

任意凸四边形区域上二阶变系数椭圆边值问题有效的谱Galerkin逼近

刘雪林, 张应洪, 施芳

贵州师范大学数学科学学院, 贵州 贵阳

收稿日期: 2023年12月25日; 录用日期: 2024年1月19日; 发布日期: 2024年1月26日

摘要

本文提出了在任意凸四边形区域上二阶变系数椭圆边值问题的一种有效谱Galerkin逼近。首先, 通过双线性等参变换和坐标变换将任意四边形区域转换到 $\tilde{D} = [-1, 1]^2$, 并建立其在 \tilde{D} 的弱形式及相应的离散格式。其次, 我们证明了弱解的存在唯一性。另外, 利用Legendre 正交多项式构造了逼近空间中一组有效的基函数, 推导出离散格式的矩阵形式。最后通过数值实验, 验证了谱Galerkin逼近任意凸四边形区域上二阶变系数椭圆边值问题的谱收敛。

关键词

任意凸四边形区域, 二阶变系数椭圆边值问题, 弱解的存在唯一性, 谱Galerkin逼近, 谱精度

Efficient Spectral Galerkin Approximation for Second Order Elliptic Boundary Value Problems with Variable Coefficient on Arbitrary Convex Quadrilateral Domain

Xuelin Liu, Yinghong Zhang, Fang Shi

School of Mathematical Sciences, Guizhou Normal University, Guiyang Guizhou

Received: Dec. 25th, 2023; accepted: Jan. 19th, 2024; published: Jan. 26th, 2024

Abstract

In this paper, an efficient spectral Galerkin approximation for second-order elliptic boundary value problems with variable coefficients on an arbitrary convex quadrilateral region is proposed. Firstly, any quadrilateral region is converted to $\tilde{D} = [-1, 1]^2$ by bilinear isoparametric transformation and coordinate transformation, and its weak form and corresponding discrete format on \tilde{D} are established. Secondly, we prove the existence and uniqueness of the weak solution. In addition, the Legendre orthogonal polynomial is used to construct a set of effective basis functions in the approximation space, and the matrix form of the discrete scheme is derived. Finally, the spectral convergence of spectral Galerkin approximation to the second-order elliptic boundary value problem with variable coefficients on arbitrary convex quadrilateral region is verified by numerical experiments.

Keywords

Arbitrary Convex Quadrilateral Region, Second Order Elliptic Boundary Value Problem with Variable Coefficient, Existence and Uniqueness of Weak Solutions, Spectral Galerkin Approximation, Spectral Convergence

Copyright © 2024 by author(s) and Hans Publishers Inc.

This work is licensed under the Creative Commons Attribution International License (CC BY 4.0).

<http://creativecommons.org/licenses/by/4.0/>



1. 引言

在很多科学和工程计算领域，我们通常要求解一些非线性的抛物问题和特征值问题，如量子力学、流体力学和传输特征值问题 [1–5] 等。这些非线性问题线性化后最终都归结为反复地求解二阶椭圆微分方程边值问题。关于二阶椭圆微分方程边值问题的数值计算已有很多成果，如有限差分法 [6, 7]、有限元法 [8–11]和谱方法 [12–15]。

众所周知，当准确解足够光滑时谱方法具有谱收敛的优点，并已经广泛的运用于微分方程的数值计算，大多数研究都是在方形或者矩形区域上分析和求解的，因此将该方法延伸到更多区域

上计算二阶问题引起了越来越多的学者的关注。安静在文献 [16] 提出了在球域上Maxwell 传输特征值问题的一种有效谱逼近, 文献 [17]中提出了任意凸四边形区域上二阶椭圆特征值问题基于高阶多项式逼近的一种数值方法, 但没有给出弱解的存在唯一性。根据我们的了解, 关于在任意凸四边形区域上二阶变系数椭圆边值问题的谱Galerkin逼近的研究较少。因此, 本文提出了在任意凸四边形区域上二阶变系数椭圆边值问题的一种有效谱Galerkin 逼近。首先, 通过双线性等参变换和坐标变换将任意四边形区域转换到, 并建立其在上的弱形式及相应的离散格式。其次, 我们证明了弱解的存在唯一性。另外, 利用Legendre正交多项式构造了逼近空间中一组有效的基函数, 推导出离散格式相应的矩阵形式。最后通过使用Matlab软件编程, 我们给出了大量的数值算例, 数值结果表明本文建立的谱Galerki逼近格式能够有效求解任意凸四边形区域上二阶变系数椭圆边值问题, 达到了谱收敛精度。

本文剩余部分组织如下: 在第2节, 我们推到了原问题在通过等参变换后的弱形式及离散格式。在第3节, 我们证明的解的存在唯一性。在第4节, 我们详细的描述了离散格式的有效实现。第5节, 我们给出了一些数值算例。最后, 在第6节给出了一些结论性的评注。

2. 变换后的弱形式及其离散格式

作为一个模型, 我们考虑下面的二阶椭圆问题:

$$\begin{cases} -\Delta u(x, y) + \alpha(x, y)u(x, y) = f(x, y), (x, y) \in \Omega \\ u(x, y) = 0, (x, y) \in \partial\Omega \end{cases} \quad (1)$$

其中 Ω 是由顶点坐标为 $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ 所构成的任意凸四边形区域, $\alpha(x, y)$ 为任意非负有界函数。

令 $H^s(\Omega)$ 和 $H_0^s(\Omega)$ 表示通常的 s 阶Sobolev空间, $\|\cdot\|_{s,\Omega}$ 和 $|\cdot|_{s,\Omega}$ 分别表示其范数和半范数。由格林公式可知(1)的弱形式为:找 $u \in H_0^1(\Omega)$, 使得

$$a(u, v) = f(v), \forall v \in H_0^1(\Omega), \quad (2)$$

其中 $a(u, v) = \int_{\Omega} \nabla u \nabla v + \alpha uv dx dy, f(v) = \int_{\Omega} f v dx dy$ 。

令

$$\begin{cases} x = x_1 + (x_2 - x_1)\bar{x} + (x_3 - x_1)\bar{y} + (x_4 - x_3 - x_2 + x_1)\bar{x}\bar{y}, \\ y = y_1 + (y_2 - y_1)\bar{x} + (y_3 - y_1)\bar{y} + (y_4 - y_3 - y_2 + y_1)\bar{x}\bar{y}, \end{cases} \quad (3)$$

显然, (3)是从一个区域 $D = \{(\bar{x}, \bar{y}) : \bar{x}, \bar{y} \in [0, 1]\}$ 到区域 Ω 上的双线性等参变换。记 $\hat{u}(\bar{x}, \bar{y}) = u(x, y), \hat{\alpha}(\bar{x}, \bar{y}) = \alpha(x, y)$ 。

为了保证凸区域上的等参变换是一对一的, 变换的Jacobi行列式需要满足下列条件:

$$J(\bar{x}, \bar{y}) = \begin{vmatrix} x_2 - x_1 + a\bar{y} & x_3 - x_1 + a\bar{x} \\ y_2 - y_1 + b\bar{y} & y_3 - y_1 + b\bar{x} \end{vmatrix} > 0 \quad (4)$$

其中 $a = x_4 - x_3 - x_2 + x_1, b = y_4 - y_3 - y_2 + y_1$.

又令

$$\begin{cases} \bar{x} = \frac{1}{2}(\xi + 1), \\ \bar{y} = \frac{1}{2}(\eta + 1). \end{cases} \quad (5)$$

显然(5)是从一个区域 $\tilde{D} = \{(\xi, \eta) : \xi, \eta \in [-1, 1]\}$ 到区域 D 上的坐标变换。记 $\tilde{u}(\xi, \eta) = \hat{u}(\bar{x}, \bar{y}), \tilde{J}(\xi, \eta) = J(\bar{x}, \bar{y}), \tilde{\alpha}(\xi, \eta) = \hat{\alpha}(\bar{x}, \bar{y})$ 。

由(3)和(5)及复合函数求导法则可得梯度算子通过变换过后的等价形式为:

$$\nabla u = \tilde{\nabla} \tilde{u} = 2[\tilde{u}_\xi \frac{y_3 - y_1 + \frac{b}{2}(\xi+1)}{j} - \tilde{u}_\eta \frac{y_2 - y_1 + \frac{b}{2}(\eta+1)}{j}, \tilde{u}_\eta \frac{x_2 - x_1 + \frac{a}{2}(\eta+1)}{j} - \tilde{u}_\xi \frac{x_3 - x_1 + \frac{a}{2}(\xi+1)}{j}] \quad (6)$$

定义 Sobolev 空间:

$$H_0^1(\tilde{D}) = \{ \tilde{u} : \int_{\tilde{D}} (|\tilde{u}_\xi \frac{y_3 - y_1 + \frac{b}{2}(\xi+1)}{j} - \tilde{u}_\eta \frac{y_2 - y_1 + \frac{b}{2}(\eta+1)}{j}|^2 + |\tilde{u}_\eta \frac{x_2 - x_1 + \frac{a}{2}(\eta+1)}{j} - \tilde{u}_\xi \frac{x_3 - x_1 + \frac{a}{2}(\xi+1)}{j}|^2) \tilde{J} d\xi d\eta < \infty, \tilde{u}(\pm 1, \eta) = 0, \tilde{u}(\xi, \pm 1) = 0 \},$$

其相应的内积和范数分别为:

$$\begin{aligned} \langle \tilde{u}, \tilde{v} \rangle &= \int_{\tilde{D}} \tilde{u} \tilde{v} \tilde{J} d\xi d\eta, \|\tilde{u}\| = \sqrt{\langle \tilde{u}, \tilde{u} \rangle}, \\ \langle \tilde{u}, \tilde{v} \rangle_1 &= \int_{\tilde{D}} (\tilde{u}_\xi \frac{y_3 - y_1 + \frac{b}{2}(\xi+1)}{j} - \tilde{u}_\eta \frac{y_2 - y_1 + \frac{b}{2}(\eta+1)}{j})(\tilde{v}_\xi \frac{y_3 - y_1 + \frac{b}{2}(\xi+1)}{j} - \tilde{v}_\eta \frac{y_2 - y_1 + \frac{b}{2}(\eta+1)}{j}) \tilde{J} d\xi d\eta \\ &+ \int_{\tilde{D}} (\tilde{u}_\eta \frac{x_2 - x_1 + \frac{a}{2}(\eta+1)}{j} - \tilde{u}_\xi \frac{x_3 - x_1 + \frac{a}{2}(\xi+1)}{j})(\tilde{v}_\eta \frac{x_2 - x_1 + \frac{a}{2}(\eta+1)}{j} - \tilde{v}_\xi \frac{x_3 - x_1 + \frac{a}{2}(\xi+1)}{j}) \tilde{J} d\xi d\eta, \\ \|\tilde{u}\|_1 &= \sqrt{\langle \tilde{u}, \tilde{u} \rangle_1}, \|\tilde{u}\|_1 = \sqrt{\langle \tilde{u}, \tilde{u} \rangle_1 + \langle \tilde{u}, \tilde{u} \rangle}. \end{aligned}$$

由(6)可得弱形式(2)通过(3)和(5)变换过后的等价形式为:找 $\tilde{u} \in H_0^1(\tilde{D})$, 使得

$$A(\tilde{u}, \tilde{v}) = F(\tilde{v}), \forall \tilde{v} \in H_0^1(\tilde{D}), \quad (7)$$

其中

$$\begin{aligned} A(\tilde{u}, \tilde{v}) &= \int_{\tilde{D}} (\tilde{u}_\xi \frac{y_3 - y_1 + \frac{b}{2}(\xi+1)}{j} - \tilde{u}_\eta \frac{y_2 - y_1 + \frac{b}{2}(\eta+1)}{j})(\tilde{v}_\xi \frac{y_3 - y_1 + \frac{b}{2}(\xi+1)}{j} - \tilde{v}_\eta \frac{y_2 - y_1 + \frac{b}{2}(\eta+1)}{j}) \\ &\tilde{J} d\xi d\eta + \int_{\tilde{D}} (\tilde{u}_\eta \frac{x_2 - x_1 + \frac{a}{2}(\eta+1)}{j} - \tilde{u}_\xi \frac{x_3 - x_1 + \frac{a}{2}(\xi+1)}{j})(\tilde{v}_\eta \frac{x_2 - x_1 + \frac{a}{2}(\eta+1)}{j} - \tilde{v}_\xi \frac{x_3 - x_1 + \frac{a}{2}(\xi+1)}{j}) \\ &\tilde{J} d\xi d\eta + \int_{\tilde{u}} \alpha \tilde{u} \tilde{v} \frac{\tilde{J}}{4} d\xi d\eta, F(\tilde{v}) = \int_{\tilde{D}} \tilde{f}(\xi, \eta) \tilde{v}(\xi, \eta) \frac{\tilde{J}(\xi, \eta)}{4} d\xi d\eta, \tilde{f}(\xi, \eta) = f(x, y). \end{aligned}$$

定义逼近空间 $X_N = P_N \times P_N \cap H_0^1(\tilde{D})$, 其中 P_N 表示次数不超过 N 的多项式空间。则(7)相应的离散格式为: 找 $u_N \in X_N$, 使得:

$$A(u_N, v_N) = F(v_N), \forall v_N \in X_N, \quad (8)$$

3. 解的存在唯一性

引理1 $A(\tilde{u}, \tilde{v})$ 为定义在 $H_0^1(\tilde{D}) \times H_0^1(\tilde{D})$ 上的连续强制的双线性形式, 即

$$\begin{aligned} |A(\tilde{u}, \tilde{v})| &\lesssim \|\tilde{u}\|_1 \|\tilde{v}\|_1, \forall \tilde{u}, \tilde{v} \in H_0^1(\tilde{D}), \\ |A(\tilde{u}, \tilde{u})| &\gtrsim \|\tilde{u}\|_1^2, \forall \tilde{u} \in H_0^1(\tilde{D}). \end{aligned}$$

证明: 由于 α 是非负有界函数, 通过Cauchy - Schwarz不等式可得:

$$\begin{aligned} |A(\tilde{u}, \tilde{v})| &= \left| \int_{\tilde{D}} \left(\tilde{u}_\xi \frac{y_3 - y_1 + \frac{b}{2}(\xi+1)}{j} - \tilde{u}_\eta \frac{y_2 - y_1 + \frac{b}{2}(\eta+1)}{j} \right) \left(\tilde{v}_\xi \frac{y_3 - y_1 + \frac{b}{2}(\xi+1)}{j} - \tilde{v}_\eta \frac{y_2 - y_1 + \frac{b}{2}(\eta+1)}{j} \right) \right. \\ &\quad \left. \tilde{J} d\xi d\eta + \int_{\tilde{D}} \left(\tilde{u}_\eta \frac{x_2 - x_1 + \frac{a}{2}(\eta+1)}{j} - \tilde{u}_\xi \frac{x_3 - x_1 + \frac{a}{2}(\xi+1)}{j} \right) \left(\tilde{v}_\eta \frac{x_2 - x_1 + \frac{a}{2}(\eta+1)}{j} - \tilde{v}_\xi \frac{x_3 - x_1 + \frac{a}{2}(\xi+1)}{j} \right) \right. \\ &\quad \left. \tilde{J} d\xi d\eta + \int_{\tilde{u}} \tilde{\alpha} \tilde{u} \tilde{v} \frac{\tilde{J}}{4} d\xi d\eta \right| = \left| \int_{\Omega} \nabla u \nabla v + \alpha uv dx dy \right| \lesssim \|u\|_{1,\Omega} \|v\|_{1,\Omega}. \end{aligned}$$

由Poincaré不等式有:

$$\begin{aligned} |A(\tilde{u}, \tilde{v})| &\lesssim \|u\|_{1,\Omega} \|v\|_{1,\Omega} \lesssim |u|_{1,\Omega} |v|_{1,\Omega} = \left(\int_{\Omega} |\nabla u|^2 dx dy \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla v|^2 dx dy \right)^{\frac{1}{2}} \\ &= \left\{ \int_{\tilde{D}} \left(\left| \tilde{u}_\xi \frac{y_3 - y_1 + \frac{b}{2}(\xi+1)}{j} - \tilde{u}_\eta \frac{y_2 - y_1 + \frac{b}{2}(\eta+1)}{j} \right|^2 + \left| \tilde{u}_\eta \frac{x_2 - x_1 + \frac{a}{2}(\eta+1)}{j} - \tilde{u}_\xi \frac{x_3 - x_1 + \frac{a}{2}(\xi+1)}{j} \right|^2 \right) \tilde{J} d\xi d\eta \right\}^{\frac{1}{2}} \\ &\quad \left\{ \int_{\tilde{D}} \left(\left| \tilde{v}_\xi \frac{y_3 - y_1 + \frac{b}{2}(\xi+1)}{j} - \tilde{v}_\eta \frac{y_2 - y_1 + \frac{b}{2}(\eta+1)}{j} \right|^2 + \left| \tilde{v}_\eta \frac{x_2 - x_1 + \frac{a}{2}(\eta+1)}{j} - \tilde{v}_\xi \frac{x_3 - x_1 + \frac{a}{2}(\xi+1)}{j} \right|^2 \right) \tilde{J} d\xi d\eta \right\}^{\frac{1}{2}} \\ &= |u|_1 |v|_1 \leq \|u\|_1 \|v\|_1. \end{aligned}$$

另一方面, 我们有

$$\begin{aligned} |A(\tilde{u}, \tilde{u})| &\geq \int_{\tilde{D}} \left(\left| \tilde{u}_\xi \frac{y_3 - y_1 + \frac{b}{2}(\xi+1)}{j} - \tilde{u}_\eta \frac{y_2 - y_1 + \frac{b}{2}(\eta+1)}{j} \right|^2 + \left| \tilde{u}_\eta \frac{x_2 - x_1 + \frac{a}{2}(\eta+1)}{j} - \right. \right. \\ &\quad \left. \left. \tilde{u}_\xi \frac{x_3 - x_1 + \frac{a}{2}(\xi+1)}{j} \right|^2 \right) \tilde{J} d\xi d\eta \gtrsim |\tilde{u}|_1^2 \end{aligned}$$

证毕.

定理1 若 $f \in L^2(\Omega)$, 则弱形式(7)和离散格式(8)分别存在唯一解 $\tilde{u} \in H_0^1(\tilde{D})$ 和 $u_N \in X_N$. 证明: 由于 $f \in L^2(\Omega)$, 则由Cauchy)Schwarz不等式及Poincaré不等式有

$$\begin{aligned} |F(\tilde{v})| &= \left| \int_{\tilde{D}} \tilde{f} \tilde{v} \tilde{J} d\xi d\eta \right| = \left| \int_{\Omega} f v dx dy \right| \\ &\leq \left(\int_{\Omega} f^2 dx dy \right)^{\frac{1}{2}} \left(\int_{\Omega} v^2 dx dy \right)^{\frac{1}{2}} \lesssim \left(\int_{\Omega} v^2 dx dy \right)^{\frac{1}{2}} \\ &\lesssim \left(\int_{\Omega} |\nabla v|^2 dx dy \right)^{\frac{1}{2}} = |v|_1. \end{aligned}$$

即 $F(\tilde{v})$ 为定义在 $H_0^1(\tilde{D})$ 上的有界线性泛函. 由引理1及Lax - Milgram定理可知弱形式(7)存在唯一解 $\tilde{u} \in H_0^1(\tilde{D})$. 类似地, 我们可证离散格式(8)存在唯一解 $u_N \in X_N$. 证毕.

4. 算法有效实现

现将描述算法的主要实现过程. 首先, 取逼近空间 X_N 的一组基函数. 令 $\varphi_i(\xi) = L_i(\xi) -$

$L_{i+2}(\xi)(i = 0, 1, 2, \dots, N-2)$, 其中 $L_i(\xi)$ 为 i 次 Legendre 多项式, 因此逼近空间 X_N 可表示为:

$$X_N = \text{span}\{\varphi_i(\xi)\varphi_j(\eta) : i, j = 0, 1, \dots, N-2\}. \quad (9)$$

将 \tilde{u}_N 用基函数展开得:

$$\tilde{u}_N = \sum_{i,j=0}^{N-2} u_{ij} \varphi_i(\xi)\varphi_j(\eta) \quad (10)$$

其中 u_{ij} 为展开系数. 令

$$U = \begin{pmatrix} u_{00} & u_{01} & \cdots & u_{0,N-2} \\ u_{10} & u_{11} & \cdots & u_{1,N-2} \\ \vdots & \vdots & \ddots & \vdots \\ u_{N-2,0} & u_{N-2,1} & \cdots & u_{N-2,N-2} \end{pmatrix} \quad (11)$$

在(8)中让 v_N 取遍逼近空间 X_N 中的所有基函数, 通过一系列的推导过程, 可整理得到:

$$AU = F \quad (12)$$

其中:

$$\begin{aligned} A &= A_{ij}^1 - A_{ij}^2 - A_{ij}^3 + A_{ij}^4 + A_{ij}^5 - A_{ij}^6 - A_{ij}^7 + A_{ij}^8 + B_{ij}, \\ A_{ij}^s &= (i_{kl}^s j_{mn}^s)_{k,l,m,n=0}^{N-2}, s = 1, 2, \dots, 8, B_{ij} = (i_{kl} j_{mn})_{k,l,m,n=0}^{N-2}, \\ i_{kl}^1 j_{mn}^1 &= 4 \int_{-1}^1 \int_{-1}^1 \varphi_l'(\xi)\varphi_n(\eta)\varphi_k'(\xi)\varphi_m(\eta)(y_3 - y_1 + b\frac{\xi+1}{2})^2 \frac{1}{J} d\xi d\eta, \\ i_{kl}^2 j_{mn}^2 &= 4 \int_{-1}^1 \int_{-1}^1 \varphi_l'(\xi)\varphi_n(\eta)\varphi_k(\xi)\varphi_m'(\eta)(y_3 - y_1 + b\frac{\xi+1}{2})(y_2 - y_1 + b\frac{\eta+1}{2}) \frac{1}{J} d\xi d\eta, \\ i_{kl}^3 j_{mn}^3 &= 4 \int_{-1}^1 \int_{-1}^1 \varphi_l(\xi)\varphi_n'(\eta)\varphi_k'(\xi)\varphi_m(\eta)(y_3 - y_1 + b\frac{\xi+1}{2})(y_2 - y_1 + b\frac{\eta+1}{2}) \frac{1}{J} d\xi d\eta, \\ i_{kl}^4 j_{mn}^4 &= 4 \int_{-1}^1 \int_{-1}^1 \varphi_l(\xi)\varphi_n'(\eta)\varphi_k(\xi)\varphi_m'(\eta)(y_2 - y_1 + b\frac{\xi+1}{2})^2 \frac{1}{J} d\xi d\eta, \\ i_{kl}^5 j_{mn}^5 &= 4 \int_{-1}^1 \int_{-1}^1 \varphi_l(\xi)\varphi_n'(\eta)\varphi_k(\xi)\varphi_m'(\eta)(x_2 - x_1 + a\frac{\xi+1}{2})^2 \frac{1}{J} d\xi d\eta, \\ i_{kl}^6 j_{mn}^6 &= 4 \int_{-1}^1 \int_{-1}^1 \varphi_l'(\xi)\varphi_n(\eta)\varphi_k(\xi)\varphi_m'(\eta)(x_3 - x_1 + a\frac{\xi+1}{2})(x_2 - x_1 + a\frac{\eta+1}{2}) \frac{1}{J} d\xi d\eta, \\ i_{kl}^7 j_{mn}^7 &= 4 \int_{-1}^1 \int_{-1}^1 \varphi_l(\xi)\varphi_n'(\eta)\varphi_k'(\xi)\varphi_m(\eta)(x_3 - x_1 + a\frac{\xi+1}{2})(x_2 - x_1 + a\frac{\eta+1}{2}) \frac{1}{J} d\xi d\eta, \\ i_{kl}^8 j_{mn}^8 &= 4 \int_{-1}^1 \int_{-1}^1 \varphi_l'(\xi)\varphi_n(\eta)\varphi_k'(\xi)\varphi_m(\eta)(x_3 - x_1 + a\frac{\xi+1}{2})^2 \frac{1}{J} d\xi d\eta, \end{aligned}$$

$$i_{kl}j_{mn} = \int_{-1}^1 \int_{-1}^1 \tilde{\alpha}(\xi, \eta) \varphi_l(\xi) \varphi_n(\eta) \varphi_k(\xi) \varphi_m(\eta) \tilde{J} d\xi d\eta,$$

$$F = \int_{-1}^1 \int_{-1}^1 \tilde{f} \varphi_m(\xi) \varphi_n(\eta) \tilde{J} d\xi d\eta.$$

5. 数值算例

为表明算法的有效性和收敛性，一系列数值算例被呈现，我们将在MATLAB2016a平台上编程测试。定义精确解与逼近解的误差如下：

$$e(u, u_N) = \left\{ \int_D (\tilde{u} - u_N)^2 \tilde{J} d\xi d\eta \right\}^{\frac{1}{2}}.$$

例1: Ω 为一般的凸四边形的情况，取 $\alpha = 1$,四个顶点坐标分别为： $A(x_1, y_1) = (-2, -1), B(x_2, y_2) = (2, -3), C(x_3, y_3) = (-1, 1.25), D(x_4, y_4) = (2, 1)$ ，如图1所示，计算如下问题的数值解：

$$\begin{cases} -\Delta u + u = f, (x, y) \in \Omega \\ u = 0, (x, y) \in \partial\Omega \end{cases}$$

该问题的精确解为： $(y + \frac{1}{2}x + 2)(x - 2)(y + \frac{1}{12}x - \frac{7}{6})(y - \frac{9}{4}x - \frac{7}{2})\sin x \sin y$ 。

对于不同的N，我们在图2中画出了精确解 u 与逼近解 u_N 之间的误差曲线 $e(u, u_N)$ 。从图2我们可以观察到，当 $N \geq 18$ 时，逼近解与精确解之间的误差达到了大约 10^{-12} 的精度。为了更直观地表明我们算法的收敛性和谱精度，我们分别在图3和图4中列出精确解与 $N = 20$ 时逼近图像，在图5和图6中分别列出 $N = 15$ 和 $N = 20$ 时精确解和误差解之间的误差图像。从图1-图6我们可以观察到我们的算法是收敛的和谱精度的。

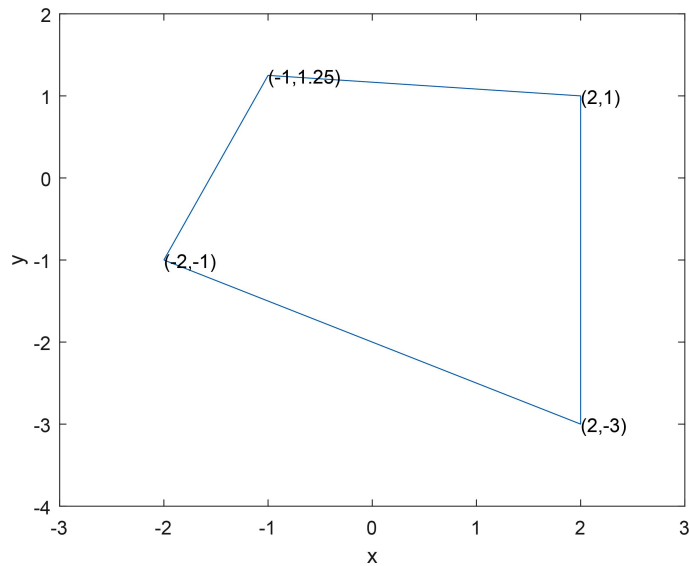


Figure 1. Generally convex quadrilateral area

图 1. 一般凸四边形区域

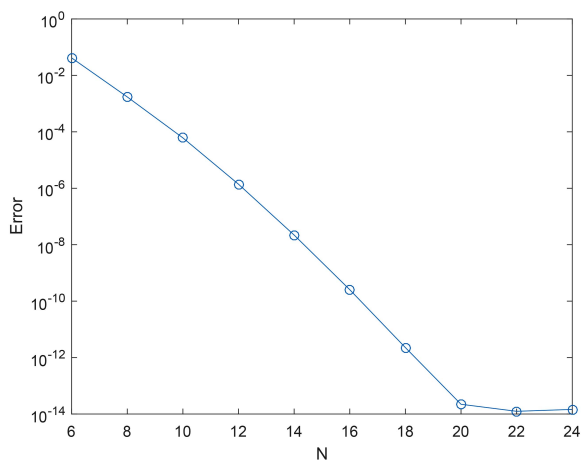


Figure 2. Error curve between approximate and analytic solutions

图 2. 逼近解与解析解之间的误差曲线

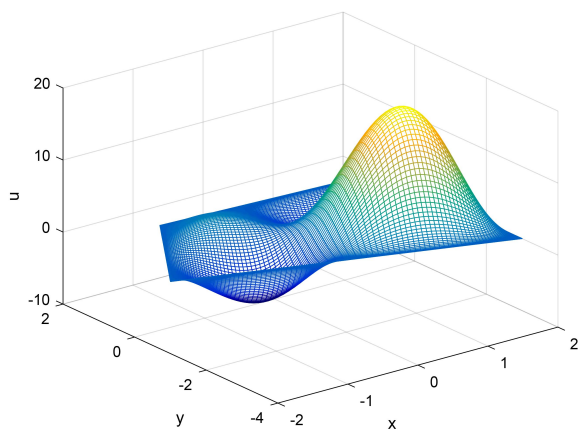


Figure 3. Exact solution of $u(x,y)$ image

图 3. 精确解 $u(x,y)$ 的图像

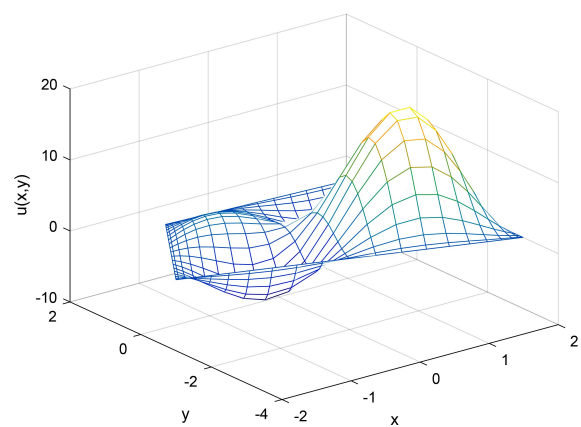


Figure 4. Image of the approximation solution at $N=20$

图 4. $N=20$ 时逼近解的图像

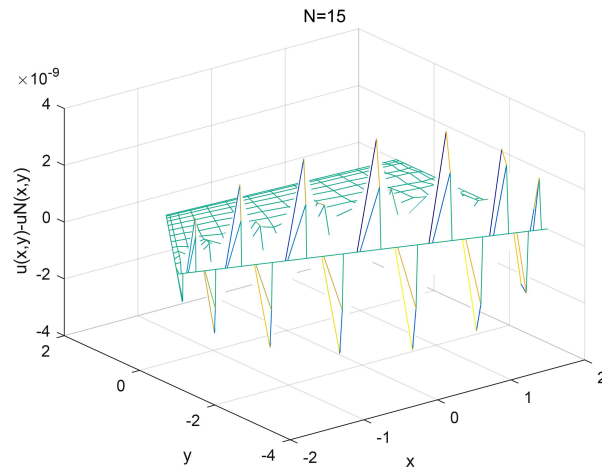


Figure 5. An image of the error between the exact solution and the approximate solution at $N=15$

图 5. 精确解与 $N=15$ 时逼近解之间的误差图像

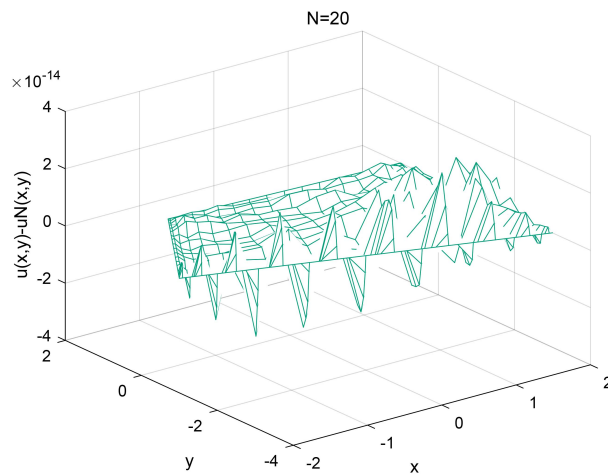


Figure 6. An image of the error between the exact solution and the approximate solution at $N=20$

图 6. 精确解与 $N=20$ 时逼近解之间的误差图像

例2: Ω 为一般的凸四边形的情况, 取 $\alpha = 5$, 四个顶点坐标分别为: $A(x_1, y_1) = (0, -1), B(x_2, y_2) = (3, 0), C(x_3, y_3) = (-1, 0), D(x_4, y_4) = (0, 1)$, 如图7所示, 计算如下问题的数值解:

$$\begin{cases} -\Delta u + \alpha u = f, (x, y) \in \Omega \\ u = 0, (x, y) \in \partial\Omega \end{cases}$$

该问题的精确解为: $(x + y + 1)(y - x - 1)(y + \frac{1}{3}x - 1)(y - \frac{1}{3}x + 1)\sin x \sin y$.

对于不同的 N , 我们在图8中画出了精确解 u 与逼近解 u_N 之间的误差曲线 $e(u, u_N)$ 。从图8我们可以观察到, 当 $N \geq 16$ 时, 逼近解与精确解之间的误差达到了大约 10^{-12} 的精度。为了更直观地表明我们算法的收敛性和谱精度, 我们分别在图9和图10中列出精确解与 $N = 20$ 时逼近图像, 在图

11和图 12中分别列出 $N = 15$ 和 $N = 20$ 时精确解和误差解之间的误差图像。从图 7-图 12 我们可以观察到我们的算法是收敛的和谱精度的。

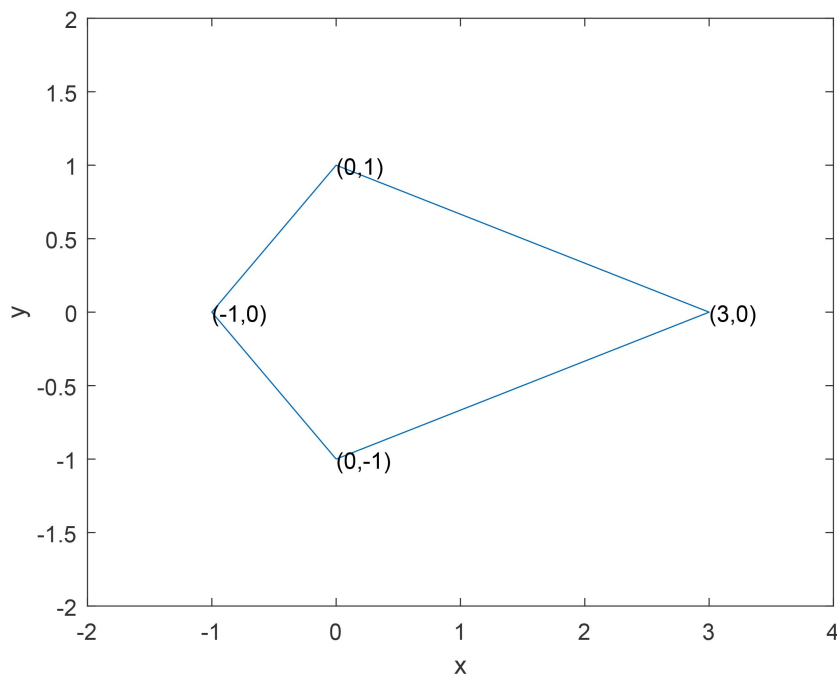


Figure 7. Generally convex quadrilateral area

图 7. 一般凸四边形区域

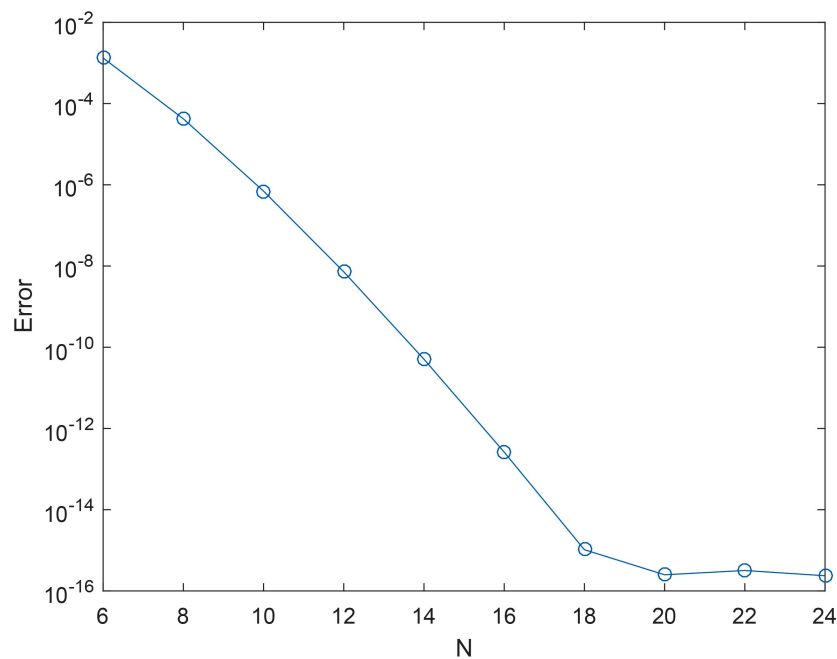


Figure 8. Error curve between approximate and analytic solutions

图 8. 逼近解与解析解之间的误差曲线

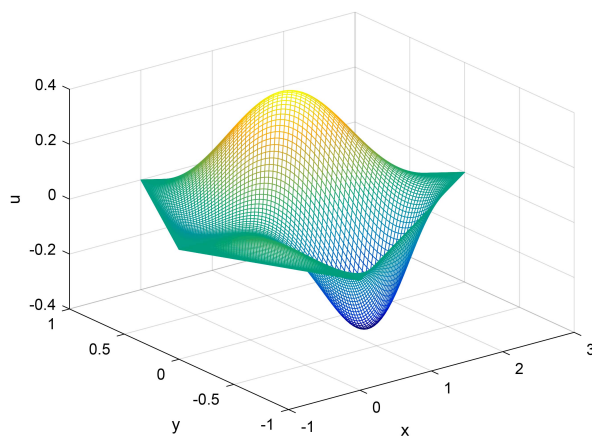


Figure 9. Exact solution of $u(x,y)$ image

图 9. 精确解 $u(x,y)$ 的图像

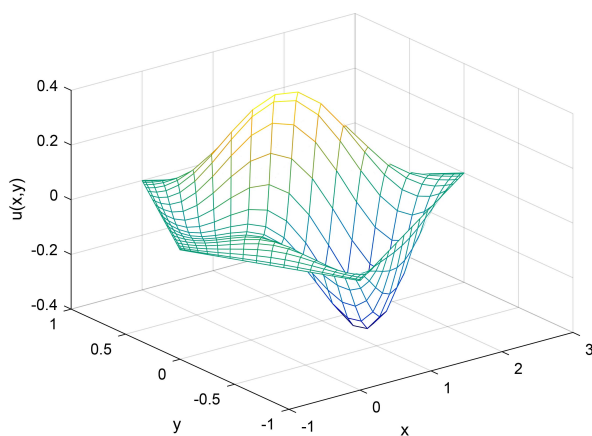


Figure 10. Image of the approximation solution at $N=20$

图 10. $N=20$ 时逼近解的图像

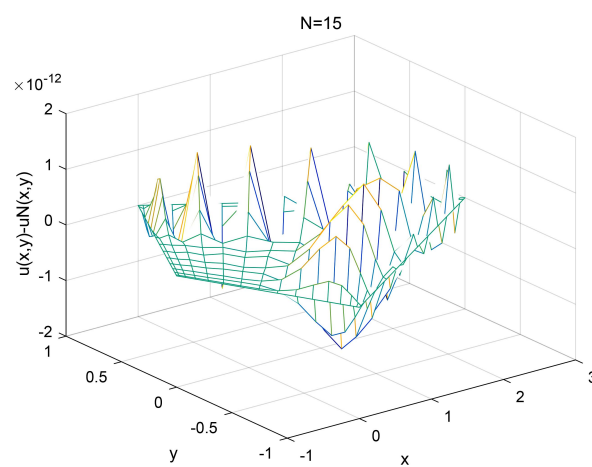


Figure 11. An image of the error between the exact solution and the approximate solution at $N=15$

图 11. 精确解与 $N=15$ 时逼近解之间的误差图像

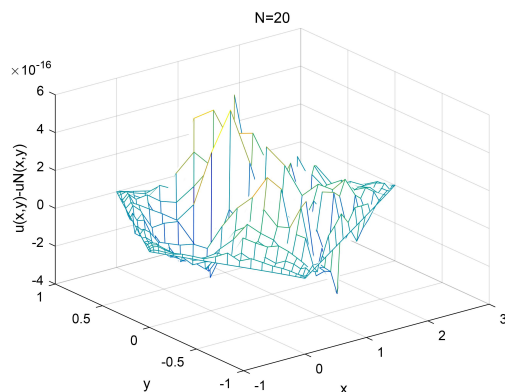


Figure 12. An image of the error between the exact solution and the approximate solution at $N=205$

图 12. 精确解与 $N=20$ 时逼近解之间的误差图像

例3: Ω 为一般的凸四边形的情况, 取 $\alpha(x, y) = xy$, 四个顶点坐标分别为: $A(x_1, y_1) = (0, -1), B(x_2, y_2) = (3, 0), C(x_3, y_3) = (-1, 0), D(x_4, y_4) = (0, 1)$, 如图 13所示, 计算如下问题的数值解:

$$\begin{cases} -\Delta u + \alpha(x, y)u = f, (x, y) \in \Omega \\ u = 0, (x, y) \in \partial\Omega \end{cases}$$

该问题的精确解为: $(x + y + 1)(y - x - 1)(y + \frac{1}{3}x - 1)(y - \frac{1}{3}x + 1)\sin x \sin y$.

对于不同的 N , 我们在图 13中画出了精确解 u 与逼近解 u_N 之间的误差曲线 $e(u, u_N)$ 。从图 14我们可以观察到, 当 $N \geq 16$ 时, 逼近解与精确解之间的误差达到了大约 10^{-12} 的精度。为了更直观地表明我们算法的收敛性和谱精度, 我们分别在图 15和图 16中列出精确解与 $N = 20$ 时逼近图像, 在图 17和图 18中分别列出 $N = 15$ 和 $N = 20$ 时精确解和误差解之间的误差图像。从图 13-图 18我们可以观察到我们的算法是收敛的和谱精度的。

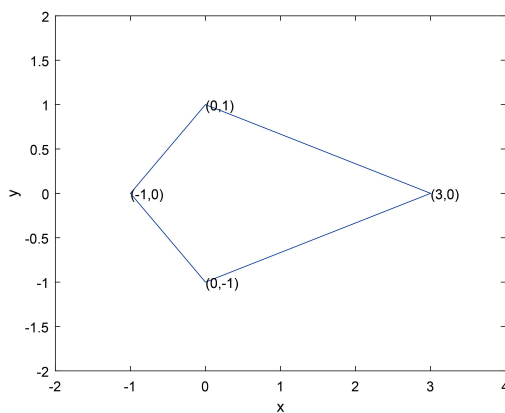


Figure 13. Generally convex quadrilateral area

图 13. 一般凸四边形区域

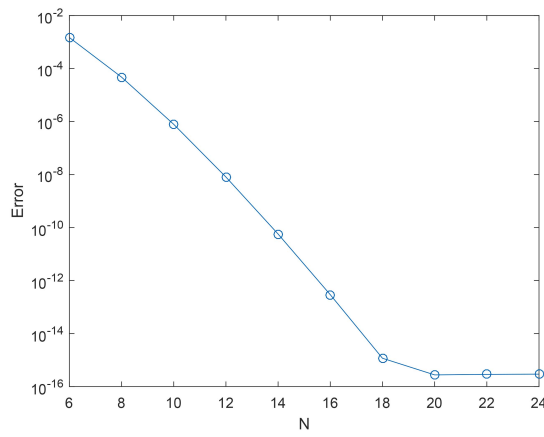


Figure 14. Error curve between approximate and analytic solutions

图 14. 逼近解与解析解之间的误差曲线

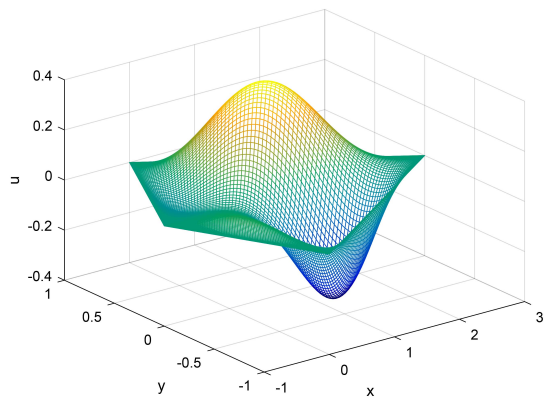


Figure 15. Exact solution of $u(x,y)$ image

图 15. 精确解 $u(x,y)$ 的图像

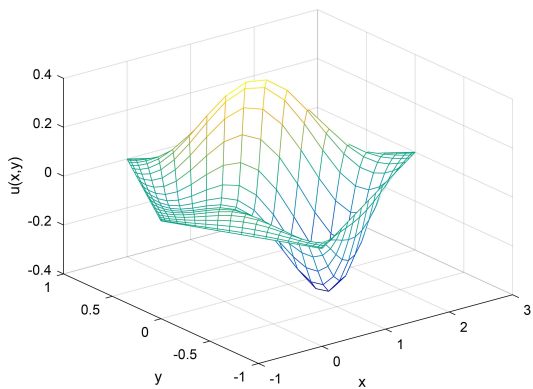


Figure 16. Image of the approximation solution at $N=20$

图 16. $N=20$ 时逼近解的图像

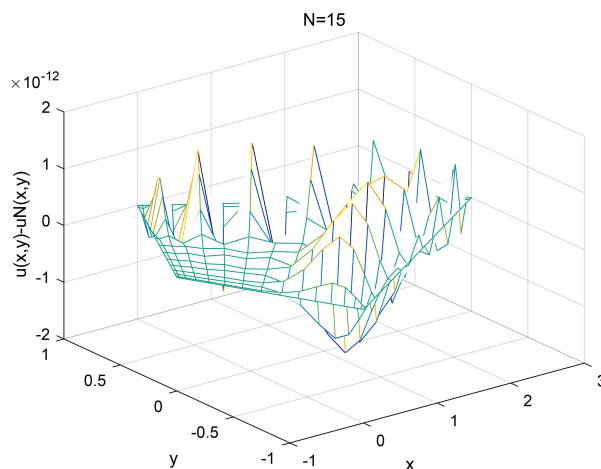


Figure 17. An image of the error between the exact solution and the approximate solution at $N=15$

图 17. 精确解与 $N=15$ 时逼近解之间的误差图像

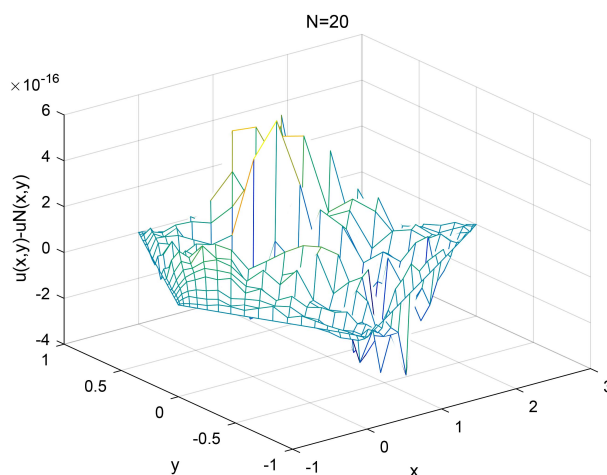


Figure 18. An image of the error between the exact solution and the approximate solution at $N=20$

图 18. 精确解与 $N=20$ 时逼近解之间的误差图像

6. 结论性评议

本文针对二阶变系数椭圆边值问题，提出了在任意凸四边形区域上一种有效谱 $Galerkin$ 逼近的数值方法。通过等参变换和仿射变换将任意凸四边形区域与 $[-1, 1]^2$ 矩形区域联系起来，再利用正交多项式逼近进行编程求解。针对二阶变系数椭圆边值问题，从理论上证明了弱解的存在唯一性。该方法不仅具有谱精度，而且还克服了有限元求解过程中区域剖分的及其基函数构造的复杂性。

另外，本文提出的算法还可以利用谱元法进一步改进和优化并且还可以应用于抛物方程，时间分数阶方程等，这是我们将来的研究目标。

参考文献

- [1] Boffi, D. (2010) Finite Element Approximation of Eigenvalue Problems. *Acta Numerica*, **19**, 1-120. <https://doi.org/10.1017/S0962492910000012>
- [2] Xu, J. and Zhou, A. (2002) Local and Parallel Finite Element Algorithms for Eigenvalue Problems. *Acta Mathematicae Applicatae Sinica*, **18**, 185-200. <https://doi.org/10.1007/s102550200018>
- [3] Hu, J., Huang, Y. and Shen, H. (2004) The Lower Approximation of Eigenvalue by Lumped Mass Finite Element Method. *Computational Mathematics (English Edition)*, **22**, 545-556.
- [4] Liu, F. and Shen, J. (2015) Stabilized Semi-Implicit Spectral Deferred Correction Methods for Allen-Cahn and Cahn-Hilliard Equations. *Mathematical Methods in the Applied Sciences*, **38**, 4564-4575. <https://doi.org/10.1002/mma.2869>
- [5] Colton, D., Päiväranta, L. and Sylvester, J. (2007) The Interior Transmission Problem. *Inverse Problems & Imaging*, **1**, 13-28. <https://doi.org/10.3934/ipi.2007.1.13>
- [6] 陈婷婷. 强异向性椭圆方程的紧致有限差分法研究[D]: [硕士学位论文]. 哈尔滨: 哈尔滨工业大学, 2021. <https://doi.org/10.27061/d.cnki.ghgdu.2021.001684>
- [7] 闵涛, 高青青. 求解二阶椭圆型方程Dirichlet问题的几种算法[J]. 科技通报, 2017, 33(12): 44-49. <https://doi.org/10.13774/j.cnki.kjtb.2017.12.009>
- [8] 林府标, 杨一都. 有限元二网格离散方案 EQ_1^{rot} 元特征值下逼近准确特征值[J]. 贵州师范大学学报(自然科学版), 2008(2): 68-74. <https://doi.org/10.16614/j.cnki.issn1004-5570.2008.02.001>
- [9] Sun, J. (2011) Iterative Methods for Transmission Eigenvalues. *SIAM Journal on Numerical Analysis*, **49**, 1860-1874. <https://doi.org/10.1137/100785478>
- [10] Perugia, I. and Schötzau, D. (2002) An hp -Analysis of the Local Discontinuous Galerkin Method for Diffusion Problems. *Journal of Scientific Computing*, **17**, 561-571. <https://doi.org/10.1023/A:1015118613130>
- [11] Cakoni, F., Colton, D., Monk, P., *et al.* (2010) The Inverse Electromagnetic Scattering Problem for Anisotropic Media. *Inverse Problems*, **26**, Article 074004. <https://doi.org/10.1088/0266-5611/26/7/074004>
- [12] Guo, B.Y. and Jia, L.H. (2010) Spectral Method on Quadrilaterals. *Mathematics of Computation*, **79**, 2237-2264. <https://doi.org/10.1090/S0025-5718-10-02329-X>
- [13] Guo, B.Y. and Wang, L.L. (2007) Error Analysis of Spectral Method on a Triangle. *Advances in Computational Mathematics*, **26**, 473-496. <https://doi.org/10.1007/s10444-005-7471-8>
- [14] Shen, J. and Tang, T. (2006) Spectral and High-Order Methods with Applications. Science Press, Beijing.
- [15] Shen, J., Tang, T. and Wang, L.L. (2011) Spectral Methods: Algorithms, Analysis and Applications. Springer, Science and Business Media. <https://doi.org/10.1007/978-3-540-71041-7>

-
- [16] An, J. and Zhang, Z.M. (2018) An Efficient Spectral-Galerkin Approximation and Error Analysis for Maxwell Transmission Eigenvalue Problems in Spherical Geometries. *Journal of Scientific Computing*, **75**, 157-181. <https://doi.org/10.1007/s10915-017-0528-2>
- [17] 郑继会. 任意凸四边形区域上二阶椭圆特征值问题基于高阶多项式逼近的一种数值方法[J]. *应用数学进展*, 2021, 10(12): 4201-4208. <https://doi.org/10.12677/AAM.2021.1012446>