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Initial Boundary Value Problem for a Class of Nonlinear Evolution Systems*

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Abstract: In this paper, we study the initial boundary value problem for a class of fourth order nonlinear wave equations the existence and uniqueness of global strong solution for the problem are obtained by means of the Galerkin method.

Keywords: Nonlinear Evolution Equations; Initial Value Problem; Global Strong Solution

一类非线性发展方程组的初边值问题*

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摘　要：本文研究了一类四阶非线性波动方程组的初边值问题，用 Glerkin 方法证明了其整体强解的存在性和唯一性。

关键词：非线性波动方程组；初边值问题；整体强解

1. 引言

有三种因素影响弹性杆内波的传播：非线性，色散及耗散。非线性使波前变陡甚至破裂；色散与耗散可减小波前斜率，制止波发生破裂，产生最终的稳态。

文[1]研究了在上述三种因素的影响下的细长弹性杆中纵向应变波的传播问题。提出并讨论了如下一类四阶拟线性波动方程

$$u_{tt} - u_{xx} - u_{xxt} - u_{xxtt} = a\left(u_x^n\right)_x \tag{1}$$

其中 $a \neq 0$ 为任意实数，a 和 n 均为材料常数，$a < 0$ 表示杆由软非线性材料(例如多数金属)构成；$a > 0$ 表示杆由硬非线性材料(例如橡胶、聚合物和少数金属)构成。文[1]在近似情况下，将(1)化为广义的KdV-Burgers方程，讨论了在单纯色散、单纯耗散效应下，软硬两种非线性材料的应变孤波，但对(1)没有进行任何讨论。文[2]研究了一类四阶非线性发展方程

$$u_{tt} - u_{xx} - u_{xxt} - u_{xxtt} = f\left(u\right)$$

的第一初边值问题

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$$u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\ 0\le x\le 1,\quad u(x,t)\big|_{x=0}=u(x,t)\big|_{x=1}=0.$$

此方程描述了单个粘弹性杆的纵振动问题，这里外力密度为依赖于位移的情形。但还未见有人研究方程组的情况。

本文考虑如下一类非线性发展方程组

$$\boldsymbol{u}_{tt}-\boldsymbol{u}_{xx}-\boldsymbol{u}_{xxt}-\boldsymbol{u}_{xxtt}=\boldsymbol{f}(\boldsymbol{u}_t)\ \ (0\le x\le 1, 0\le t\le T) \tag{1.1}$$

的第一初边值问题

$$\boldsymbol{u}(x,0)=\boldsymbol{u}_0(x),\ \boldsymbol{u}_t(x,0)=\boldsymbol{u}_1(x),\ 0\le x\le 1, \tag{1.2}$$

$$\boldsymbol{u}(x,t)\big|_{x=0}=\boldsymbol{u}(x,t)\big|_{x=1}=\boldsymbol{0}. \tag{1.3}$$

其中 **0** 表示零矢量，

$$\boldsymbol{u}=\boldsymbol{u}(x,t)=(u_1,u_2,\cdots,u_N)^{\mathrm{T}},\quad \boldsymbol{f}(\boldsymbol{u}_t)=\left(f_1(\boldsymbol{u}_t),f_2(\boldsymbol{u}_t),\cdots,f_N(\boldsymbol{u}_t)\right)^{\mathrm{T}},$$

$$\boldsymbol{u}_0(x)=\left(u_{01}(x),u_{02}(x),\cdots,u_{0N}(x)\right)^{\mathrm{T}},\quad \boldsymbol{u}_1(x)=\left(u_{11}(x),u_{12}(x),\cdots,u_{1N}(x)\right)^{\mathrm{T}}$$

(这里“T”表示转置)。

本文所考虑的方程组(1.1)是多条粘弹性杆耦合在一起的振动情形，此方程组可看成线性粘弹性杆受依赖于速度的外力 $\boldsymbol{f}$ 作用下纵向形变波的传播模型方程。无论从理论上还是从实际上比文[2]所讨论的单个方程更复杂。

首先对 Galerkin 方法加以介绍：

1) 在一个可分的函数空间中取一组基，本文是取负 Laplace 算子的特征函数作为一组基，这样作先验估计更方便。

2) 构造原问题近似解并建立 Galerkin 逼近格式，该逼近格式一般是关于近似解的非线性常微分方程组。

3) 对近似解及其关于空间变量，关于时间变量的导数作出相应的先验估计。

4) 在近似解相应先验估计的基础上，由列紧性原理取弱极限可得原问题的整体强解。

假设 $\boldsymbol{f}$ 满足：$\boldsymbol{f}(\boldsymbol{0})=\boldsymbol{0}$，$\boldsymbol{f}\in C^1$ 及 Jacobi 矩阵 $\dfrac{\partial\boldsymbol{f}}{\partial\boldsymbol{u}}$ 半有界，即 $\exists c_0>0,\forall\boldsymbol{\xi}\in R^n$ 满足

$$\left(\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{u}}\boldsymbol{\xi},\boldsymbol{\xi}\right)\le c_0(\boldsymbol{\xi},\boldsymbol{\xi}). \tag{1.4}$$

记 $\Omega=(0,1),(\boldsymbol{u},\boldsymbol{v})=\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\mathrm{d}x,|\boldsymbol{u}|^2_{L_2(\Omega)}=(\boldsymbol{u},\boldsymbol{u}),$

$$[\boldsymbol{u},\boldsymbol{v}]=\int_0^t(\boldsymbol{u},\boldsymbol{v})\mathrm{d}t\quad(0\le t\le T),\ \|\boldsymbol{u}\|^2_{L_2(\Omega)}=[\boldsymbol{u},\boldsymbol{u}],\ |\boldsymbol{u}|=\left(\sum_{i=1}^N u_i^2\right)^{\frac{1}{2}},$$

本文用 Galerkin 方法研究问题(1.1)~(1.3)整体强解的存在性与唯一性。

2. 先验估计

设 $\{\omega_j\,|\,j=1,2,\cdots\}$ 为问题 $-\omega_j''(x)=\lambda_j\omega_j(x)$，$\omega_j(0)=\omega_j(1)=0$ 的特征函数，则 $\{\omega_j(x)\}$ 在 $L^2(\Omega)$ 中构成正交完备系，$\omega_j(x)\in C^\infty(\bar{\Omega})\cap H_0^1(\Omega)$ 且 $\omega_j(x)$ 的线性组合在 $H_0^1(\Omega)$ 中稠，$\{\omega_j(x)\}$ 在 $H^2(\Omega)$ 中的闭线性扩张为 $H^2(\Omega)\cap H_0^1(\Omega)$ (见[3])，对初值作假设

$$\boldsymbol{u}_0(x)\in H^2(\Omega)\cap H_0^1(\Omega),\ \boldsymbol{u}_1(x)\in H^2(\Omega)\cap H_0^1(\Omega).$$

设问题(1.1)~(1.3)的近似解为

$$u_{mi}=u_{mi}\left(x,t\right)=\sum_{j=1}^{m}a_{mij}\left(t\right)\omega_j\left(x\right),$$
$$u_{mi}\left(x,0\right)=\sum_{j=1}^{m}a_{mij}\omega_j,u_{mit}\left(x,0\right)=\sum_{j=1}^{m}b_{mij}\omega_j$$
$$\left(m=1,2,\cdots,\quad i=1,2,\cdots,N\right)$$

由 Galerkin 方法，该近似解应满足如下非线性常微分方程组的初值问题

$$\left(u_{mitt},\omega_s\right)-\left(u_{mixx},\omega_s\right)-\left(u_{mixxt},\omega_s\right)-\left(u_{mixxtt},\omega_s\right)=\left(f_i\left(\boldsymbol{u}_{mt}\right),\omega_s\right),\tag{2.1}$$

$$\left(u_{mi}\left(x,0\right),\omega_s\right)=\left(u_{0i}\left(x\right),\omega_s\right)\quad\left(i=1,2,\cdots,N;s=1,2\cdots,m\right),\tag{2.2}$$

$$\left(u_{mit}\left(x,0\right),\omega_s\right)=\left(u_{1i}\left(x\right),\omega_s\right)\quad\left(i=1,2,\cdots,N;s=1,2\cdots,m\right).\tag{2.3}$$

由 $\boldsymbol{u}_0\left(x\right)\in H^2\left(\Omega\right)\cap H_0^1\left(\Omega\right)$ 及 $\boldsymbol{u}_1\left(x\right)\in H^2\left(\Omega\right)\cap H_0^1\left(\Omega\right)$, 初值 a_{mij},b_{mij} 应这样选取，使在 $\boldsymbol{u}_0\left(x\right)$, $\boldsymbol{u}_1\left(x\right)$ 各自所在的空间里分别有

$$\boldsymbol{u}_m\left(x,0\right)\to\boldsymbol{u}_0\left(x\right),\boldsymbol{u}_{mt}\left(x,0\right)\to\boldsymbol{u}_1\left(x\right)\ \left(m\to\infty\right).$$

引理 1：设 $\boldsymbol{u}_0\left(x\right)\in H_0^1\left(\Omega\right)$, $\boldsymbol{u}_1\left(x\right)\in H_0^1\left(\Omega\right)$，条件(1.4)成立并选取初值 a_{mij},b_{mij} 使得当 $m\to+\infty$ 时，$\boldsymbol{u}_m\left(x,0\right)$ 在 $H_0^1\left(\Omega\right)$ 中强收敛于 $\boldsymbol{u}_0\left(x\right),\boldsymbol{u}_{mt}\left(x,0\right)$ 在 $H_0^1\left(\Omega\right)$ 中强收敛于 $\boldsymbol{u}_1\left(x\right)$。则对任一 $T>0$ 及(2.1)~(2.3)的任意解 $\boldsymbol{u}_m\left(x,t\right)$ 均有估计：

$$\left|\boldsymbol{u}_m\right|_{L_2(\Omega)}^2+\left|\boldsymbol{u}_{mx}\right|_{L_2(\Omega)}^2+\left|\boldsymbol{u}_{mt}\right|_{L_2(\Omega)}^2+\left|\boldsymbol{u}_{mxt}\right|_{L_2(\Omega)}^2\le E_1,\quad\left\|\boldsymbol{u}_{mxt}\right\|_{L_2(\Omega)}^2\le E_2\quad\left(0\le t\le T\right).$$

式中 E_1,E_2 及以下诸引理中的 $E_i\left(i=1,2,\cdots,5\right)$ 均为与 m 无关的正常数。

证明：方程(2.1)两边同乘 $a'_{mis}\left(t\right)$, 得

$$\left(u_{mitt},a'_{mis}\left(t\right)\omega_s\right)-\left(u_{mixx},a'_{mis}\left(t\right)\omega_s\right)-\left(u_{mixxt},a'_{mis}\left(t\right)\omega_s\right)-\left(u_{mixxtt},a'_{mis}\left(t\right)\omega_s\right)=\left(f_i\left(\boldsymbol{u}_{mt}\right),a'_{mis}\left(t\right)\omega_s\right),$$

对 s 从 1 到 m 求和得

$$\left(u_{mitt},\sum_{s=1}^{m}a'_{mis}\left(t\right)\omega_s\right)-\left(u_{mixx},\sum_{s=1}^{m}a'_{mis}\left(t\right)\omega_s\right)-\left(u_{mixxt},\sum_{s=1}^{m}a'_{mis}\left(t\right)\omega_s\right)-\left(u_{mixxtt},\sum_{s=1}^{m}a'_{mis}\left(t\right)\omega_s\right)=\left(f_i\left(\boldsymbol{u}_{mt}\right),\sum_{s=1}^{m}a'_{mis}\left(t\right)\omega_s\right),$$

即

$$\left(u_{mitt,}u_{mit}\right)-\left(u_{mixx,}u_{mit}\right)-\left(u_{mixxt,}u_{mit}\right)-\left(u_{mixxtt,}u_{mit}\right)=\left(f_i\left(\boldsymbol{u}_{mt}\right),u_{mit}\right),$$

再对 i 从 1 到 N 求和得

$$\left(\boldsymbol{u}_{mtt,}\boldsymbol{u}_{mt}\right)-\left(\boldsymbol{u}_{mxx,}\boldsymbol{u}_{mt}\right)-\left(\boldsymbol{u}_{mxxt,}\boldsymbol{u}_{mt}\right)-\left(\boldsymbol{u}_{mxxtt,}\boldsymbol{u}_{mt}\right)=\left(\boldsymbol{f}(\boldsymbol{u}_{mt}),\boldsymbol{u}_{mt}\right),$$

两边加上 $\left(\boldsymbol{u}_m,\boldsymbol{u}_{mt}\right)$，分部积分可得

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\boldsymbol{u}_m,\boldsymbol{u}_m\right)+\left(\boldsymbol{u}_{mt},\boldsymbol{u}_{mt}\right)+\left(\boldsymbol{u}_{mx},\boldsymbol{u}_{mx}\right)+\left(\boldsymbol{u}_{mxt},\boldsymbol{u}_{mxt}\right)\right)+2\left(\boldsymbol{u}_{mxt},\boldsymbol{u}_{mxt}\right)=2\left(\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right),\boldsymbol{u}_{mt}\right)+2\left(\boldsymbol{u}_m,\boldsymbol{u}_{mt}\right).\tag{2.4}$$

由

$$\left(\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right),\boldsymbol{u}_{mt}\right)=\left(\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right)-\boldsymbol{f}\left(\boldsymbol{0}\right),\boldsymbol{u}_{mt}\right)=\left(\frac{\partial\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right)}{\partial\boldsymbol{u}_{mt}}\Big|_{\theta\boldsymbol{u}_{mt}}\boldsymbol{u}_{mt},\boldsymbol{u}_{mt}\right)\le c_0\left(\boldsymbol{u}_{mt},\boldsymbol{u}_{mt}\right),\quad\left(0<\theta<1\right)\tag{2.5}$$

$$2\left(\boldsymbol{u}_m,\boldsymbol{u}_{mt}\right)\le\left|\boldsymbol{u}_m\right|_{L_2(\Omega)}^2+\left|\boldsymbol{u}_{mt}\right|_{L_2(\Omega)}^2,\tag{2.6}$$

将(2.5)、(2.6)代入(2.4)可得

$$\frac{\mathrm{d}}{\mathrm{d}t}\big((\boldsymbol{u}_m,\boldsymbol{u}_m)+(\boldsymbol{u}_{mt},\boldsymbol{u}_{mt})+(\boldsymbol{u}_{mx},\boldsymbol{u}_{mx})+(\boldsymbol{u}_{mxt},\boldsymbol{u}_{mxt})\big)+2(\boldsymbol{u}_{mxt},\boldsymbol{u}_{mxt})\le M_1|\boldsymbol{u}_m|^2_{L_2(\Omega)}+M_1|\boldsymbol{u}_{mt}|^2_{L_2(\Omega)}\quad(M_1\text{为与}\boldsymbol{u}_m\text{无关正常数}),$$

对满足 $0<t\le T$ 的任意 t，从 0 到 t 积分得

$$(\boldsymbol{u}_m,\boldsymbol{u}_m)+(\boldsymbol{u}_{mt},\boldsymbol{u}_{mt})+(\boldsymbol{u}_{mx},\boldsymbol{u}_{mx})+(\boldsymbol{u}_{mxt},\boldsymbol{u}_{mxt})+2[\boldsymbol{u}_{mxt},\boldsymbol{u}_{mxt}]\le(\boldsymbol{u}_m(x,0),\boldsymbol{u}_m(x,0))+(\boldsymbol{u}_{mt}(x,0),\boldsymbol{u}_{mt}(x,0))$$
$$+(\boldsymbol{u}_{mx}(x,0),\boldsymbol{u}_{mx}(x,0))+(\boldsymbol{u}_{mxt}(x,0),\boldsymbol{u}_{mxt}(x,0))+M_1\int_0^t\left(|\boldsymbol{u}_m|^2_{L_2(\Omega)}+|\boldsymbol{u}_{mt}|^2_{L_2(\Omega)}\right)\mathrm{d}t,$$

由已知条件得当 $m\to+\infty$ 时，

$$(\boldsymbol{u}_m(x,0),\boldsymbol{u}_m(x,0))+(\boldsymbol{u}_{mt}(x,0),\boldsymbol{u}_{mt}(x,0))+(\boldsymbol{u}_{mx}(x,0),\boldsymbol{u}_{mx}(x,0))+(\boldsymbol{u}_{mxt}(x,0),\boldsymbol{u}_{mxt}(x,0))$$
$$\to(\boldsymbol{u}_0(x),\boldsymbol{u}_0(x))+(\boldsymbol{u}_1(x),\boldsymbol{u}_1(x))+(\boldsymbol{u}_{0x}(x),\boldsymbol{u}_{0x}(x))+(\boldsymbol{u}_{1x}(x),\boldsymbol{u}_{1x}(x)),$$

$$\therefore(\boldsymbol{u}_m,\boldsymbol{u}_m)+(\boldsymbol{u}_{mt},\boldsymbol{u}_{mt})+(\boldsymbol{u}_{mx},\boldsymbol{u}_{mx})+(\boldsymbol{u}_{mxt},\boldsymbol{u}_{mxt})+2[\boldsymbol{u}_{mxt},\boldsymbol{u}_{mxt}]\le M_2+M_1\int_0^t\left(|\boldsymbol{u}_m|^2_{L_2(\Omega)}+|\boldsymbol{u}_{mt}|^2_{L_2(\Omega)}\right)\mathrm{d}t$$
$$(M_2\text{为与}\boldsymbol{u}_m\text{无关正常数}),$$

由 Gronwall 不等式即得

$$|\boldsymbol{u}_m|^2_{L_2(\Omega)}+|\boldsymbol{u}_{mx}|^2_{L_2(\Omega)}+|\boldsymbol{u}_{mt}|^2_{L_2(\Omega)}+|\boldsymbol{u}_{mxt}|^2_{L_2(\Omega)}\le E_1,\quad\|\boldsymbol{u}_{mxt}\|^2_{L_2(\Omega)}\le E_2$$

引理证毕!

由 Sobolev 嵌入定理得

推论：$|\boldsymbol{u}_m|_\infty\le\text{const}\quad(0\le t\le T)$，$|\boldsymbol{u}_{mt}|_\infty\le\text{const}\quad(0\le t\le T)$,

引理 2：设引理 1 的条件成立，并且 $\boldsymbol{u}_0(x)\in H^2(\Omega)\cap H_0^1(\Omega)$， $\boldsymbol{u}_1(x)\in H^2(\Omega)\cap H_0^1(\Omega)$，选取 a_{mij},b_{mij} 使得当 $m\to+\infty$ 时， $\boldsymbol{u}_m(x,0)$ 在 $H^2(\Omega)\cap H_0^1(\Omega)$ 中强收敛于 $\boldsymbol{u}_0(x)$， $\boldsymbol{u}_{mt}(x,0)$ 在 $H^2(\Omega)\cap H_0^1(\Omega)$ 中强收敛于 $\boldsymbol{u}_1(x)$，则对问题(2.1)~(2.3)的解 $\boldsymbol{u}_m(x,t)$ 均有估计 $|\boldsymbol{u}_{mxx}|^2_{L_2(\Omega)}+|\boldsymbol{u}_{mxxt}|^2_{L_2(\Omega)}\le E_3$。

证明：方程(2.1)两边同乘 $\lambda_s a'_{mis}(t)$,

$$(u_{mitt},\lambda_s a'_{mis}(t)\omega_s)-(u_{mixx},\lambda_s a'_{mis}(t)\omega_s)-(u_{mixxt},\lambda_s a'_{mis}(t)\omega_s)-(u_{mixxtt},\lambda_s a'_{mis}(t)\omega_s)=(f_i(\boldsymbol{u}_{mt}),\lambda_s a'_{mis}(t)\omega_s),$$

对 s 从 1 到 m 求和得

$$\left(u_{mitt},\sum_{s=1}^m\lambda_s a'_{mis}(t)\omega_s\right)-\left(u_{mixx},\sum_{s=1}^m\lambda_s a'_{mis}(t)\omega_s\right)-\left(u_{mixxt},\sum_{s=1}^m\lambda_s a'_{mis}(t)\omega_s\right)-\left(u_{mixxtt},\sum_{s=1}^m\lambda_s a'_{mis}(t)\omega_s\right)$$
$$=\left(f_i(\boldsymbol{u}_{mt}),\sum_{s=1}^m\lambda_s a'_{mis}(t)\omega_s\right),$$

$$\left(u_{mitt},-\sum_{s=1}^m a'_{mis}(t)\omega''_s\right)-\left(u_{mixx},-\sum_{s=1}^m a'_{mis}(t)\omega''_s\right)-\left(u_{mixxt},-\sum_{s=1}^m a'_{mis}(t)\omega''_s\right)-\left(u_{mixxtt},-\sum_{s=1}^m a'_{mis}(t)\omega''_s\right)$$
$$=\left(f_i(\boldsymbol{u}_{mt}),-\sum_{s=1}^m a'_{mis}(t)\omega''_s\right),$$

即

$$(u_{mitt},-u_{mixxt})-(u_{mixx},-u_{mixxt})-(u_{mixxt},-u_{mixxt})-(u_{mixxtt},-u_{mixxt})=(f_i(\boldsymbol{u}_{mt}),-u_{mixxt}),$$

对 i 从 1 到 N 求和得

$$(\boldsymbol{u}_{mtt},-\boldsymbol{u}_{mxxt})-(\boldsymbol{u}_{mxx},-\boldsymbol{u}_{mxxt})-(\boldsymbol{u}_{mxxt},-\boldsymbol{u}_{mxxt})-(\boldsymbol{u}_{mxxtt},-\boldsymbol{u}_{mxxt})=(\boldsymbol{f}(\boldsymbol{u}_{mt}),-\boldsymbol{u}_{mxxt}),$$

分部积分得

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\boldsymbol{u}_{mxt},\boldsymbol{u}_{mxt}\right)+\left(\boldsymbol{u}_{mxx},\boldsymbol{u}_{mxx}\right)+\left(\boldsymbol{u}_{mxxt},\boldsymbol{u}_{mxxt}\right)\right)+2\left(\boldsymbol{u}_{mxxt},\boldsymbol{u}_{mxxt}\right)=2\left(\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right),-\boldsymbol{u}_{mxxt}\right),$$

$$\because\left(\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right),-\boldsymbol{u}_{mxxt}\right)=\left(\frac{\partial\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right)}{\partial\boldsymbol{u}_{mt}}\boldsymbol{u}_{mxt},\boldsymbol{u}_{mxt}\right)\le c_0\left(\boldsymbol{u}_{mxt},\boldsymbol{u}_{mxt}\right),$$

$$\therefore\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\boldsymbol{u}_{mxt},\boldsymbol{u}_{mxt}\right)+\left(\boldsymbol{u}_{mxx},\boldsymbol{u}_{mxx}\right)+\left(\boldsymbol{u}_{mxxt},\boldsymbol{u}_{mxxt}\right)\right)+2\left(\boldsymbol{u}_{mxxt},\boldsymbol{u}_{mxxt}\right)\le c_0\left(\boldsymbol{u}_{mxt},\boldsymbol{u}_{mxt}\right).$$

关于 t 从 0 到 $t\left(0<t\le T\right)$ 积分得

$$\begin{aligned}&\left(\boldsymbol{u}_{mxt},\boldsymbol{u}_{mxt}\right)+\left(\boldsymbol{u}_{mxx},\boldsymbol{u}_{mxx}\right)+\left(\boldsymbol{u}_{mxxt},\boldsymbol{u}_{mxxt}\right)+2\left[\boldsymbol{u}_{mxxt},\boldsymbol{u}_{mxxt}\right]\le\left(\boldsymbol{u}_{mxt}\left(x,0\right),\boldsymbol{u}_{mxt}\left(x,0\right)\right)+\left(\boldsymbol{u}_{mxx}\left(x,0\right),\boldsymbol{u}_{mxx}\left(x,0\right)\right)\\&+\left(\boldsymbol{u}_{mxxt}\left(x,0\right),\boldsymbol{u}_{mxxt}\left(x,0\right)\right)+c_0\int_0^t\left(\boldsymbol{u}_{mxt},\boldsymbol{u}_{mxt}\right)\mathrm{d}t,\end{aligned}$$

由已知条件得当 $m\to+\infty$ 时，

$$\begin{aligned}&\left(\boldsymbol{u}_{mxt}\left(x,0\right),\boldsymbol{u}_{mxt}\left(x,0\right)\right)+\left(\boldsymbol{u}_{mxx}\left(x,0\right),\boldsymbol{u}_{mxx}\left(x,0\right)\right)+\left(\boldsymbol{u}_{mxxt}\left(x,0\right),\boldsymbol{u}_{mxxt}\left(x,0\right)\right)\\&\to\left(\boldsymbol{u}_{1x}\left(x\right),\boldsymbol{u}_{1x}\left(x\right)\right)+\left(\boldsymbol{u}_{0xx}\left(x\right),\boldsymbol{u}_{0xx}\left(x\right)\right)+\left(\boldsymbol{u}_{1xx}\left(x\right),\boldsymbol{u}_{1xx}\left(x\right)\right),\end{aligned}$$

因此

$$\left(\boldsymbol{u}_{mxt}\left(x,0\right),\boldsymbol{u}_{mxt}\left(x,0\right)\right)+\left(\boldsymbol{u}_{mxx}\left(x,0\right),\boldsymbol{u}_{mxx}\left(x,0\right)\right)+\left(\boldsymbol{u}_{mxxt}\left(x,0\right),\boldsymbol{u}_{mxxt}\left(x,0\right)\right)$$

能用一与 m 无关的正常数界住，由 Gronwall 不等式知

$$\left|\boldsymbol{u}_{mxx}\right|_{L_2(\Omega)}^2+\left|\boldsymbol{u}_{mxxt}\right|_{L_2(\Omega)}^2\le E_3.$$

引理证毕！

引理 3：在引理 2 的条件下有

$$\left|\boldsymbol{u}_{mtt}\right|_{L_2(\Omega)}^2+\left|\boldsymbol{u}_{mxtt}\right|_{L_2(\Omega)}^2\le E_4.$$

证明：对方程(2.1)两端对 t 求导得

$$\left(u_{mittt},\omega_s\right)-\left(u_{mixxt},\omega_s\right)-\left(u_{mixxtt},\omega_s\right)-\left(u_{mixttt},\omega_s\right)=\left(\frac{\mathrm{d}}{\mathrm{d}t}f_i\left(\boldsymbol{u}_{mt}\right),\omega_s\right),\tag{2.7}$$

两端同乘 $a''_{mis}\left(t\right)$ 得

$$\left(u_{mittt},a''_{mis}\left(t\right)\omega_s\right)-\left(u_{mixxt},a''_{mis}\left(t\right)\omega_s\right)-\left(u_{mixxtt},a''_{mis}\left(t\right)\omega_s\right)-\left(u_{mixxttt},a''_{mis}\left(t\right)\omega_s\right)=\left(\frac{\mathrm{d}}{\mathrm{d}t}f_i\left(\boldsymbol{u}_{mt}\right),a''_{mis}\left(t\right)\omega_s\right),$$

对 s 从 1 到 m 求和得

$$\left(u_{mittt},\sum_{s=1}^m a''_{mis}\left(t\right)\omega_s\right)-\left(u_{mixxt},\sum_{s=1}^m a''_{mis}\left(t\right)\omega_s\right)-\left(u_{mixxtt},\sum_{s=1}^m a''_{mis}\left(t\right)\omega_s\right)-\left(u_{mixxttt},\sum_{s=1}^m a''_{mis}\left(t\right)\omega_s\right)=\left(\frac{\mathrm{d}}{\mathrm{d}t}f_i\left(\boldsymbol{u}_{mt}\right),\sum_{s=1}^m a''_{mis}\left(t\right)\omega_s\right),$$

即

$$\left(u_{mittt},u_{mitt}\right)-\left(u_{mixxt},u_{mitt}\right)-\left(u_{mixxtt},u_{mitt}\right)-\left(u_{mixxttt},u_{mitt}\right)=\left(\frac{\mathrm{d}}{\mathrm{d}t}f_i\left(\boldsymbol{u}_{mt}\right),u_{mitt}\right),$$

对 i 从 1 到 N 作和得

$$\left(\boldsymbol{u}_{mttt},\boldsymbol{u}_{mtt}\right)-\left(\boldsymbol{u}_{mxxt},\boldsymbol{u}_{mtt}\right)-\left(\boldsymbol{u}_{mxxtt},\boldsymbol{u}_{mtt}\right)-\left(\boldsymbol{u}_{mxxttt},\boldsymbol{u}_{mtt}\right)=\left(\frac{\mathrm{d}}{\mathrm{d}t}f\left(\boldsymbol{u}_{mt}\right),\boldsymbol{u}_{mtt}\right),$$

即

$$\left(\boldsymbol{u}_{mttt},\boldsymbol{u}_{mtt}\right)-\left(\boldsymbol{u}_{mxxt},\boldsymbol{u}_{mtt}\right)-\left(\boldsymbol{u}_{mxxtt},\boldsymbol{u}_{mtt}\right)-\left(\boldsymbol{u}_{mxxttt},\boldsymbol{u}_{mtt}\right)=\left(\frac{\partial f\left(\boldsymbol{u}_{mt}\right)}{\partial \boldsymbol{u}_{mt}}\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right),$$

分部积分有

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left|\boldsymbol{u}_{mtt}\right|_{L_2(\Omega)}^2+\left|\boldsymbol{u}_{mxt}\right|_{L_2(\Omega)}^2+\left|\boldsymbol{u}_{mxtt}\right|_{L_2(\Omega)}^2\right]+\left|\boldsymbol{u}_{mxtt}\right|_{L_2(\Omega)}^2=\left(\frac{\partial \boldsymbol{f}\left(\boldsymbol{u}_{mt}\right)}{\partial \boldsymbol{u}_{mt}}\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right),$$

即

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left|\boldsymbol{u}_{mtt}\right|_{L_2(\Omega)}^2+\left|\boldsymbol{u}_{mxt}\right|_{L_2(\Omega)}^2+\left|\boldsymbol{u}_{mxtt}\right|_{L_2(\Omega)}^2\right]+2\left|\boldsymbol{u}_{mxtt}\right|_{L_2(\Omega)}^2=2\left(\frac{\partial \boldsymbol{f}\left(\boldsymbol{u}_{mt}\right)}{\partial \boldsymbol{u}_{mt}}\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right)\le 2c_0\left(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right)$$

$\forall 0<t\le T$, 两边从 0 到 t 积分得

$$\begin{aligned}&\left|\boldsymbol{u}_{mtt}(x,t)\right|_{L_2(\Omega)}^2+\left|\boldsymbol{u}_{mxt}(x,t)\right|_{L_2(\Omega)}^2+\left|\boldsymbol{u}_{mxtt}(x,t)\right|_{L_2(\Omega)}^2+2\left\|\boldsymbol{u}_{mxtt}(x,t)\right\|_{L_2(\Omega)}^2\\&\le\left|\boldsymbol{u}_{mtt}(x,0)\right|_{L_2(\Omega)}^2+\left|\boldsymbol{u}_{mxt}(x,0)\right|_{L_2(\Omega)}^2+\left|\boldsymbol{u}_{mxtt}(x,0)\right|_{L_2(\Omega)}^2+2c_0\int_0^t\left|\boldsymbol{u}_{mtt}(x,t)\right|_{L_2(\Omega)}^2\mathrm{d}t,\end{aligned}\tag{2.8}$$

当 $m\to\infty$, $\left|\boldsymbol{u}_{mxt}(x,0)\right|_{L_2(\Omega)}^2\to\left|\boldsymbol{u}_{1x}(x)\right|_{L_2(\Omega)}^2$, $\left|\boldsymbol{u}_{mxt}(x,0)\right|_{L_2(\Omega)}^2$ 能用一与 m 无关的正常数界住。

下面证明 $\left|\boldsymbol{u}_{mtt}(x,0)\right|_{L_2(\Omega)}^2+\left|\boldsymbol{u}_{mxtt}(x,0)\right|_{L_2(\Omega)}^2$ 也能用一与 m 无关的正常数界住。

方程组(2.1)两边同乘 $a''_{mis}(t)$, 得

$$\left(u_{mitt},a''_{mis}(t)\omega_s\right)-\left(u_{mixx},a''_{mis}(t)\omega_s\right)-\left(u_{mixxt},a''_{mis}(t)\omega_s\right)-\left(u_{mixxtt},a''_{mis}(t)\omega_s\right)=\left(f_i\left(\boldsymbol{u}_{mt}\right),a''_{mis}(t)\omega_s\right),$$

对 s 从 1 到 m 求和得

$$\left(u_{mitt},\sum_{s=1}^{m}a''_{mis}(t)\omega_s\right)-\left(u_{mixx},\sum_{s=1}^{m}a''_{mis}(t)\omega_s\right)-\left(u_{mixxt},\sum_{s=1}^{m}a''_{mis}(t)\omega_s\right)-\left(u_{mixxtt},\sum_{s=1}^{m}a''_{mis}(t)\omega_s\right)=\left(f_i\left(\boldsymbol{u}_{mt}\right),\sum_{s=1}^{m}a''_{mis}(t)\omega_s\right),$$

即

$$\left(u_{mitt},u_{mitt}\right)-\left(u_{mixx},u_{mitt}\right)-\left(u_{mixxt},u_{mitt}\right)-\left(u_{mixxtt},u_{mitt}\right)=\left(f_i\left(\boldsymbol{u}_{mt}\right),u_{mitt}\right),$$

再对 i 从 1 到 N 求和得

$$\left(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right)-\left(\boldsymbol{u}_{mxx},\boldsymbol{u}_{mtt}\right)-\left(\boldsymbol{u}_{mxxt},\boldsymbol{u}_{mtt}\right)-\left(\boldsymbol{u}_{mxxtt},\boldsymbol{u}_{mtt}\right)=\left(\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right),\boldsymbol{u}_{mtt}\right),$$

分部积分得

$$\left(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right)-\left(\boldsymbol{u}_{mxx},\boldsymbol{u}_{mtt}\right)-\left(\boldsymbol{u}_{mxxt},\boldsymbol{u}_{mtt}\right)+\left(\boldsymbol{u}_{mxtt},\boldsymbol{u}_{mxtt}\right)=\left(\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right),\boldsymbol{u}_{mtt}\right),$$

即

$$\begin{aligned}\left(\boldsymbol{u}_{mtt}(x,t),\boldsymbol{u}_{mtt}(x,t)\right)+\left(\boldsymbol{u}_{mxtt}(x,t),\boldsymbol{u}_{mxtt}(x,t)\right)&=\left(\boldsymbol{u}_{mxx}(x,t),\boldsymbol{u}_{mtt}(x,t)\right)+\left(\boldsymbol{u}_{mxxt}(x,t),\boldsymbol{u}_{mtt}(x,t)\right)\\&\quad+\left(\boldsymbol{f}\left(\boldsymbol{u}_{mt}(x,t)\right),\boldsymbol{u}_{mtt}(x,t)\right),\end{aligned}$$

令 $t=0$ 得

$$\begin{aligned}\left(\boldsymbol{u}_{mtt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)+\left(\boldsymbol{u}_{mxtt}(x,0),\boldsymbol{u}_{mxtt}(x,0)\right)&=\left(\boldsymbol{u}_{mxx}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)+\left(\boldsymbol{u}_{mxxt}(x,0),\boldsymbol{u}_{mtt}(x,0)\right)\\&\quad+\left(\boldsymbol{f}\left(\boldsymbol{u}_{mt}(x,0)\right),\boldsymbol{u}_{mtt}(x,0)\right),\end{aligned}$$

即

$$\left|\boldsymbol{u}_{mtt}(x,0)\right|_{L_2(\Omega)}^2+\left|\boldsymbol{u}_{mxtt}(x,0)\right|_{L_2(\Omega)}^2\le\left(\left|\boldsymbol{u}_{mxx}(x,0)\right|_{L_2(\Omega)}+\left|\boldsymbol{u}_{mxxt}(x,0)\right|_{L_2(\Omega)}+\left|f\left(\boldsymbol{u}_{mt}(x,0)\right)\right|_{L_2(\Omega)}\right)\cdot\left|\boldsymbol{u}_{mtt}(x,0)\right|_{L_2(\Omega)},\tag{2.9}$$

由已知条件当 $m\to\infty$,

$$\left|\boldsymbol{u}_{mxx}(x,0)\right|_{L_2(\Omega)}+\left|\boldsymbol{u}_{mxxt}(x,0)\right|_{L_2(\Omega)}\to\left|\boldsymbol{u}_{0xx}(x)\right|_{L_2(\Omega)}+\left|\boldsymbol{u}_{1xx}(x)\right|_{L_2(\Omega)},$$

$\left|\boldsymbol{u}_{mxx}(x,0)\right|_{L_2(\Omega)}+\left|\boldsymbol{u}_{mxxt}(x,0)\right|_{L_2(\Omega)}$ 能用一与 m 无关的正常数界住。又当 $m\to\infty$, $\boldsymbol{u}_{mt}(x,0)$ 在 $H^2(\Omega)\cap H_0^1(\Omega)$ 中强收敛于 $\boldsymbol{u}_1(x)$，而 $\Omega=(0,1)$ 为 R^1 的子集，由 Sobolev 嵌入定理，当 $m\to+\infty$, $\boldsymbol{u}_{mt}(x,0)$ 在 Ω 中一致收敛于 $\boldsymbol{u}_1(x)$，由 $\boldsymbol{f}\in C^1$ 当 $m\to+\infty$ 时，

$$\left|f\left(\boldsymbol{u}_{mt}(x,0)\right)\right|_{L_2(\Omega)}\to\left|f\left(\boldsymbol{u}_1(x)\right)\right|_{L_2(\Omega)}.$$

因此 $\left|f\left(\boldsymbol{u}_{mt}(x,0)\right)\right|_{L_2(\Omega)}$ 能用一与 m 无关的正常数界住。由(2.9)知

$$\left|\boldsymbol{u}_{mtt}(x,0)\right|_{L_2(\Omega)}^2+\left|\boldsymbol{u}_{mxtt}(x,0)\right|_{L_2(\Omega)}^2\le C\quad\left(C\text{为与}m\text{无关正常数}\right),$$

由(2.8)和 Gronwall 不等式得

$$\left|\boldsymbol{u}_{mtt}\right|_{L_2(\Omega)}^2+\left|\boldsymbol{u}_{mxtt}\right|_{L_2(\Omega)}^2\le E_4\left(E_4\text{为与}m\text{无关的正常数}\right).$$

引理证毕!

引理 4：在引理 2 的条件下有估计 $\left|\boldsymbol{u}_{mxxtt}\right|_{L_2(\Omega)}^2\le E_5$。

证明：方程(2.1)两边同乘 $\lambda_s a''_{mis}(t)$ 得

$$\left(u_{mitt},\lambda_s a''_{mis}(t)\omega_s\right)-\left(u_{mixx},\lambda_s a''_{mis}(t)\omega_s\right)-\left(u_{mixxt},\lambda_s a''_{mis}(t)\omega_s\right)-\left(u_{mixxtt},\lambda_s a''_{mis}(t)\omega_s\right)=\left(f_i(\boldsymbol{u}_{mt}),\lambda_s a''_{mis}(t)\omega_s\right),$$

再对 s 从 1 到 m 求和得

$$\begin{aligned}&\left(u_{mitt},\sum_{s=1}^m\lambda_s a''_{mis}(t)\omega_s\right)-\left(u_{mixx},\sum_{s=1}^m\lambda_s a''_{mis}(t)\omega_s\right)-\left(u_{mixxt},\sum_{s=1}^m\lambda_s a''_{mis}(t)\omega_s\right)-\left(u_{mixxtt},\sum_{s=1}^m\lambda_s a''_{mis}(t)\omega_s\right)\\&=\left(f_i(\boldsymbol{u}_{mt}),\sum_{s=1}^m\lambda_s a''_{mis}(t)\omega_s\right),\end{aligned}$$

即

$$\left(u_{mitt},u_{mixxtt}\right)-\left(u_{mixx},u_{mixxtt}\right)-\left(u_{mixxt},u_{mixxtt}\right)-\left(u_{mixxtt},u_{mixxtt}\right)=\left(f_i(\boldsymbol{u}_{mt}),u_{mixxtt}\right),$$

对 i 从 1 到 N 求和得

$$\left(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mxxtt}\right)-\left(\boldsymbol{u}_{mxx},\boldsymbol{u}_{mxxtt}\right)-\left(\boldsymbol{u}_{mxxt},\boldsymbol{u}_{mxxtt}\right)-\left(\boldsymbol{u}_{mxxtt},\boldsymbol{u}_{mxxtt}\right)=\left(\boldsymbol{f}(\boldsymbol{u}_{mt}),\boldsymbol{u}_{mxxtt}\right),$$

即

$$\left(\boldsymbol{u}_{mxxtt},\boldsymbol{u}_{mxxtt}\right)=\left(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mxxtt}\right)-\left(\boldsymbol{u}_{mxx},\boldsymbol{u}_{mxxtt}\right)-\left(\boldsymbol{u}_{mxxt},\boldsymbol{u}_{mxxtt}\right)-\left(\boldsymbol{f}(\boldsymbol{u}_{mt}),\boldsymbol{u}_{mxxtt}\right),$$

$$\left|\boldsymbol{u}_{mxxtt}\right|_{L_2(\Omega)}^2\le\left(\left|\boldsymbol{u}_{mtt}\right|_{L_2(\Omega)}+\left|\boldsymbol{u}_{mxx}\right|_{L_2(\Omega)}+\left|\boldsymbol{u}_{mxxt}\right|_{L_2(\Omega)}+\left|\boldsymbol{f}(\boldsymbol{u}_{mt})\right|_{L_2(\Omega)}\right)\left|\boldsymbol{u}_{mxxtt}\right|_{L_2(\Omega)},\tag{2.10}$$

下面证明 $\forall 0\le t\le T,\left|\boldsymbol{f}(\boldsymbol{u}_{mt})\right|_{L_2(\Omega)}$ 能用一与 m 和 t 无关的正常数界住。

由引理 1，2，3 知 $\forall 0\le t\le T$，$\left|\boldsymbol{u}_{mtt}(x,t)\right|_{L_2(\Omega)}$，$\left|\boldsymbol{u}_{mxx}(x,t)\right|_{L_2(\Omega)}$，$\left|\boldsymbol{u}_{mxxt}(x,t)\right|_{L_2(\Omega)}$ 能用与 m 和 t 无关的正常数界住，又由引理 1~引理 3 知 $\forall 0\le t\le T$，$\left|\boldsymbol{u}_{mt}\right|_{H^2(\Omega)}\le E_6$ (E_6 为与 m 和 t 无关正常数)，由 Sobolev 嵌入定理得 $\forall 0\le t\le T$，$\left|\boldsymbol{u}_{mt}\right|_{C(\bar{\Omega})}\le E_7$ (E_7 为与 m 和 t 无关正常数，$C(\bar{\Omega})$ 表示在 $\bar{\Omega}$ 上连续函数的全体，$|\cdot|_{C(\bar{\Omega})}$ 表示在 $\bar{\Omega}$ 上的最大模)。

由 $\boldsymbol{f}\in C^1$ 知 $\left|\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right)\right|_{L_2(\Omega)}$ 能用一与 m 和 t 无关的正常数界住。

由(2.10)可推出 $\forall 0\le t\le T$，$\left|\boldsymbol{u}_{mxxtt}\right|^2_{L_2(\Omega)}\le E_5\left(E_5\text{为与}m\text{和}t\text{无关的正常数}\right)$。
引理证毕！

定义：函数 $\boldsymbol{u}(x,t)$ 称为问题(1.1)~(1.3)在 $\Omega\times[0,T]$ 上的强解，若对任一 $T>0$，均有

i) $\boldsymbol{u}(x,t)\in L^\infty\left(0,T,H^2(\Omega)\cap H_0^1(\Omega)\right)$，$\boldsymbol{u}_t(x,t)\in L^\infty\left(0,T,H^2(\Omega)\cap H_0^1(\Omega)\right)$。

ii) 对一切 $\varphi(x,t)\in C\left([0,T];L_2(\Omega)\right)$ 成立 $\int_0^T\left(u_{itt}-u_{ixx}-u_{ixxt}-u_{ixxtt}-f_i\left(\boldsymbol{u}_t\right),\varphi\right)\mathrm{d}t=0\ \ (i=1,2,\cdots,N)$。

iii) $u_i(x,0)=u_{0i}(x)$ 于 $H^2(\Omega)\cap H_0^1(\Omega)$，$u_{it}(x,0)=u_{1i}(x)$ 于 $H^2(\Omega)\cap H_0^1(\Omega)$ $(i=1,2,\cdots,N)$。这里 $\boldsymbol{u}_0(x)=\left(u_{01}(x),\cdots,u_{0N}(x)\right)^{\mathrm{T}}$，$\boldsymbol{u}_1(x)=\left(u_{11}(x),\cdots,u_{1N}(x)\right)^{\mathrm{T}}$。

定理 1：设条件(1.4)成立且 $\boldsymbol{u}_0(x)\in H^2(\Omega)\cap H_0^1(\Omega)$，$\boldsymbol{u}_1(x)\in H^2(\Omega)\cap H_0^1(\Omega)$，则问题(1.1)~(1.3)存在上述意义下的整体强解 $\boldsymbol{u}(x,t)$。

证明：由引理 1~引理 4 知 $\left(f_i\left(\boldsymbol{u}_{mt}\right),\omega_s\right)$ 有界，因此由常微分方程理论知方程组(2.1)~(2.3)有整体解 $u_{mi}(x,t)$，由引理 1~引理 4 知：$\left\{\boldsymbol{u}_m(x,t)\right\}$，$\left\{\boldsymbol{u}_{mt}(x,t)\right\}$，$\left\{\boldsymbol{u}_{mtt}(x,t)\right\}$ 于空间 $L^\infty\left(0,T,H^2(\Omega)\cap H_0^1(\Omega)\right)$ 中关于 m 一致有界，由列紧性原理知存在 u_{mi} 的一个子列(仍记为 u_{mi})使当 $m\to\infty$ 时，$u_{mi}(x,t)\to u_i(x,t)$ 于 $L^\infty\left(0,T,H^2(\Omega)\cap H_0^1(\Omega)\right)$ 中弱∗收敛，$u_{mit}(x,t)\to u_{it}(x,t)$ 于 $L^\infty\left(0,T,H^2(\Omega)\cap H_0^1(\Omega)\right)$ 中弱∗收敛，$u_{mitt}(x,t)\to u_{itt}(x,t)$ 于 $L^\infty\left(0,T,H^2(\Omega)\cap H_0^1(\Omega)\right)$ 中弱∗收敛。

而由 $\left\{\boldsymbol{u}_{mt}(x,t)\right\},\left\{\boldsymbol{u}_{mtt}(x,t)\right\},\left\{\boldsymbol{u}_{mxt}(x,t)\right\}$ 都于

$$L^\infty\left(0,T;L_2(\Omega)\right)\subset L^2\left(0,T;L_2(\Omega)\right)=L_2\left(Q_T\right)\quad\left(Q_T=\Omega\times[0,T]\right)$$

中有界可知 $\left\{u_{mit}(x,t)\right\}$ 于 $H^1\left(Q_T\right)$ 中有界，由 Sobolev 嵌入定理从而有子序列(仍记为 $\left\{u_{mit}(x,t)\right\}$)使当 $m\to\infty$ 时，$u_{mit}(x,t)\to u_{it}(x,t)\quad(i=1,2,\cdots,N)$ 于 $L_2\left(Q_T\right)$ 中强收敛，且于 Q_T 中几乎处处收敛。由以上证明知 $\left(\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right),\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right)\right)=\left|\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right)\right|^2_{L_2(\Omega)}\le\mathrm{const}$，由[4](p. 11，引理 1.3)，当 $m\to\infty$ 时，$\boldsymbol{f}\left(\boldsymbol{u}_{mt}\right)\to\boldsymbol{f}\left(\boldsymbol{u}_t\right)$ 在 $L_2\left(Q_T\right)$ 中弱收敛，从而 $\boldsymbol{f}_i\left(\boldsymbol{u}_{mt}\right)$ 在 $L_2\left(Q_T\right)$ 中弱收敛于 $\boldsymbol{f}_i\left(\boldsymbol{u}_t\right)\ (i=1,2,\cdots,N)$。任取 $d_{si}(t)\in C\ (i=1,2,\cdots,N)$，在(2.1)两端同乘 $d_{si}(t)\ (s=1,2,\cdots,N',\ i=1,2,\cdots,N)$ 对 $s=1,2,\cdots,N'\ (N'\le m)$ 求和，关于 t 在[0,T]上积分，令 $m\to\infty$ 取极限，由上面弱*收敛及弱收敛结果得

$$\int_0^T\left(u_{itt}-u_{ixx}-u_{ixxt}-u_{ixxtt},\sum_{s=1}^{N'}d_{si}(t)\omega_s\right)\mathrm{d}t=\int_0^T\left(f_i\left(\boldsymbol{u}_t\right),\sum_{s=1}^{N'}d_{si}(t)\omega_s\right)\mathrm{d}t.$$

$\because\left\{\omega_s(x)\right\}$ 在 $L_2(\Omega)$ 中稠，$\sum_{s=1}^{N'}d_{si}(t)\omega_s(x)\ (N'=1,2,\cdots)$ 在 $C\left([0,T];L_2(\Omega)\right)$ 中稠密，因此对任意 $\varphi(x,t)\in C\left([0,T];L_2(\Omega)\right)$，成立

$$\int_0^T\left(u_{itt}-u_{ixx}-u_{ixxt}-u_{ixxtt},\varphi(x,t)\right)\mathrm{d}t=\int_0^T\left(f_i\left(\boldsymbol{u}_t\right),\varphi(x,t)\right)\mathrm{d}t\quad(i=1,2,\cdots,N),$$

因此强解定义中的(i)(ii)都满足。

下面验证初始条件 $u_i(x,0)=u_{0i}(x)\quad(i=1,2,\cdots,N)$。$u_{mi}(x,t)\to u_i(x,t)$ 与 $L^\infty\left(0,T,H^2(\Omega)\cap H_0^1(\Omega)\right)$ 中弱∗收敛，$u_{mit}(x,t)\to u_{it}(x,t)$ 与 $L^\infty\left(0,T,H^2(\Omega)\cap H_0^1(\Omega)\right)$ 中弱∗收敛，因此 $u_{mi}(x,t)\in C\left([0,T],H^2(\Omega)\cap H_0^1(\Omega)\right)$ 且 $u_i(x,t)\in C\left([0,T];H^2(\Omega)\cap H_0^1(\Omega)\right)\quad(i=1,2,\cdots,N)$，故当 $m\to\infty$ 时，$u_{mi}(x,0)$ 弱收敛到 $u_i(x,0)$ 于 $H^2(\Omega)\cap H_0^1(\Omega)$ 中，又已知当 $m\to\infty$，$u_{mi}(x,0)\to u_{0i}(x)$ 在 $H^2(\Omega)\cap H_0^1(\Omega)$ 中强收敛，这样得到 $u_i(x,0)=u_{0i}(x)\quad(i=1,2,\cdots,N)$。

再证 $\boldsymbol{u}(x,t)$ 满足初始条件 $u_{it}(x,0)=u_{1i}(x)\quad(i=1,2,\cdots,N)$。当 $m\to\infty$ 时，由 $u_{mit}(x,t)\to u_{it}(x,t)$ $L^\infty\left(0,T,H^2(\Omega)\cap H_0^1(\Omega)\right)$ 中弱∗收敛，$u_{mitt}(x,t)\to u_{itt}(x,t)$ 与 $L^\infty\left(0,T,H^2(\Omega)\cap H_0^1(\Omega)\right)$ 中弱

∗收敛，$u_{mit}(x,t)\in C([0,T],H^2(\Omega)\cap H_0^1(\Omega))$ 且 $u_{it}(x,t)\in C([0,T];H^2(\Omega)\cap H_0^1(\Omega))$ $(i=1,2,\cdots,N)$，因此当 $m\to\infty$ 时 $u_{mit}(x,0)$ 弱收敛到 $u_{it}(x,0)$ 于 $H^2(\Omega)\cap H_0^1(\Omega)$ 中，又因为当 $m\to+\infty$，$u_{mit}(x,0)\to u_{1i}(x)$ $(i=1,2,\cdots,N)$ 在 $H^2(\Omega)\cap H_0^1(\Omega)$ 中强收敛，从而得到 $u_{it}(x,0)=u_{1i}(x)$ $(i=1,2,\cdots,N)$。

3. 唯一性

定理 2：若定理 1 的条件满足，问题(1)~(3)的强解是唯一的。

证明：设 $\boldsymbol{u}$，$\boldsymbol{v}$ 为问题(1.1)~(1.3)的两个强解，令 $\boldsymbol{\omega}=\boldsymbol{u}-\boldsymbol{v}$，则 $\boldsymbol{\omega}$ 满足 $\boldsymbol{\omega}_{tt}-\boldsymbol{\omega}_{xx}-\boldsymbol{\omega}_{xxt}-\boldsymbol{\omega}_{xxtt}=\boldsymbol{f}(\boldsymbol{u}_t)-\boldsymbol{f}(\boldsymbol{v}_t)$ 及齐初始条件和齐边值条件：$\boldsymbol{\omega}(x,0)=\mathbf{0},\ \boldsymbol{\omega}_t(x,0)=\mathbf{0},\boldsymbol{\omega}(0,t)=\boldsymbol{\omega}(1,t)=\mathbf{0}.$

两边用 $\boldsymbol{\omega}_t$ 做内积得

$$(\boldsymbol{\omega}_{tt},\boldsymbol{\omega}_t)-(\boldsymbol{\omega}_{xx},\boldsymbol{\omega}_t)-(\boldsymbol{\omega}_{xxt},\boldsymbol{\omega}_t)-(\boldsymbol{\omega}_{xxtt},\boldsymbol{\omega}_t)=(\boldsymbol{f}(\boldsymbol{u}_t)-\boldsymbol{f}(\boldsymbol{v}_t),\boldsymbol{\omega}_t)=\left(\frac{\partial\boldsymbol{f}(\boldsymbol{u}_t)}{\partial\boldsymbol{u}_t}\Big|_{\boldsymbol{v}_t+\theta\boldsymbol{\omega}_t}\cdot\boldsymbol{\omega}_t,\boldsymbol{\omega}_t\right)\le C_0(\boldsymbol{\omega}_t,\boldsymbol{\omega}_t)\ \ (0<\theta<1),$$

分部积分得

$$\frac{\mathrm{d}}{\mathrm{d}t}\big((\boldsymbol{\omega}_t,\boldsymbol{\omega}_t)+(\boldsymbol{\omega}_x,\boldsymbol{\omega}_x)+(\boldsymbol{\omega}_{xt},\boldsymbol{\omega}_{xt})\big)+2(\boldsymbol{\omega}_{xt},\boldsymbol{\omega}_{xt})\le 2C_0(\boldsymbol{\omega}_t,\boldsymbol{\omega}_t),$$

两边同时加上 $(\boldsymbol{\omega},\boldsymbol{\omega}_t)$ 左边将其化为 $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\omega},\boldsymbol{\omega})$，右边将其估计为 $(\boldsymbol{\omega},\boldsymbol{\omega}_t)\le(\boldsymbol{\omega},\boldsymbol{\omega})+(\boldsymbol{\omega}_t,\boldsymbol{\omega}_t)$ 经计算得

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}(\boldsymbol{\omega},\boldsymbol{\omega})+(\boldsymbol{\omega}_t,\boldsymbol{\omega}_t)+(\boldsymbol{\omega}_x,\boldsymbol{\omega}_x)+(\boldsymbol{\omega}_{xt},\boldsymbol{\omega}_{xt})\right)+2(\boldsymbol{\omega}_{xt},\boldsymbol{\omega}_{xt})\\&\le 2C_0(\boldsymbol{\omega}_t,\boldsymbol{\omega}_t)+(\boldsymbol{\omega},\boldsymbol{\omega}_t)\le M\big((\boldsymbol{\omega},\boldsymbol{\omega})+(\boldsymbol{\omega}_t,\boldsymbol{\omega}_t)\big)\\&\quad(M\text{为与}\boldsymbol{\omega}\text{无关的正常数}),\end{aligned}$$

$\forall 0\le t\le T$，从 0 到 t 积分，且注意到 $\boldsymbol{\omega}$ 满足齐初始条件，得

$$\frac{1}{2}|\boldsymbol{\omega}|^2_{L_2(\Omega)}+|\boldsymbol{\omega}_t|^2_{L_2(\Omega)}+|\boldsymbol{\omega}_x|^2_{L_2(\Omega)}+|\boldsymbol{\omega}_{xt}|^2_{L_2(\Omega)}+2[\boldsymbol{\omega}_{xt},\boldsymbol{\omega}_{xt}]\le M\int_0^t\left(|\boldsymbol{\omega}|^2_{L_2(\Omega)}+|\boldsymbol{\omega}_t|^2_{L_2(\Omega)}+|\boldsymbol{\omega}_x|^2_{L_2(\Omega)}+|\boldsymbol{\omega}_{xt}|^2_{L_2(\Omega)}\right)\mathrm{d}t,$$

由 Gronwall 不等式得 $|\boldsymbol{\omega}|^2_{L_2(\Omega)}+|\boldsymbol{\omega}_t|^2_{L_2(\Omega)}+|\boldsymbol{\omega}_x|^2_{L_2(\Omega)}+|\boldsymbol{\omega}_{xt}|^2_{L_2(\Omega)}\equiv 0,\ \boldsymbol{\omega}=0$ 即 $\boldsymbol{u}=\boldsymbol{v}$。

定理证毕！

4. 结论

本文讨论的四阶非线性偏微分方程组(1.1)描述了多条粘弹性杆的耦合振动问题，所用的方法是 Galerkin 方法。首先选取负拉普拉斯算子的特征函数作为一组基，构造原问题的近似解并建立关于近似解的 Galerkin 逼近格式，在引理 1~引理 4 中对近似解作出一系列的先验估计，在先验估计的基础上通过取弱极限得到原问题的整体强解，最后证明了整体强解的唯一性。

在常见的文献中 Galerkin 方法一般用于讨论单个方程，而本文成功地将 Galerkin 方法应用于方程组的情形。在对方程组进行讨论时，非线性项的 Jacobi 矩阵半有界这一条件很重要，本文在引理 1~引理 4 中对近似解作先验估计时多次用到这一条件。

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