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General Solution of Stress and Deformation at Crack-Tip for Orthotropic Plate

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Abstract

The basic equation of elastic mechanics at plane stress state is determined by the mechanical characteristic of orthotropic plate. The stress boundary problem about the orthotropic plate with a crack is solved by using the complex function method. To take Mode I crack problem for the typical example, the general solution method is discussed for the stress and deformation near the crack-tip. And the singular stress and strain fields are derived for the orthotropic plate. The crack opening displacement is yet obtained to have a relation with the material characteristics.

Keywords

Orthotropic Plate, Crack-Tip, Complex Function, Stress and Strain, Crack Opening Displacement

正交异性板裂纹端部应力及变形通解

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摘　要

本文根据正交异性板的力学性能，确定平面应力弹性力学的基本方程，利用复变函数方法求解含裂纹正交异性板的应力边值问题。采用I型裂纹作为典型实例，讨论了裂纹端部应力与变形的一般解法，推导出正交异性板的奇异应力和应变场，并得到裂纹张开位移与材料特性的关系。

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关键词

正交异性板，裂纹端部，复变函数，应力和应变，裂纹张开位移

1. 引言

断裂力学经历一个世纪的发展形成了完整的理论体系，并被广泛应用到机械工程、船舶工程、航空航天工程和土木建筑等工程领域。断裂力学涉及的基本理论是关于裂纹尖端奇异应力场及位移场，对于一般各向同性材料已获得完备的解答和通用公式[1] [2] [3]。目前断裂力学依然是固体力学中十分活跃的研究领域，且不断涌现出许多新课题。随着先进复合材料的工程应用日益扩张，各向异性材料断裂力学的理论发展显得更加突出。早在半个世纪前，列赫尼茨基推广复变函数法解决各向异性材料弹性力学问题，其求解方法和理论具有开创性，并为复合材料断裂力学奠定了理论基础[4] [5] [6]。利用复变函数方法在解决复合材料应力边值问题中已取得显著结果，且理论和应用研究不断增加[7] [8]。本文以常见的 I 型裂纹或张开型裂纹板为例(如图 1 所示)，推导出典型正交异性材料的裂纹端部应力场和应变场的通解，并确定出裂纹张开位移与材料特征参数的关系。

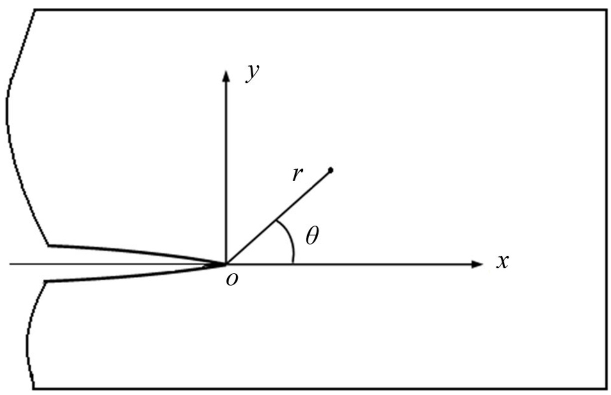

Figure 1. Opening mode crack plate and axes
图 1. 张开型裂纹板及坐标系

2. 复变函数基本理论

在解决弹性力学平面边值问题时，利用复变函数及其坐标变换法得到了一些经典的结果，若采用实函数求解就十分困难。复变函数理论中，常用的基本变量是复数 z，即：$z = x + iy$，$\bar{z} = x - iy$ $(i^2 = -1)$。为了解决复合材料中的应力边值问题，需要扩大复变函数的适应范围。现在引入另一个复变量 w，并将复数 w 用直角坐标和极坐标表示如下：

$$w = x + ihy = r\cos\theta + ihr\sin\theta = r\left(\cos\theta + ih\sin\theta\right) \tag{1}$$

式中系数 h 取为实常数(且令：$h > 0$)。再用字母 β 表示与角度 θ 相关联，且令两个角度之间具有以下的转化关系：

$$\cos\theta = \lambda^2\cos\beta,\quad h\sin\theta = \lambda^2\sin\beta \tag{2}$$

由此可推出以下关系式：

$$\cos^2\theta + h^2\sin^2\theta = \lambda^4\left(\cos^2\beta + \sin^2\beta\right) = \lambda^4,\quad \tan\beta = h\tan\theta$$

$$\lambda^2 = \sqrt{\cos^2\theta + h^2\sin^2\theta},\quad \lambda^4\left(h^2\cos^2\beta + \sin^2\beta\right) = h^2$$

从而确定出 β 和 λ 与 θ 的函数关系为：

$$\beta = \arctan\left(h\tan\theta\right),\quad \lambda = \sqrt[4]{\cos^2\theta + h^2\sin^2\theta} \tag{3}$$

复变量 w 和共轭复变量 $\bar{w}$ 可表示为：

$$\left.\begin{aligned} w &= x + ihy = r\left(\cos\theta + ih\sin\theta\right) = r\lambda^2\left(\cos\beta + i\sin\beta\right) = r\lambda^2 e^{i\beta} \\ \bar{w} &= x - ihy = r\left(\cos\theta - ih\sin\theta\right) = r\lambda^2\left(\cos\beta - i\sin\beta\right) = r\lambda^2 e^{-i\beta} \end{aligned}\right\} \tag{4}$$

由此也可推出以下表达式：

$$\left.\begin{aligned} \sqrt{w} &= \sqrt{r}\lambda e^{i\beta/2} = \sqrt{r}\lambda\left(\cos\frac{\beta}{2} + i\sin\frac{\beta}{2}\right) \\ \frac{1}{\sqrt{w}} &= \frac{1}{\sqrt{r}}\frac{1}{\lambda}e^{-i\beta/2} = \frac{1}{\sqrt{r}}\frac{1}{\lambda}\left(\cos\frac{\beta}{2} - i\sin\frac{\beta}{2}\right) \end{aligned}\right\} \tag{5}$$

对于任意的复变函数 $\Phi(w)$，用实函数 $P(x,y)$ 和 $Q(x,y)$ 表示实部和虚部。即有：

$$\Phi = \Phi(w) = \mathrm{Re}\,\Phi + i\,\mathrm{Im}\,\Phi = P + iQ = P(x,y) + iQ(x,y) \tag{6}$$

可推导出复变函数的偏导数为：

$$\frac{\partial\Phi}{\partial x} = \frac{d\Phi}{dw}\frac{\partial w}{\partial x} = \Phi' = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x},\quad \frac{\partial\Phi}{\partial y} = \frac{d\Phi}{dw}\frac{\partial w}{\partial y} = ih\Phi' = \frac{\partial P}{\partial y} + i\frac{\partial Q}{\partial y}$$

$$\left.\begin{aligned} &\mathrm{Re}\,\Phi' = \frac{\partial P}{\partial x} = \frac{1}{h}\frac{\partial Q}{\partial y},\quad \mathrm{Im}\,\Phi' = \frac{\partial Q}{\partial x} = -\frac{1}{h}\frac{\partial P}{\partial y} \\ &\mathrm{Re}\,\Phi'' = \frac{\partial^2 P}{\partial x^2} = -\frac{1}{h^2}\frac{\partial^2 P}{\partial y^2},\quad \mathrm{Im}\,\Phi'' = \frac{\partial^2 Q}{\partial x^2} = -\frac{1}{h^2}\frac{\partial^2 Q}{\partial y^2} \\ &\mathrm{Re}\,\Phi''' = \frac{\partial^3 P}{\partial x^3} = -\frac{1}{h^3}\frac{\partial^3 Q}{\partial y^3},\quad \mathrm{Im}\,\Phi''' = \frac{\partial^3 Q}{\partial x^3} = \frac{1}{h^3}\frac{\partial^3 P}{\partial y^3} \end{aligned}\right\} \tag{7}$$

这些导数关系将在下面表达式中运用。

3. 平面裂纹问题解法

求解弹性力学平面应力问题的惯用方法就是将应力分量作为基本变量，满足平衡微分方程和应变协调方程，并根据应力边界条件选择合理的应力函数进行求解。为了满足平衡微分方程(忽略体力)，一般利用应力函数 $F(x,y)$ 表示应力如下：

$$\sigma_x = \frac{\partial^2 F}{\partial y^2},\quad \sigma_y = \frac{\partial^2 F}{\partial x^2},\quad \tau_{xy} = -\frac{\partial^2 F}{\partial x\partial y} \tag{8}$$

复合材料层合板的铺层可认为是正交异性材料，其板内的弹性力学分析可按照平面应力状态处理。正交铺设层合板的工艺比较简单且实用，也是复合材料力学分析的典型结构。为了叙述简单，现将参考

坐标轴(x,y)沿着材料主轴方向(1,2)放置，则在平面应力状态下的正交异性材料本构关系为：

$$\varepsilon_x = S_{11}\sigma_x + S_{12}\sigma_y,\quad \varepsilon_y = S_{12}\sigma_x + S_{22}\sigma_y,\quad \gamma_{xy} = S_{66}\tau_{xy} \tag{9}$$

式中，$S_{11} = \frac{1}{E_1}, S_{22} = \frac{1}{E_2}, S_{12} = -\frac{\mu_{12}}{E_1} = -\frac{\mu_{21}}{E_2}, S_{66} = \frac{1}{G_{12}}$。对于一般平面裂纹问题(如图 1 所示)，并考虑到 T 应力，可选择应力函数如下：

$$F = A_1 P + A_2 y \frac{\partial Q}{\partial x} + \frac{T}{2} y^2 \tag{10}$$

偏导数为：

$$\frac{\partial F}{\partial x} = A_1 \frac{\partial P}{\partial x} + A_2 y \frac{\partial^2 Q}{\partial x^2},\quad \frac{\partial F}{\partial y} = A_1 \frac{\partial P}{\partial y} + A_2 \frac{\partial Q}{\partial x} + A_2 y \frac{\partial^2 Q}{\partial x \partial y} + Ty$$

$$\frac{\partial^2 F}{\partial x^2} = A_1 \frac{\partial^2 P}{\partial x^2} + A_2 y \frac{\partial^3 Q}{\partial x^3},\quad \frac{\partial^2 F}{\partial y^2} = A_1 \frac{\partial^2 P}{\partial y^2} + 2A_2 \frac{\partial^2 Q}{\partial x \partial y} + A_2 y \frac{\partial^3 Q}{\partial x \partial y^2} + T$$

$$\frac{\partial^2 F}{\partial x \partial y} = A_1 \frac{\partial^2 P}{\partial x \partial y} + A_2 \frac{\partial^2 Q}{\partial x^2} + A_2 y \frac{\partial^3 Q}{\partial x^2 \partial y}$$

再利以上偏导数和式(7)，可将式(8)的应力分量确定为：

$$\left.\begin{aligned} \sigma_x &= -A_1 h^2 \operatorname{Re}\Phi'' + 2A_2 h \operatorname{Re}\Phi'' - A_2 h^2 y \operatorname{Im}\Phi''' + T \\ \sigma_y &= A_1 \operatorname{Re}\Phi'' + A_2 y \operatorname{Im}\Phi''' \\ \tau_{xy} &= A_1 h \operatorname{Im}\Phi'' - A_2 \operatorname{Im}\Phi'' - A_2 h y \operatorname{Re}\Phi''' \end{aligned}\right\} \tag{11}$$

对于 I 型裂纹，当 $y = 0$ 时，$\tau_{xy} = 0$。因此可设 $A_2 = A_1 h$，将上式简化为：

$$\left.\begin{aligned} \sigma_x &= A_1 h^2 \operatorname{Re}\Phi'' - A_1 h^3 y \operatorname{Im}\Phi''' + T \\ \sigma_y &= A_1 \operatorname{Re}\Phi'' + A_1 h y \operatorname{Im}\Phi''' \\ \tau_{xy} &= -A_1 h^2 y \operatorname{Re}\Phi''' \end{aligned}\right\} \tag{12}$$

代入式(9)后可得应变分量表达式为：

$$\left.\begin{aligned} \varepsilon_x &= A_1\left(S_{11}h^2 + S_{12}\right)\operatorname{Re}\Phi'' - A_1\left(S_{11}h^2 - S_{12}\right) h y \operatorname{Im}\Phi''' + S_{11}T \\ \varepsilon_y &= A_1\left(S_{12}h^2 + S_{22}\right)\operatorname{Re}\Phi'' - A_1\left(S_{12}h^2 - S_{22}\right) h y \operatorname{Im}\Phi''' + S_{12}T \\ \gamma_{xy} &= -A_1 S_{66} h^2 y \operatorname{Re}\Phi''' \end{aligned}\right\} \tag{13}$$

一般用 u 和 v 分别表示沿 x 和 y 方向的位移，平面问题的几何方程为：

$$\varepsilon_x = \frac{\partial u}{\partial x},\quad \varepsilon_y = \frac{\partial v}{\partial y},\quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{14}$$

现将位移分量按如下形式表示：

$$\left.\begin{aligned} u &= B_1 \frac{\partial P}{\partial x} + B_2 y \frac{\partial^2 Q}{\partial x^2} + B_3 x \\ v &= C_1 \frac{\partial Q}{\partial x} + C_2 y \frac{\partial^2 P}{\partial x^2} + C_3 y \end{aligned}\right\} \tag{15}$$

把上式代入几何方程后，再利用式(7)的微分关系，可将应变分量转化为：

$$\left.\begin{aligned}\varepsilon_x &= B_1\operatorname{Re}\Phi'' + B_2 y\operatorname{Im}\Phi''' + B_3\\ \varepsilon_y &= \left(C_1 h + C_2\right)\operatorname{Re}\Phi'' - C_2 h y\operatorname{Im}\Phi''' + C_3\\ \gamma_{xy} &= \left(C_1 - B_1 h + B_2\right)\operatorname{Im}\Phi'' + \left(C_2 + B_2 h\right) y\operatorname{Re}\Phi'''\end{aligned}\right\} \tag{16}$$

由于式(13)和式(16)确定的应变必须相等，比较后可得到待定常数如下：

$$B_1 = A_1\left(S_{11}h^2 + S_{12}\right),\ B_2 = -A_1\left(S_{11}h^2 - S_{12}\right)h,\ B_3 = S_{11}T$$

$$C_1 = 2A_1 S_{11} h^3,\ C_2 = A_1\left(S_{12}h^2 - S_{22}\right),\ C_3 = S_{12}T$$

$$h^2 = \frac{2S_{12} + S_{66}}{2S_{11}} = \sqrt{\frac{S_{22}}{S_{11}}} \tag{17}$$

把常数代入式(15)，再利用式(7)，可得位移的表达式为：

$$\left.\begin{aligned}u &= A_1\left(S_{11}h^2 + S_{12}\right)\operatorname{Re}\Phi' - A_1\left(S_{11}h^2 - S_{12}\right) h y\operatorname{Im}\Phi'' + S_{11}Tx\\ v &= 2A_1 S_{11} h^3\operatorname{Im}\Phi' + A_1\left(S_{12}h^2 - S_{22}\right) y\operatorname{Re}\Phi'' + S_{12}Ty\end{aligned}\right\} \tag{18}$$

4. 应力场与位移场

正交异性板 I 型裂纹问题可借鉴常规材料 I 型断裂力学理论求解。对于图 1 所示的一般裂纹板，主要考虑受到拉伸载荷时裂纹端部的奇异应力场与位移场。因此需要寻求合理的应力函数满足边界条件，首先讨论以下的复变函数及其导数：

$$\Phi = \frac{4}{3}\sqrt{w^3} = \frac{4}{3}\sqrt{\left(r\lambda^2 e^{i\beta}\right)^3} = \frac{4}{3}\sqrt{r^3}\lambda^3\exp\left(i\frac{3\beta}{2}\right) = \frac{4}{3}\sqrt{r^3}\lambda^3\left(\cos\frac{3\beta}{2} + i\sin\frac{3\beta}{2}\right)$$

$$\Phi' = 2\sqrt{w} = 2\sqrt{r\lambda^2 e^{i\beta}} = 2\sqrt{r}\lambda\exp\frac{i\beta}{2} = 2\sqrt{r}\lambda\left(\cos\frac{\beta}{2} + i\sin\frac{\beta}{2}\right)$$

$$\Phi'' = \frac{1}{\sqrt{w}} = \frac{1}{\sqrt{r}}\frac{1}{\lambda}\exp\left(-\frac{i\beta}{2}\right) = \frac{1}{\sqrt{r}}\frac{1}{\lambda}\left(\cos\frac{\beta}{2} - i\sin\frac{\beta}{2}\right)$$

$$\Phi''' = -\frac{1}{2\sqrt{w^3}} = -\frac{1}{2\sqrt{r^3}}\frac{1}{\lambda^3}\exp\left(-i\frac{3\beta}{2}\right) = -\frac{1}{2\sqrt{r}}\frac{1}{r\lambda^3}\left(\cos\frac{3\beta}{2} - i\sin\frac{3\beta}{2}\right)$$

把复变函数的实部和虚部代入式(12)，注意到 $hy = r\lambda^2\sin\beta$，可将应力表达为：

$$\left.\begin{aligned}\sigma_x &= \frac{A_1}{\sqrt{r}}\frac{h^2}{\lambda}\cos\frac{\beta}{2} - \frac{A_1}{2\sqrt{r}}\frac{h^2}{\lambda}\sin\beta\sin\frac{3\beta}{2} + T\\ \sigma_y &= \frac{A_1}{\sqrt{r}}\frac{1}{\lambda}\cos\frac{\beta}{2} + \frac{A_1}{2\sqrt{r}}\frac{1}{\lambda}\sin\beta\sin\frac{3\beta}{2}\\ \tau_{xy} &= \frac{A_1}{2\sqrt{r}}\frac{h}{\lambda}\sin\beta\cos\frac{3\beta}{2}\end{aligned}\right\} \tag{19}$$

当 $\theta = \pm\pi$ 时，$\beta = \pm\pi$，显然有 $\sigma_y = \tau_{xy} = 0$，即满足裂纹面自由条件。在裂纹前沿区，$\theta = 0$，则有 $\beta = 0$，$\lambda = 1$，再利用应力强度因子 K_I 的定义可得：

$$K_I = \lim_{r\to 0}\sqrt{2\pi r}\,\sigma_y\Big|_{\theta=0} = \sqrt{2\pi}A_1,\quad A_1 = \frac{K_I}{\sqrt{2\pi}}$$

由此可确定出裂纹端部的奇异应力场如下：

$$\left.\begin{aligned}\sigma_x &= \frac{K_I}{\sqrt{2\pi r}}\frac{h^2}{\lambda}\cos\frac{\beta}{2}\left(1-\sin\frac{\beta}{2}\sin\frac{3\beta}{2}\right)+T\\ \sigma_y &= \frac{K_I}{\sqrt{2\pi r}}\frac{1}{\lambda}\cos\frac{\beta}{2}\left(1+\sin\frac{\beta}{2}\sin\frac{3\beta}{2}\right)\\ \tau_{xy} &= \frac{K_I}{\sqrt{2\pi r}}\frac{h}{\lambda}\sin\frac{\beta}{2}\cos\frac{\beta}{2}\cos\frac{3\beta}{2}\end{aligned}\right\} \tag{20}$$

当 $h=1$ 时，$\lambda=1$，则此式就成为各向同性材料的裂纹端部应力场。把应力分量代入本构关系式(9)，可求得裂纹端部的奇异应变场为：

$$\left.\begin{aligned}\varepsilon_x &= \frac{K_I}{\sqrt{2\pi r}}\frac{S_{11}}{\lambda}\cos\frac{\beta}{2}\left[h^2+\frac{S_{12}}{S_{22}}-\left(h^2-\frac{S_{12}}{S_{22}}\right)\sin\frac{\beta}{2}\sin\frac{3\beta}{2}\right]+S_{11}T\\ \varepsilon_y &= \frac{K_I}{\sqrt{2\pi r}}\frac{S_{22}}{\lambda}\cos\frac{\beta}{2}\left[1+\frac{h^2S_{12}}{S_{22}}+\left(1-\frac{h^2S_{12}}{S_{22}}\right)\sin\frac{\beta}{2}\sin\frac{3\beta}{2}\right]+S_{12}T\\ \gamma_{xy} &= \frac{K_I}{\sqrt{2\pi r}}\frac{hS_{66}}{\lambda}\sin\frac{\beta}{2}\cos\frac{\beta}{2}\cos\frac{3\beta}{2}\end{aligned}\right\} \tag{21}$$

下面再确定正交异性材料裂纹端部位移场。根据上面确定的复变函数及其常数，可将式(18)的位移分量转化为：

$$\left.\begin{aligned}u &= K_I\sqrt{\frac{2r}{\pi}}\lambda\cos\frac{\beta}{2}\left[\left(S_{11}h^2+S_{12}\right)+\left(S_{11}h^2-S_{12}\right)\sin^2\frac{\beta}{2}\right]+S_{11}Tx\\ v &= K_I\sqrt{\frac{2r}{\pi}}\frac{\lambda}{h}\sin\frac{\beta}{2}\left[2S_{11}h^4+\left(S_{12}h^2-S_{22}\right)\cos^2\frac{\beta}{2}\right]+S_{12}Ty\end{aligned}\right\} \tag{22}$$

再利用前面给出的材料弹性关系及式(17)可知：

$$S_{11}=\frac{1}{E_1},\ \frac{S_{12}}{S_{11}}=-\mu_{12},\ h^4=\frac{S_{22}}{S_{11}}$$

可将位移表达式(22)转化为：

$$\left.\begin{aligned}u &= \frac{K_I}{E_1}\sqrt{\frac{2r}{\pi}}\lambda\cos\frac{\beta}{2}\left[2h^2-\left(h^2+\mu_{12}\right)\cos^2\frac{\beta}{2}\right]+\frac{T}{E_1}x\\ v &= \frac{hK_I}{E_1}\sqrt{\frac{2r}{\pi}}\lambda\sin\frac{\beta}{2}\left[2h^2-\left(h^2+\mu_{12}\right)\cos^2\frac{\beta}{2}\right]-\frac{\mu_{12}T}{E_1}y\end{aligned}\right\} \tag{23}$$

当 $\theta=\pm\pi$ 时($y=0$)， $\beta=\pm\pi$ ， $\lambda=1$，由此确定出裂纹上下面的相对位移 Δ 。即可求得一般裂纹尖端区的裂纹张开位移为：

$$\Delta=v^+\big|_{\theta=\pi}-v^-\big|_{\theta=-\pi}=\frac{8h^3K_I}{E_1}\sqrt{\frac{r}{2\pi}} \tag{24}$$

显然，裂纹张开位移 Δ 与正交异性材料参数 h 的三次方相关。当 $h=1$ 时，则上式就表示各向同性材料裂纹端部的张开位移。

5. 结论

随着先进复合材料的工程应用日益增大，各向异性材料断裂力学的理论发展显得更加突出。本文根

据正交异性板的力学性能确定平面应力弹性力学的基本方程，利用复变函数方法求解含裂纹正交异性板的应力边值问题。推导出正交异性板的奇异应力场和应变场的通解，并确定了裂纹张开位移与材料特征参数的关系。本文结果也适合一般各向同性材料，具有明确的理论意义和工程应用的潜力。

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