i=1}^{p}\left(B_{ij}T_i^H(n)A_{ij}+SA_{ij}^H T_i(n)B_{ij}^H S\right)W_j(s)\right.\right.\\
&\left.\left.-\frac{\sum_{i=1}^{p}tr\left(T_i^H(n)M_i(n)\right)}{\tau(n)}W_j^H(n)W_j(s)-\frac{\sum_{j=1}^{q}tr\left(W_j^H(n-1)N_j(n)\right)}{\tau(n-1)}W_j^H(n-1)W_j(s)\right)\right)\\
&= \frac{1}{\sum_{i=1}^{p}\left\|V_i(n+1)\right\|}\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(\frac{1}{2}\sum_{i=1}^{p}\left(T_i^H(n)A_{ij}W_j(s)B_{ij}+T_i(n)\left(A_{ij}W_j(s)B_{ij}\right)^H\right)\right.\right.\\
&\left.\left.-\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)M_i(n)\right)\right)}{\tau(n)}W_j^H(n)W_j(s)-\frac{\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n-1)N_j(n)\right)\right)}{\tau(n-1)}W_j^H(n-1)W_j(s)\right)\right)\\
&= \frac{1}{\sum_{i=1}^{p}\left\|V_i(n+1)\right\|}\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(\sum_{i=1}^{p}T_i^H(n)A_{ij}W_j(s)B_{ij}-\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)M_i(n)\right)\right)}{\tau(n)}W_j^H(n)W_j(s)\right.\right.\\
&\left.\left.-\frac{\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n-1)N_j(n)\right)\right)}{\tau(n-1)}W_j^H(n-1)W_j(s)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\sum_{i=1}^{p}\|V_i(n+1)\|}\Bigg(\sum_{j=1}^{q}\operatorname{Re}\Bigg(tr\Bigg(\sum_{i=1}^{p}T_i^H(n)A_{ij}\Bigg(U_j(s+1)+\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)M_i(s)\right)\right)}{\sigma(s)}\tilde{U}_j(s)\\
&+\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(s-1)M_i(s)\right)\right)}{\sigma(s-1)}\tilde{U}_j(s-1)\Bigg)B_{ij}\Bigg)\Bigg)\\
&-\frac{\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n-1)N_j(n)\right)\right)}{\tau(n-1)}\operatorname{Re}\left(tr\left(\sum_{j=1}^{q}W_j^H(n-1)W_j(s)\right)\right)\Bigg)\\
&= \frac{1}{\sum_{i=1}^{p}\|V_i(n+1)\|}\Bigg(\sum_{j=1}^{q}\|U_j(s+1)\|\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)T_i(s+1)\right)\right)\\
&+\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(s)M_i(s)\right)\right)}{\sigma(s)}\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)T_i(s)\right)\right)\\
&+\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(s-1)M_i(s)\right)\right)}{\sigma(s-1)}\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)T_i(s-1)\right)\right)\\
&-\frac{\sum_{j=1}^{q}tr\left(W_j^H(n-1)N_j(n)\right)}{\tau(n-1)}tr\left(\sum_{j=1}^{q}W_j^H(n-1)W_j(s)\right)\Bigg)\\
&= \frac{\sum_{j=1}^{q}\|U_j(s+1)\|}{\sum_{i=1}^{p}\|V_i(n+1)\|}\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)T_i(s+1)\right)\right)\\
&-\frac{\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n-1)W_j(s)\right)\right)}{\sum_{i=1}^{p}\|V_i(n+1)\|}\frac{\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n-1)N_j(n)\right)\right)}{\tau(n-1)}
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
&\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n+1)T_i(s)\right)\right)\\
&=\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(\left(\sum_{j=1}^{q}A_{ij}\tilde{U}_j(n+1)B_{ij}\right)^H T_i(s)\right)\right)\\
&=\frac{1}{\sum_{j=1}^{q}\|U_j(n+1)\|}\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(\left(\sum_{j=1}^{q}A_{ij}U_j(n+1)B_{ij}\right)^H T_i(s)\right)\right)
\end{aligned}
$$

$$=\frac{1}{\sum_{j=1}^{q}\left\|U_j(n+1)\right\|^2}\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(\left(\left(A_{ij}\left(W_j(n)-\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)M_i(n)\right)\right)}{\sigma(n)}\tilde{U}_j(n)\right.\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left.-\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n-1)M_i(n)\right)\right)}{\sigma(n-1)}\tilde{U}_j(n-1)\right)B_{ij}\right)^H T_i(s)\right)\right)$$

$$=\frac{1}{\sum_{j=1}^{q}\left\|U_j(n+1)\right\|^2}\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(\left(A_{ij}W_j(n)B_{ij}-\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)M_i(n)\right)\right)}{\sigma(n)}T_i(n)\right.\right.\right.$$

$$\left.\left.\left.-\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n-1)M_i(n)\right)\right)}{\sigma(n-1)}T_i(n-1)\right)^H T_i(s)\right)\right)$$

$$=\frac{1}{\sum_{j=1}^{q}\left\|U_j(n+1)\right\|^2}\left(\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(\sum_{j=1}^{q}B_{ij}^H W_j^H(n)A_{ij}^H T_i(s)\right)\right)\right.$$

$$-\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)M_i(n)\right)\right)}{\sigma(n)}\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)T_i(s)\right)\right)$$

$$\left.-\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n-1)M_i(n)\right)\right)}{\sigma(n-1)}\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n-1)T_i(s)\right)\right)\right)$$

$$=\frac{1}{\sum_{j=1}^{q}\left\|U_j(n+1)\right\|^2}\left(\sum_{i=1}^{p}\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n)\left(A_{ij}^H T_i(s)B_{ij}^H\right)\right)\right)\right.$$

$$\left.-\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n-1)M_i(n)\right)\right)}{\sigma(n-1)}\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n-1)T_i(s)\right)\right)\right)$$

$$=\frac{1}{\sum_{j=1}^{q}\left\|U_j(n+1)\right\|^2}\left(\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n)\left(A_{ij}^H T_i(s)B_{ij}^H+S\left(A_{ij}^H T_i(s)B_{ij}^H\right)^H S\right)\right)\right)\right.$$

$$\left.-\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n-1)M_i(n)\right)\right)}{\sigma(n-1)}\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n-1)T_i(s)\right)\right)\right)$$

$$
\begin{aligned}
&=\frac{1}{\sum_{j=1}^{q}\left\|U_j(n+1)\right\|}\left(\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n)\left(A_{ij}^H\left(V_i(s+1)+\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(s)M_i(s)\right)\right)}{\tau(s)}\tilde{V}_i(s)\right.\right.\right.\right.\right.\\
&\quad\left.+\frac{\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(s-1)N_j(s)\right)\right)}{\tau(s-1)}\tilde{V}_i(s-1)\right)B_{ij}^H+S\left(A_{ij}^H\left(V_i(s+1)+\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(s)M_i(s)\right)\right)}{\tau(s)}\tilde{V}_i(s)\right.\right.\\
&\quad\left.\left.\left.\left.\left.+\frac{\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(s-1)N_j(s)\right)\right)}{\tau(s-1)}\tilde{V}_i(s-1)\right)B_{ij}^H\right)^H S\right)\right)\right)\\
&\quad\left.-\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n-1)M_i(n)\right)\right)}{\sigma(n-1)}\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n-1)T_i(s)\right)\right)\right)\\
&=\frac{1}{\sum_{j=1}^{q}\left\|U_j(n+1)\right\|}\left(\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n)\left(W_j(s+1)\sum_{i=1}^{p}\left\|V_i(s+1)\right\|\right.\right.\right.\right.\\
&\quad\left.\left.\left.\left.+\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(s)M_i(s)\right)\right)}{\tau(s)}W_j(s)+\frac{\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(s-1)N_j(s)\right)\right)}{\tau(s-1)}W_j(s-1)\right)\right)\right)\\
&\quad\left.-\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n-1)M_i(n)\right)\right)}{\sigma(n-1)}\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n-1)T_i(s)\right)\right)\right)\\
&=\frac{\sum_{i=1}^{p}\left\|V_i(s+1)\right\|}{\sum_{j=1}^{q}\left\|U_j(n+1)\right\|}\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n)W_j(s+1)\right)\right)\\
&\quad-\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n-1)M_i(n)\right)\right)}{\sum_{j=1}^{q}\left\|U_j(n+1)\right\|}\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n-1)T_i(s)\right)\right)}{\sigma(n-1)}
\end{aligned}
\tag{3.5}
$$

当 $s=n-1$，由于 $s<n$，根据上述(3.1)~(3.3)，可得

$$\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(R_i^H(n+1)T_i(n-1)\right)\right)=0.$$

$$
\begin{aligned}
&\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n+1)W_j(n-1)\right)\right)\\
&=\frac{\sum_{j=1}^{q}\|U_j(n)\|}{\sum_{i=1}^{p}\|V_i(n+1)\|}\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)T_i(n)\right)\right)\\
&\quad-\frac{\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n-1)W_j(n-1)\right)\right)}{\sum_{i=1}^{p}\|V_i(n+1)\|}\frac{\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n-1)N_j(n)\right)\right)}{\tau(n-1)}\\
&=\frac{\sum_{j=1}^{q}\|U_j(n)\|}{\sum_{i=1}^{p}\|V_i(n+1)\|}\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)T_i(n)\right)\right)-\frac{\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n-1)N_j(n)\right)\right)}{\sum_{i=1}^{p}\|V_i(n+1)\|}\\
&=\frac{\sum_{j=1}^{q}\|U_j(n)\|}{\sum_{i=1}^{p}\|V_i(n+1)\|}\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)T_i(n)\right)\right)-\frac{\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n-1)\sum_{i=1}^{p}A_{ij}^H T_i(n)B_{ij}^H\right)\right)}{\sum_{i=1}^{p}\|V_i(n+1)\|}\\
&=\frac{\sum_{j=1}^{q}\|U_j(n)\|}{\sum_{i=1}^{p}\|V_i(n+1)\|}\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)T_i(n)\right)\right)-\frac{\sum_{j=1}^{q}\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(\left(A_{ij}W_j(n-1)B_{ij}\right)^H T_i(n)\right)\right)}{\sum_{i=1}^{p}\|V_i(n+1)\|}\\
&=\frac{\sum_{j=1}^{q}\|U_j(n)\|}{\sum_{i=1}^{p}\|V_i(n+1)\|}\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)T_i(n)\right)\right)-\frac{1}{\sum_{i=1}^{p}\|V_i(n+1)\|}\sum_{i=1}^{p}\operatorname{Re}\Bigg(tr\Bigg(\Bigg(\Bigg(\sum_{j=1}^{q}\|U_j(n)\|T_i(n)\\
&\quad+\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n-1)M_i(n-1)\right)\right)}{\sigma(n-1)}T_i(n-1)\\
&\quad+\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n-2)M_i(n-1)\right)\right)}{\sigma(n-2)}T_i(n-2)\Bigg)^H T_i(n)\Bigg)\Bigg)\\
&=\frac{\sum_{j=1}^{q}\|U_j(n)\|}{\sum_{i=1}^{p}\|V_i(n+1)\|}\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)T_i(n)\right)\right)-\frac{\sum_{j=1}^{q}\|U_j(n)\|}{\sum_{i=1}^{p}\|V_i(n+1)\|}\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)T_i(n)\right)\right)\\
&=0.
\end{aligned}
$$

$$\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n+1)T_i(n-1)\right)\right)$$

$$=\frac{\sum_{i=1}^{p}\|V_i(n)\|}{\sum_{j=1}^{q}\|U_j(n+1)\|}\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n)W_j(n)\right)\right)$$

$$-\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n-1)M_i(n)\right)\right)}{\sum_{j=1}^{q}\|U_j(n+1)\|}\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n-1)T_i(n-1)\right)\right)}{\sigma(n-1)}$$

$$=\frac{\sum_{i=1}^{p}\|V_i(n)\|}{\sum_{j=1}^{q}\|U_j(n+1)\|}\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n)W_j(n)\right)\right)-\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n-1)M_i(n)\right)\right)}{\sum_{j=1}^{q}\|U_j(n+1)\|}$$

$$=\frac{\sum_{i=1}^{p}\|V_i(n)\|}{\sum_{j=1}^{q}\|U_j(n+1)\|}\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n)W_j(n)\right)\right)-\frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n-1)\left(\sum_{j=1}^{q}A_{ij}W_j(n)B_{ij}\right)\right)\right)}{\sum_{j=1}^{q}\|U_j(n+1)\|}$$

$$=\frac{\sum_{i=1}^{p}\|V_i(n)\|}{\sum_{j=1}^{q}\|U_j(n+1)\|}\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n)W_j(n)\right)\right)$$

$$-\frac{1}{\sum_{j=1}^{q}\|U_j(n+1)\|}\sum_{i=1}^{p}\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(\left(A_{ij}^H T_i(n-1)B_{ij}^H\right)W_j(n)\right)\right)$$

$$=\frac{\sum_{i=1}^{p}\|V_i(n)\|}{\sum_{j=1}^{q}\|U_j(n+1)\|}\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n)W_j(n)\right)\right)$$

$$-\frac{1}{\sum_{j=1}^{q}\|U_j(n+1)\|}\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(\frac{1}{2}\sum_{i=1}^{p}\left(A_{ij}^H T_i(n-1)B_{ij}^H+S\left(A_{ij}^H T_i(n-1)B_{ij}^H\right)^H S\right)^H W_j(n)\right)\right)$$

$$=\frac{\sum_{i=1}^{p}\|V_i(n)\|}{\sum_{j=1}^{q}\|U_j(n+1)\|}\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n)W_j(n)\right)\right)$$

$$-\frac{1}{\sum_{j=1}^{q}\|U_j(n+1)\|}\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(\left(\left(\sum_{i=1}^{p}\|V_i(n)\|W_j(n)+\frac{\operatorname{Re}\sum_{i=1}^{p}tr\left(T_i^H(n-1)M_i(n-1)\right)}{\tau(n-1)}W_j(n-1)\right.\right.\right.\right.$$

$$\left.\left.\left.\left.+\frac{\operatorname{Re}\sum_{j=1}^{q}tr\left(W_j^H(n-2)N_j(n-1)\right)}{\tau(n-2)}W_j(n-2)\right)^H W_j(n)\right)\right)$$

$$
\begin{aligned}
&= \frac{\sum_{i=1}^{p}\|V_i(n)\|}{\sum_{j=1}^{q}\|U_j(n+1)\|}\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n)W_j(n)\right)\right) - \frac{\sum_{i=1}^{p}\|V_i(n)\|}{\sum_{j=1}^{q}\|U_j(n+1)\|}\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n)W_j(n)\right)\right) \\
&= 0.
\end{aligned}
$$

当 $s = n$ 时，通过归纳假设可得

$$
\begin{aligned}
&\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(R_i^H(n+1)T_i(n)\right)\right) \\
&= \sum_{i=1}^{p}\operatorname{Re}\left(tr\left(\left(R_i(n)-\alpha(n)T_i(n)\right)^H T_i(n)\right)\right) \\
&= \sum_{i=1}^{p}\operatorname{Re}\left(tr\left(R_i^H(n)T_i(n)\right)\right) - \frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)R_i(n)\right)\right)}{\sum_{i=1}^{p}\|T_i(n)\|^2}\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)T_i(n)\right)\right) \\
&= \sum_{i=1}^{p}\operatorname{Re}\left(tr\left(R_i^H(n)T_i(n)\right)\right) - \sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)R_i(n)\right)\right) \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n+1)W_j(n)\right)\right) \\
&= \frac{1}{\sum_{i=1}^{p}\|V_i(n+1)\|}\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(\left(\left(\frac{1}{2}\sum_{i=1}^{p}\left(A_{ij}^H T_i(n) B_{ij}^H + S\left(A_{ij}^H T_i(n) B_{ij}^H\right)^H S\right)\right.\right.\right.\right. \\
&\quad \left.\left.\left.\left. - \frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)M_i(n)\right)\right)}{\tau(n)}W_j(n) - \frac{\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n-1)N_j(n)\right)\right)}{\tau(n-1)}W_j(n-1)\right)W_j(n)\right)\right) \\
&= \frac{1}{\sum_{i=1}^{p}\|V_i(n+1)\|}\left(\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)\sum_{j=1}^{q}A_{ij}W_j(n)B_{ij}\right)\right) - \frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)M_i(n)\right)\right)}{\tau(n)}\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n)W_j(n)\right)\right)\right) \\
&= \frac{1}{\sum_{i=1}^{p}\|V_i(n+1)\|}\left(\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)M_i(n)\right)\right) - \frac{\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)M_i(n)\right)\right)}{\sum_{j=1}^{q}\|W_j(n)\|^2}\sum_{j=1}^{q}\operatorname{Re}\left(tr\left(W_j^H(n)W_j(n)\right)\right)\right) \\
&= \frac{1}{\sum_{i=1}^{p}\|V_i(n+1)\|}\left(\sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)M_i(n)\right)\right) - \sum_{i=1}^{p}\operatorname{Re}\left(tr\left(T_i^H(n)M_i(n)\right)\right)\right) \\
&= 0.
\end{aligned}
$$

$$\sum_{i=1}^{p}\text{Re}\left(tr\left(T_i^H(n+1)T_i(n)\right)\right)$$

$$=\frac{1}{\sum_{j=1}^{q}\left\|U_j(n+1)\right\|}\sum_{i=1}^{p}\text{Re}\left(tr\left(\left(\left(\sum_{j}^{q}A_{ij}W_j(n)B_{ij}-\frac{\sum_{i=1}^{p}\text{Re}\left(tr\left(T_i^H(n)M_i(n)\right)\right)}{\sigma(n)}T_i(n)\right.\right.\right.\right.$$

$$\left.\left.\left.\left.-\frac{\sum_{i=1}^{p}\text{Re}\left(tr\left(T_i^H(n-1)M_i(k)\right)\right)}{\sigma(n-1)}T_i(n-1)\right)^H T_i(n)\right)\right)$$

$$=\frac{1}{\sum_{j=1}^{q}\left\|U_j(n+1)\right\|}\left(\sum_{i=1}^{p}\text{Re}\left(tr\left(M_i^H(n)T_i(n)\right)\right)-\frac{\sum_{i=1}^{p}\text{Re}\left(tr\left(T_i^H(n)M_i(n)\right)\right)}{\sum_{i=1}^{p}\left\|T_i(n)\right\|^2}\sum_{i=1}^{p}\text{Re}\left(tr\left(T_i^H(n)T_i(n)\right)\right)\right)$$

$$=\frac{1}{\sum_{j=1}^{q}\left\|U_j(n+1)\right\|}\left(\sum_{i=1}^{p}\text{Re}\left(tr\left(M_i^H(n)T_i(n)\right)\right)-\sum_{i=1}^{p}\text{Re}\left(tr\left(M_i^H(n)T_i(n)\right)\right)\right)$$

$$=0.$$

因此，对于次数 $n+1$ 成立，数学归纳法证明完成。 □

定理 3.2 假设矩阵方程组(1.1)是相容的，在没有舍入误差的情况下，对任意初始矩阵 $X_j(1)\in\mathbb{PC}^{n\times n}$，根据算法 3.1，可以在有限步迭代得到矩阵方程组(1.1)的 perhermitian 解。

证明 首先，定义空间 $\mathbb{C}^{m\times l}\times\mathbb{C}^{m\times l}\times\cdots\times\mathbb{C}^{m\times l}$ 的实内积为 $\langle T_i,T_j\rangle=\text{Re}\left(tr\left(T_j^H T_i\right)\right)$，其中 $T_i,T_j\in\mathbb{C}^{m\times l},i,j=1,2,\cdots,p$。如果 $T_i(k)\neq 0,k=1,2,\cdots,pml$，那么 $\{T_1(k),T_2(k),\cdots,T_p(k)\}$ 是该空间的一组正交基。根据(3.1)式可以得到 $R_i(pml+1)=0$，即 $X_j(pml+1),j=1,2,\cdots,q$ 是矩阵方程(1.1)的 perhermitian 解。

定理 3.3 算法 3.1 中残量范数具有以下性质

$$\sum_{i=1}^{p}\left\|R_i(k+1)\right\|^2\leq\sum_{i=1}^{p}\left\|R_i(k)\right\|^2.$$

证明：

$$\begin{aligned}\sum_{i=1}^{p}\left\|R_i(k+1)\right\|^2&=\sum_{i=1}^{p}\text{Re}\left(tr\left(R_i^H(k+1)R_i(k+1)\right)\right)\\&=\sum_{i=1}^{p}\text{Re}\left(tr\left(\left(R_i(k)-\alpha(k)T_i(k)\right)^H\left(R_i(k)-\alpha(k)T_i(k)\right)\right)\right)\\&=\sum_{i=1}^{p}\text{Re}\left(tr\left(\left(R_i^H(k)-\alpha(k)T_i^H(k)\right)\left(R_i(k)-\alpha(k)T_i(k)\right)\right)\right)\\&=\sum_{i=1}^{p}\text{Re}\left(tr\left(R_i^H(k)R_i(k)\right)\right)+\alpha^2(k)\sum_{i=1}^{p}\text{Re}\left(tr\left(T_i^H(k)T_i(k)\right)\right)\\&\quad-2\alpha(k)\sum_{i=1}^{p}\text{Re}\left(tr\left(R_i^H(k)T_i(k)\right)\right)\end{aligned}$$

$$=\sum_{i=1}^{p}\left\|R_i(k)\right\|^2+\alpha^2(k)\sigma(k)-2\alpha^2(k)\sigma(k)$$
$$=\sum_{i=1}^{p}\left\|R_i(k)\right\|^2-\alpha^2(k)\sigma(k)\le\sum_{i=1}^{p}\left\|R_i(k)\right\|^2.$$

□

定理 3.3 表明，如果 $\sum_{i=1}^{p}\left\|R_i(k)\right\|^2\neq 0$ 且 $\sum_{i=1}^{p}\mathrm{Re}\left(tr\left(R_i^H(k)T_i(k)\right)\right)\neq 0$，那么 $\sum_{i=1}^{p}\left\|R_i(k)\right\|^2$ 是严格单调递减的，所以算法 3.1 是收敛的。

3.3. 最小范数 Perhermitian 解

接下来考虑 Sylvester 矩阵方程组(1.1)的最佳逼近 perhermitian 解，即最小范数 perhermitian 解。

引理 3.1 [19] [24] [25] [26]设线性矩阵方程 $Ax=b$ 有解 $x^*\in\mathcal{R}\left(A^{\mathrm{T}}\right)$，其中 $\mathcal{R}\left(A^{\mathrm{T}}\right)$ 表示 A^{T} 的列空间，则 x^* 是 $Ax=b$ 的唯一最小范数解。

定理 3.4 设矩阵方程组(1.1)是相容的，初始值取 $X_j(1)=\sum_{i=1}^{p}\left(A_{ij}^H E_i B_{ij}^H+S\left(A_{ij}^H E_i B_{ij}^H\right)^H S\right)$，$Z_j(1)=\sum_{i=1}^{p}\left(A_{ij}^H F_i B_{ij}^H+S\left(A_{ij}^H F_i B_{ij}^H\right)^H S\right)$，$j=1,2,\cdots,q$，其中 E_i 和 F_i 为任意的 $\mathbb{C}^{m\times l}$ 矩阵，特别地，取 $X_j(1)=0$，$Z_j(1)=0$，S 是适当维数的反射矩阵。如果矩阵方程组(1.1)有 perhermitian 解，那么算法 3.1 经过有限步迭代求出的解是矩阵方程(1.1)的唯一的最小范数 perhermitian 解 X_j^*，$j=1,2,\cdots,q$。

证明：根据 Kronecker 积，将 $X_j(1)=\sum_{i=1}^{p}\left(A_{ij}^H E_i B_{ij}^H+S\left(A_{ij}^H E_i B_{ij}^H\right)^H S\right)$ 改写为

$$\underbrace{\begin{pmatrix}\left(vec\left(X_1(1)\right)\right)^H\\ \vdots\\ \left(vec\left(X_q(1)\right)\right)^H\end{pmatrix}}_{\mathcal{X}^*(1)}=\underbrace{\begin{pmatrix}B_{11}\otimes A_{11}^H & SA_{11}^H\otimes SB_{11} & \cdots & B_{p1}\otimes A_{p1}^H & SA_{p1}^H\otimes SB_{p1}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ B_{1q}\otimes A_{1q}^H & SA_{1q}^H\otimes SB_{1q} & \cdots & B_{pq}\otimes A_{pq}^H & SA_{pq}^H\otimes SB_{pq}\end{pmatrix}}_{\mathcal{A}^H}\underbrace{\begin{pmatrix}\left(vec\left(E_1\right)\right)^H\\ \left(vec\left(E_1^H\right)\right)^H\\ \vdots\\ \left(vec\left(E_p\right)\right)^H\\ \left(vec\left(E_p^H\right)\right)^H\end{pmatrix}}_{\mathcal{M}}$$

因此，如果通过上述选择 $X_j(1)$，那么通过算法 3.1 得到的 $X_j(k)$ 满足

$\mathcal{X}^*(k)=\left(\left(vec\left(X_1(k)\right)\right)^H,\cdots,\left(vec\left(X_q(k)\right)\right)^H\right)^H\in\mathcal{R}\left(\mathcal{A}^H\right)$，其中 $\mathcal{R}\left(\mathcal{A}^H\right)$ 表示 $\mathcal{A}^H$ 的列空间。由引理 3.1 可知，如果 $X_j(1)=\sum_{i=1}^{p}\left(A_{ij}^H E_i B_{ij}^H+S\left(A_{ij}^H E_i B_{ij}^H\right)^H S\right)$，$Z_j(1)=\sum_{i=1}^{p}\left(A_{ij}^H F_i B_{ij}^H+S\left(A_{ij}^H F_i B_{ij}^H\right)^H S\right)$，作为初始解，特别是 $X_j(1)=0$，$Z_j(1)=0$，那么可以通过算法 3.1 来求解矩阵方程组(1.1)的唯一最小范数 perhermitian 解。

□

4. 数值算例

在这个部分，我们给出两个算例来说明所提出的算法的有效性。当 $r(k)=\sqrt{\sum_{i=1}^{p}\left\|R_i(k)\right\|^2}\le 10^{-10}$ 时，停止迭代，并且 $X_j(k)$ 被视为矩阵方程组(1.1)的唯一最小范数 perhermitian 解。

例 4.1 给定 Sylvester 矩阵方程为

$$A_{11}X_1B_{11}+A_{12}X_2B_{12}=C_1,$$

其中

$$A_{11}=\begin{pmatrix}6+5i & 5+6i & 1+6i\\ 3+7i & 10+4i & 2+i\\ 2+i & 9+10i & 3+2i\\ 3+9i & 4+6i & 6+3i\end{pmatrix},\ B_{11}=\begin{pmatrix}8+5i & 9+2i & 9+9i & 1+3i\\ 6+3i & 7+i & 2+4i & 6+4i\\ 2+7i & 4+3i & 1+10i & 4+7i\end{pmatrix},$$

$$A_{12}=\begin{pmatrix}4+3i & 10+2i & 4+i\\ 2+5i & 1+5i & 8+7i\\ 3+7i & 4+i & 2+10i\\ 10+2i & 6+8i & 10+4i\end{pmatrix},\ B_{12}=\begin{pmatrix}6+2i & 9+4i & 8+3i & 6+4i\\ 10+8i & 9+9i & 5+3i & 8+9i\\ 10+2i & 2+3i & 6+9i & 1+i\end{pmatrix},$$

$$C_1=10^2\times\begin{pmatrix}1.3500+2.8400i & 1.6000+2.9600i & 0.1800+2.6500i & 0.3600+2.5500i\\ 0.7200+3.1900i & 0.4100+2.2600i & -0.6400+3.4700i & -0.1700+2.0000i\\ 0.6300+3.2400i & 0.7500+2.5500i & -0.8800+2.7500i & 0.1600+2.4200i\\ 1.2000+4.0300i & 1.1800+3.8500i & 0.1000+4.1700i & 0.1300+3.0800i\end{pmatrix},$$

求其唯一最小范数 perhermitian 解。

选取初始矩阵

$$X_1(1)=10^3\times\begin{pmatrix}0.5979+0.0000i & 0.9305+0.2703i & 0.2987-0.2502i\\ 0.9305-0.2703i & 1.0413+0.0000i & 0.4424-0.5889i\\ 0.2987+0.2502i & 0.4424+0.5889i & 0.1692+0.0000i\end{pmatrix},$$

$$X_1(2)=10^3\times\begin{pmatrix}0.8504+0.0000i & 0.9667-0.3004i & 0.7801+0.0635i\\ 0.9667+0.3004i & 1.1728+0.0000i & 0.9089+0.4224i\\ 0.7801-0.0635i & 0.9089-0.4224i & 0.5345+0.0000i\end{pmatrix},$$

根据算法 3.1，经过 24 步迭代终止，得到该矩阵方程的唯一最小范数 perhermitian 解：

$$X_1(24)=10^3\times\begin{pmatrix}1.1948+0.0000i & 1.8609+0.5406i & 0.5974-0.5004i\\ 1.8609-0.5406i & 2.0817+0.0000i & 0.8847-1.1779i\\ 0.5974+0.5004i & 0.8847+1.1779i & 0.3373+0.0000i\end{pmatrix},$$

$$X_2(24)=10^3\times\begin{pmatrix}1.6998+0.0000i & 1.9334-0.6008i & 1.5601+0.1270i\\ 1.9334+0.6008i & 2.3446+0.0000i & 1.8178+0.8648i\\ 1.5601-0.1270i & 1.8178-0.8648i & 1.0680+0.0000i\end{pmatrix},$$

并且，相应残差的范数 $r=2.4009\times10^{-13}<10^{-10}$。

例 4.2 给定 Sylvester 矩阵方程组为

$$\begin{cases}A_{11}X_1B_{11}+A_{12}X_2B_{12}=C_1\\ A_{21}X_1B_{21}+A_{22}X_2B_{22}=C_2\end{cases},$$

其中

$$A_{11}=\begin{pmatrix}10+2i & 9+6i & 1+8i\\ 8+4i & 5+4i & 4+i\\ 10+6i & 9+i & 9+7i\\ 6+10i & 6+8i & 5+3i\end{pmatrix},B_{11}=\begin{pmatrix}4+i & 9+8i & 4+6i & 4+3i\\ 6+9i & 6+5i & 7+5i & 6+3i\\ 5+2i & 6+10i & 6+5i & 3+4i\end{pmatrix},$$

$$A_{12}=\begin{pmatrix}9+9i & 4+3i & 8+9i\\ 3+10i & 4+4i & 8+4i\\ 10+3i & 8+6i & 3+7i\\ 4+5i & 7+4i & 9+7i\end{pmatrix},B_{12}=\begin{pmatrix}5+6i & 5+6i & 5+6i & 10+3i\\ 8+4i & 10+3i & 1+10i & 1+9i\\ 1+2i & 10+2i & 2+6i & 1+3i\end{pmatrix},$$

$$A_{21}=\begin{pmatrix}2+5i & 9+5i & 2+i\\ 10+8i & 2+i & 7+10i\\ 3+4i & 10+8i & 4+7i\\ 2+9i & 6+5i & 4+9i\end{pmatrix},B_{21}=\begin{pmatrix}3+3i & 5+3i & 10+6i & 5+7i\\ 6+4i & 8+4i & 10+5i & 3+10i\\ 8+9i & 7+2i & 6+10i & 4+4i\end{pmatrix},$$

$$A_{22}=\begin{pmatrix}5+7i & 4+9i & 4+3i\\ 7+5i & 3+4i & 10+i\\ 2+6i & 4+3i & 2+9i\\ 8+9i & 7+3i & 5+10i\end{pmatrix},B_{22}=\begin{pmatrix}5+5i & 4+4i & 3+7i & 10+2i\\ 4+2i & 5+i & 1+2i & 1+10i\\ 5+9i & 4+10i & 1+9i & 2+i\end{pmatrix},$$

$$C_1=10^2\times\begin{pmatrix}0.2800+3.4100i & 1.0800+4.8400i & -0.4600+4.1600i & 0.6200+3.1800i\\ 0.1100+2.4200i & 1.1900+3.7100i & -0.4700+3.1100i & 0.1000+2.7500i\\ 1.7100+3.4200i & 1.8500+5.5200i & 0.2500+4.1600i & 0.9900+2.9800i\\ 0.2200+3.0700i & 0.9400+4.8200i & -0.8600+3.9600i & -0.0700+3.1600i\end{pmatrix},$$

$$C_2=10^2\times\begin{pmatrix}0.1300+2.6800i & 0.4800+2.6700i & -0.1400+2.9500i & -0.8900+2.9500i\\ 0.3500+3.8800i & 1.1600+3.4500i & -0.0900+4.5500i & 0.2000+3.1100i\\ -0.8700+3.2400i & -0.1600+2.9700i & -0.9700+3.4000i & -0.9800+3.3600i\\ -1.0200+4.0100i & -0.3100+3.6600i & -1.8800+4.3100i & -0.6600+3.9000i\end{pmatrix},$$

求其唯一最小范数 perhermitian 解。

选取初始矩阵

$$X_1(1)=10^3\times\begin{pmatrix}2.2322+0.0000i & 2.4950-0.4177i & 1.8794-0.1577i\\ 2.4950+0.4177i & 2.8320+0.0000i & 2.2000+0.1784i\\ 1.8794+0.1577i & 2.2000-0.1784i & 1.5266+0.0000i\end{pmatrix},$$

$$X_2(1)=10^3\times\begin{pmatrix}2.3482+0.0000i & 1.6111-0.3890i & 1.7942-0.0223i\\ 1.6111+0.3890i & 1.2291+0.0000i & 1.2886+0.3338i\\ 1.7942+0.0223i & 1.2886-0.3338i & 1.4593+0.0000i\end{pmatrix},$$

根据算法 3.1，经过 19 步迭代终止，得到该矩阵方程的唯一最小范数 perhermitian 解：

$$X_1(19)=10^3\times\begin{pmatrix}4.4633+0.0000i & 4.9899-0.8353i & 3.7589-0.3154i\\ 4.9899+0.8353i & 5.6630+0.0000i & 4.4000+0.3567i\\ 3.7589+0.3154i & 4.4000-0.3567i & 3.0522+0.0000i\end{pmatrix},$$

$$X_2(19) = 10^3 \times \begin{pmatrix} 4.6953 + 0.0000i & 3.2223 - 0.7781i & 3.5884 - 0.0445i \\ 3.2223 + 0.7781i & 2.4572 + 0.0000i & 2.5773 + 0.6675i \\ 3.5884 + 0.0445i & 2.5773 - 0.6675i & 2.9175 + 0.0000i \end{pmatrix},$$

并且，相应残差的范数 $r = 4.4335 \times 10^{-12} < 10^{-10}$ 。

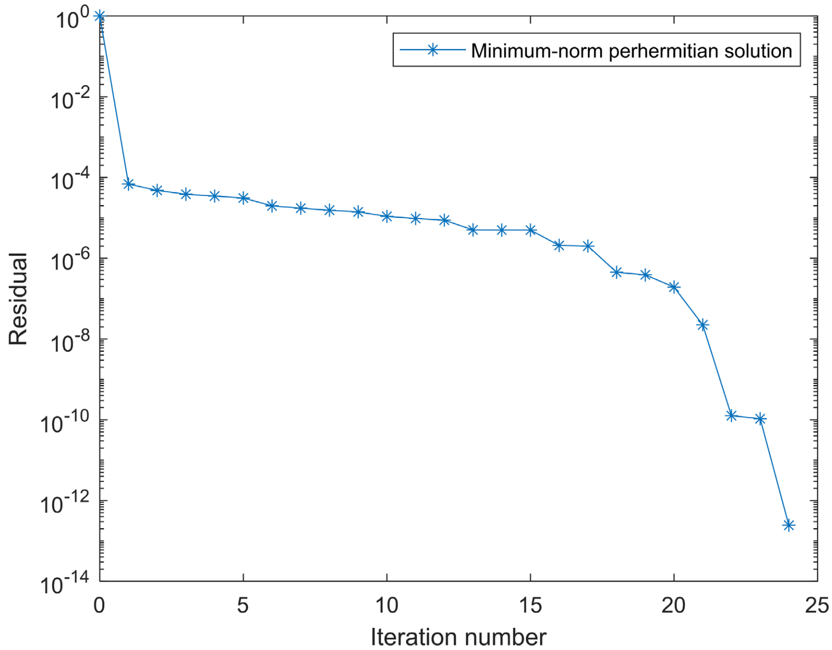

Figure 1. Convergence curves for Example 4.1
图 1. 例 4.1 的收敛曲线

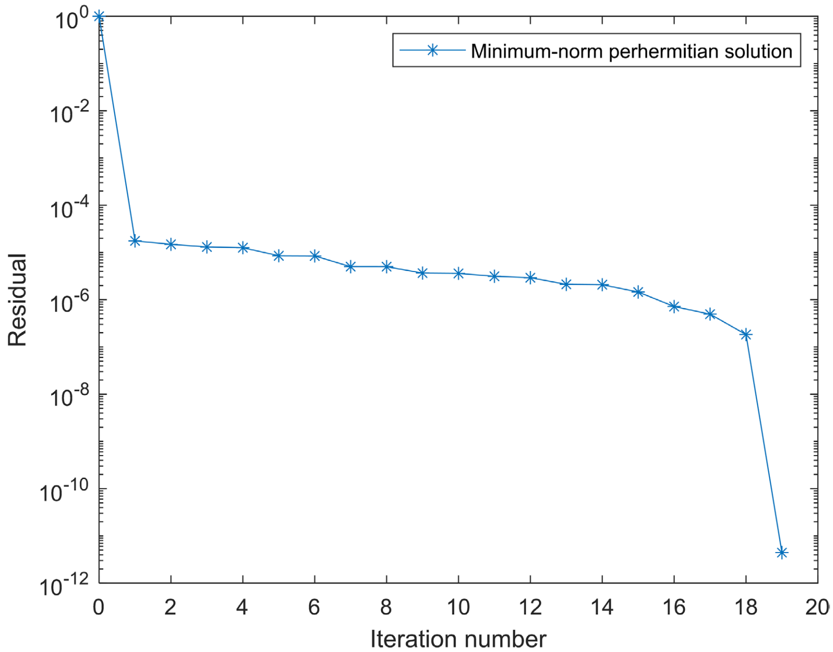

Figure 2. Convergence curves for Example 4.2
图 2. 例 4.2 的收敛曲线

上述两个例子表明，我们的方法能够有效的获得 Sylvester 矩阵方程组的唯一最小范数 perhermitian 解。此外，从图 1 和图 2 可以看出，算法在数值上是非常可靠的。

5. 总结

在本文中，我们利用 BCR 算法来求解 Sylvester 矩阵方程组(1.1)的 perhermitian 解。我们证明了算法是收敛的，在不存在舍入误差的情况下，可以在有限步内得到唯一的最小范数 perhermitian 解。最后，通过两个算例说明了算法的可行性和有效性。

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