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涉及差分与平移的亚纯函数的有理函数的收敛指数

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摘 要

设c是一个非零有穷复数，f是一个有穷级超越亚纯函数，R是一个非常数有理函数，本文研究$f(z)-R(z)$，$f(z+c)-R(z)$ 及$\Delta_c f(z)-R(z)$的零点收敛指数与f的级之间的关系。由此推广了Chen, Zhang-Chen, Chen-Zheng 等人的结果。

关键词

亚纯函数，差分，平移，收敛指数

The Exponents of Convergence of Rational Functions of Meromorphic Functions Concerning Differences and Shifts

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Abstract

Let c be a nonzero finite complex number, let f be a transcendental meromorphic function of finite order, and let R be a nonconstant rational function. It is studied that the relationship between the exponent of convergence of zeros of $f(z)-R(z)$, $f(z+c)-R(z)$, and $\Delta_c f(z)-R(z)$ and the order of f. This improves the results of Chen, Zhang-Chen and Chen-Zheng.

Keywords

Meromorphic Functions, Differences, Shifts, The Exponent of Convergence

1. 引言及主要结果

本文中, 亚纯函数指的是在整个复平面上的亚纯函数. 以下将使用值分布论中的标准记号$T(r,f)$, $m(r,f)$, $N(r,f)$, $S(r,f)$, $\cdots$(参见 [1] [2] [3] [4]), 其中$S(r,f)$表示任一函数f满足$S(r,f)=o\{T(r,f)\}$, $r\to\infty$, $r\notin E$, E是一个对数测度有穷的r值集.

设a是复平面上的亚纯函数(可以恒为∞), 如果a满足$T(r,a)=S(r,f)$, 称a是f的小函数. 本文用$\rho(f)$, $\lambda(f)$ 和$\lambda(\frac{1}{f})$分别表示f的级, f的零点和极点收敛指数,

$$\rho(f)=\overline{\lim_{r\to\infty}}\frac{\log^+ T(r,f)}{\log r},$$

$$\lambda(f)=\overline{\lim_{r\to\infty}}\frac{\log^+ N(r,\frac{1}{f})}{\log r},$$

$$\lambda(\frac{1}{f})=\overline{\lim_{r\to\infty}}\frac{\log^+ N(r,f)}{\log r}.$$

d是一个复数, 若$\lambda(f-d)<\rho(f)$, 称d 为f的Borel例外值. 设f是一个复平面上非常数亚纯函数, c是一个非零有穷复数, f的差分为$\Delta_c f(z)=f(z+c)-f(z)$.

最近几十年来, 有关亚纯函数涉及差分与平移的不动点, 亚纯函数与其平移和差分的不动点与级的关系等都有了许多新的研究成果, 如 [5–14]. 2000 年, Fang [12]研究了亚纯函数的导数的不动

点的值分布, 证明了

定理A 设f是一个超越亚纯函数, 若f的所有零点和极点重级均≥ 2, 则f'有无穷多个不动点.

2008年, Chen-Shon [10]研究了超越整函数和亚纯函数的零点和不动点. 2013年, Chen [8]研究了亚纯函数与其平移和差分的不动点跟级之间的关系, 证明了

定理B 设f是复平面上的一个满足$\lambda(\frac{1}{f}) < \rho(f)$的有穷级亚纯函数, c是一个非零有穷复数, 满足$f(z+c) \not\equiv f(z)+c$, 则有

$$\max\{\tau(f(z)), \tau(\Delta_c f(z))\} = \rho(f),$$
$$\max\{\tau(f(z)), \tau(f(z+c))\} = \rho(f),$$
$$\max\{\tau(\Delta_c f(z)), \tau(f(z+c))\} = \rho(f).$$

其中$\tau(f)$表示f不动点的收敛指数(下同).

2016年, Zhang-Chen [14]指出定理B中的条件$\lambda(\frac{1}{f}) < \rho(f)$不能去掉, 并将条件改为$\lambda\left(f(z)-d\right) < \rho(f)$, d是一个有穷复数, 定理B依然成立, 证明了

定理C 设d是一个有穷复数, f是复平面上的一个满足$\lambda\left(f(z)-d\right) < \rho(f)$ 的有穷级亚纯函数, c是一个非零有穷复数, 则有

$$\max\{\tau(f(z)), \tau(\Delta_c f(z))\} = \rho(f),$$
$$\max\{\tau(f(z)), \tau(f(z+c))\} = \rho(f),$$
$$\max\{\tau(\Delta_c f(z)), \tau(f(z+c))\} = \rho(f).$$

2003年, Bergweiler-Pang [6]推广并改进了定理A, 证明了

定理D 设f是一个有穷级超越亚纯函数, $P \not\equiv 0$是一个多项式, 若f的所有零点重级≥ 2, 去掉有限个例外, 则$f'-P$有无穷多个零点.

定理E 设f是一个超越亚纯函数, $R \not\equiv 0$是一个有理函数, 若f的所有零点和极点重级均≥ 2, 去掉有限个例外, 则$f'-R$有无穷多个零点.

2019年, Chen-Zheng [7]推广了定理C, 证明了

定理F 设c是一个非零有穷复数, f是一个有穷级超越亚纯函数, $m \in \mathbf{N}_+$, $P(z) = p_m z^m + p_{m-1}z^{m-1} + \cdots + p_0$是一个非常数多项式, $p_i \in \mathbb{C}, i = 0, 1, \cdots, m$, 且$p_m \neq 0$, 若$f$有一个Borel例外值$d \in \mathbb{C}$,则有

$$\max\{\lambda(f(z)-P(z)), \lambda(\Delta_c f(z)-P(z))\} = \rho(f),$$
$$\max\{\lambda(f(z)-P(z)), \lambda(f(z+c)-P(z))\} = \rho(f),$$
$$\max\{\lambda(\Delta_c f(z)-P(z)), \lambda(f(z+c)-P(z))\} = \rho(f).$$

自然会问, 对于有穷级超越亚纯函数与有理函数的零点收敛指数是否也有类似于定理F的结论.

本文给出了肯定的回答, 证明了

定理1 设c是一个非零有穷复数, f是一个有穷级超越亚纯函数,

$$R(z)=\frac{a_pz^p+a_{p-1}z^{p-1}+\cdots+a_0}{b_qz^q+b_{q-1}z^{q-1}+\cdots+b_0}$$

是一个非常数有理函数, 其中$a_p\neq 0,a_{p-1},\cdots,a_0,b_q\neq 0,b_{q-1},\cdots,b_0$是有穷复数, p,q是非负整数, 且$p+q\geq 1$. 若f有一个Borel例外值$d\in\mathbb{C}$, 则有

$$\max\{\lambda(f(z)-R(z)),\lambda(\Delta_cf(z)-R(z))\}=\rho(f),$$
$$\max\{\lambda(f(z)-R(z)),\lambda(f(z+c)-R(z))\}=\rho(f),$$
$$\max\{\lambda(\Delta_cf(z)-R(z)),\lambda(f(z+c)-R(z))\}=\rho(f).$$

定理2 设c是一个非零有穷复数, f是一个有穷级超越亚纯函数,

$$R(z)=\frac{a_pz^p+a_{p-1}z^{p-1}+\cdots+a_0}{b_qz^q+b_{q-1}z^{q-1}+\cdots+b_0}$$

是一个非常数有理函数, 其中$a_p\neq 0,a_{p-1},\cdots,a_0,b_q\neq 0,b_{q-1},\cdots,b_0$是有穷复数, p,q是非负整数, 且$p+q\geq 1$. 若f有一个Borel例外值$d=\infty$, $\Delta_cf(z)$是超越亚纯函数, 则有

$$\max\{\lambda(f(z)-R(z)),\lambda(\Delta_cf(z)-R(z))\}=\rho(f),$$
$$\max\{\lambda(f(z)-R(z)),\lambda(f(z+c)-R(z))\}=\rho(f),$$
$$\max\{\lambda(\Delta_cf(z)-R(z)),\lambda(f(z+c)-R(z))\}=\rho(f).$$

例1 设$f(z)=e^z+R(z)$, $R(z)$是有理函数, c是满足$e^c=1$的非零常数. 则$\Delta_cf(z)=R(z+c)-R(z)$, $\rho(f)=1$, 但$\lambda(f(z)-R(z))=0,\lambda(\Delta_cf(z)-R(z))=0,\lambda(f(z+c)-R(z))=0$. 因此$\max\{\lambda(f(z)-R(z)),\lambda(f(z+c)-R(z)),\lambda(\Delta_cf(z)-R(z))\}<\rho(f)$.

例1说明定理2中的条件$\Delta_cf(z)$是超越亚纯函数是必需的.

2. 一些引理

为了证明本文的结果, 需要如下几个引理.

引理2.1 [11] [14]设f是复平面上的一个有穷级亚纯函数, c是一个给定的非零有穷复数, d 是一个有穷复数, 则对于任意的$\varepsilon>0$, 有

$$T(r,f(z+c))=T(r,f)+O(r^{\rho(f)-1+\varepsilon})+O(\log r),$$
$$N(r,f(z+c))=N(r,f)+O(r^{\rho(f)-1+\varepsilon})+O(\log r),$$
$$N\left(r,\frac{1}{f(z+c)-d}\right)=N\left(r,\frac{1}{f(z)-d}\right)+O(r^{\rho(f)-1+\varepsilon})+O(\log r).$$

引理2.2 [2]设f是复平面上的一个亚纯函数, 则对于f的所有不可约有理函数

$$R(z,f(z))=\frac{\sum\limits_{i=0}^{p}a_i(z)f^i(z)}{\sum\limits_{j=0}^{q}b_j(z)f^j(z)},$$

其中$a_i(z),i=0,1,\cdots,p,b_j(z),j=0,1,\cdots,q$是$f$的小亚纯函数, 则有

$$T(r,R(z,f(z)))=\max\{p,q\}T(r,f)+S(r,f).$$

引理2.3 [14]设f是复平面上的一个亚纯函数, 满足$\overline{N}(r,f)+\overline{N}(r,\frac{1}{f})=S(r,f)$.令

$$F(z)=\frac{a_0(z)f^p(z)+a_1(z)f^{p-1}(z)+\cdots+a_p(z)}{b_0(z)f^q(z)+b_1(z)f^{q-1}(z)+\cdots+b_q(z)},$$

其中$a_i(z),i=0,1,\cdots,p,b_j(z),j=0,1,\cdots,q$是$f$的小亚纯函数, 且$a_0b_0a_p\not\equiv 0$, 若$q\le p,T(r,F)\ge T(r,f)+S(r,f)$, 则有

$$\lambda(F)=\rho(f).$$

引理2.4 [3]设$f_1(z),f_2(z),\cdots,f_n(z)$是复平面上的亚纯函数, $g_1(z),g_2(z),\cdots,g_n(z)$是整函数, 满足以下条件

(1)$\sum\limits_{j=1}^{n}f_j(z)e^{g_j(z)}\equiv 0$;

(2)对于$1\le j<k\le n,g_j(z)-g_k(z)$ 不为常数;

(3)对于$1\le j\le n,1\le h<k\le n$,

$$T(r,f_j)=o\{T(r,e^{g_h-g_k})\},r\to\infty.$$

则对于$j=1,2,\cdots,n$, 有$f_j(z)\equiv 0$.

引理2.5 [7] [14]设H是复平面上的一个亚纯函数, h是满足$\deg h\ge 1$的多项式, c是一个非零有穷复数, 若$\rho(H)<\rho(e^h)$, 则有

$$T(r,H)=S(r,e^h),\quad T(r,H(z+c))=S(r,e^h),\quad T(r,e^{h(z+c)-h(z)})=S(r,e^h).$$

对于$j=1,2,\cdots,n,T(r,H(z+jc))=S(r,e^h)$. 对于$k\in\mathbf{N}_+,s\in\mathbf{N},k>s,T(r,e^{h(z+kc)-h(z+sc)})=S(r,e^h)$.

3. 定理的证明

定理1的证明

假设$\lambda(f(z)-R(z))<\rho(f)$, 下证$\lambda(\Delta_c f(z)-R(z))=\lambda(f(z+c)-R(z))=\rho(f)$. 令

$$F_1(z)=\frac{f(z)-R(z)}{f(z)-d}. \tag{3.1}$$

由引理2.2得$T(r,F_1)=T(r,f)+S(r,f)$,则$\rho(F_1)=\rho(f)<\infty$, 因此有

$$\lambda(\frac{1}{F_1})=\lambda(f(z)-d)<\rho(f)=\rho(F_1),$$

$$\lambda(F_1)=\lambda(f(z)-R(z))<\rho(f)=\rho(F_1),$$

这意味着0和∞是F_1的Borel例外值. 由Hadamard因子分解定理, 得

$$F_1(z)=A_1(z)e^{B_1(z)}. \tag{3.2}$$

其中$A_1(z)\not\equiv 0$是满足$\rho(A_1)<\rho(F_1)=\rho(f)$的亚纯函数. $B_1(z)$是满足$\rho(f)=\rho(F_1)=\deg B_1(z)\geq 1$的多项式.

由(3.1)(3.2)得

$$f(z)=\frac{R(z)-d}{1-F_1(z)}+d=\frac{R(z)-d}{1-A_1(z)e^{B_1(z)}}+d. \tag{3.3}$$

由(3.3)得

$$\begin{aligned}\Delta_c f(z)=&f(z+c)-f(z)\\=&\frac{R(z+c)-d}{1-A_1(z+c)e^{B_1(z+c)}}-\frac{R(z)-d}{1-A_1(z)e^{B_1(z)}}\\=&\frac{\left[(d-R(z+c))A_1(z)+(R(z)-d)A_1(z+c)e^{B_1(z+c)-B_1(z)}\right]e^{B_1(z)}+R(z+c)-R(z)}{\left[A_1(z)A_1(z+c)e^{B_1(z+c)-B_1(z)}\right]e^{2B_1(z)}+\left[-A_1(z+c)e^{B_1(z+c)-B_1(z)}-A_1(z)\right]e^{B_1(z)}+1}\\=&\frac{D_1(z)e^{B_1(z)}+R(z+c)-R(z)}{D_3(z)e^{2B_1(z)}+D_2(z)e^{B_1(z)}+1},\end{aligned} \tag{3.4}$$

其中

$$\begin{aligned}D_1(z)&=(d-R(z+c))A_1(z)+(R(z)-d)A_1(z+c)e^{B_1(z+c)-B_1(z)},\\D_2(z)&=-A_1(z+c)e^{B_1(z+c)-B_1(z)}-A_1(z),\\D_3(z)&=A_1(z)A_1(z+c)e^{B_1(z+c)-B_1(z)}\not\equiv 0.\end{aligned}$$

$$\begin{aligned}R(z+c)-R(z)&=\frac{a_p(z+c)^p+a_{p-1}(z+c)^{p-1}+\cdots+a_0}{b_q(z+c)^q+b_{q-1}(z+c)^{q-1}+\cdots+b_0}-\frac{a_pz^p+a_{p-1}z^{p-1}+\cdots+a_0}{b_qz^q+b_{q-1}z^{q-1}+\cdots+b_0}\\&=\frac{a_pb_q(C_p^1-C_q^1)c\cdot z^{p+q-1}+\cdots+(a_pc^p+\cdots+a_1c)b_0-(b_qc^q+\cdots+b_1c)a_0}{\left[b_q(z+c)^q+b_{q-1}(z+c)^{q-1}+\cdots+b_0\right]\left[b_qz^q+b_{q-1}z^{q-1}+\cdots+b_0\right]}\not\equiv 0.\end{aligned}$$

由(3.4)得

$$\Delta_c f(z) - R(z) = \frac{D_1(z)e^{B_1(z)} + R(z+c) - R(z)}{D_3(z)e^{2B_1(z)} + D_2(z)e^{B_1(z)} + 1} - R(z)$$
$$= \frac{-R(z)D_3(z)e^{2B_1(z)} + (D_1(z) - R(z)D_2(z))e^{B_1(z)} + R(z+c) - 2R(z)}{D_3(z)e^{2B_1(z)} + D_2(z)e^{B_1(z)} + 1}$$
$$= \frac{-R(z)D_3(z)e^{2B_1(z)} + D_4(z)e^{B_1(z)} + R(z+c) - 2R(z)}{D_3(z)e^{2B_1(z)} + D_2(z)e^{B_1(z)} + 1}, \tag{3.5}$$

其中

$$D_4(z) = D_1(z) - R(z)D_2(z) = (d - R(z+c) + R(z))A_1(z) + (2R(z) - d)e^{B_1(z+c)-B_1(z)}.$$

$$R(z+c) - 2R(z) = \frac{a_p(z+c)^p + a_{p-1}(z+c)^{p-1} + \cdots + a_0}{b_q(z+c)^q + b_{q-1}(z+c)^{q-1} + \cdots + b_0} - 2\frac{a_p z^p + a_{p-1}z^{p-1} + \cdots + a_0}{b_q z^q + b_{q-1}z^{q-1} + \cdots + b_0}$$
$$= \frac{-a_p b_q z^{p+q} + \cdots + (a_p c^p + \cdots + a_1 c)b_0 - 2(b_q c^q + \cdots + b_1 c)a_0 - a_0 b_0}{[b_q(z+c)^q + b_{q-1}(z+c)^{q-1} + \cdots + b_0]\,[b_q z^q + b_{q-1}z^{q-1} + \cdots + b_0]} \not\equiv 0.$$

由(3.4)，$R(z+c) - R(z) \not\equiv 0$与(3.5)，$R(z)D_3(z) \not\equiv 0$可以看出$\Delta_c f(z)$ 与$\Delta_c f(z) - R(z)$是$e^{B_1(z)}$的有理函数．因为$\rho(A_1) < \rho(e^{B_1})$，由引理2.5得，系数$D_i(z)(i = 1,2,3,4)$是$e^{B_1(z)}$的小函数．显然$D_i(z) \not\equiv 0(i = 1,2,3,4)$，由引理2.4 得

$$D_1(z)e^{B_1(z)} + R(z+c) - R(z) \not\equiv 0.$$

则由(3.4)(3.5)得

$$T(r, \Delta_c f(z)) \geq T(r, e^{B_1(z)}) + S(r, e^{B_1(z)}),$$
$$T(r, \Delta_c f(z) - R(z)) \geq T(r, e^{B_1(z)}) + S(r, e^{B_1(z)}). \tag{3.6}$$

由(3.5)(3.6)与引理2.3得

$$\lambda(\Delta_c f(z) - R(z)) = \rho(e^{B_1(z)}) = \rho(f), \tag{3.7}$$

即

$$\max\{\lambda(f(z) - R(z)), \lambda(\Delta_c f(z) - R(z))\} = \rho(f).$$

由(3.3)得

$$f(z+c) - R(z) = \frac{R(z+c) - d}{1 - A_1(z+c)e^{B_1(z+c)}} + d - R(z)$$
$$= \frac{(R(z) - d)A_1(z+c)e^{B_1(z+c)} + R(z+c) - R(z)}{1 - A_1(z+c)e^{B_1(z+c)}}. \tag{3.8}$$

因为$(R(z)-d)A_1(z+c)+(R(z+c)-R(z))A_1(z+c)=(R(z+c)-d)A_1(z+c)\not\equiv 0$, 所以$f(z+c)-R(z)$可以看作$e^{B_1(z+c)}$的不可约有理函数. 因为$\rho(A_1(z+c))=\rho(A_1)<\rho(e^{B_1})=\rho(e^{B_1(z+c)})$, 由引理2.5得

$$T(r,A_1(z+c))=S(r,e^{B_1(z+c)}). \tag{3.9}$$

由(3.8)(3.9)与引理2.2得

$$T(r,f(z+c)-R(z))=T(r,e^{B_1(z+c)})+S(r,e^{B_1(z+c)}).$$

由引理2.3得

$$\lambda(f(z+c)-R(z))=\rho(e^{B_1(z+c)})=\rho(e^{B_1})=\rho(f), \tag{3.10}$$

即

$$\max\{\lambda(f(z)-R(z)),\lambda(f(z+c)-R(z))\}=\rho(f).$$

假设$\lambda(f(z+c)-R(z))<\rho(f)$, 下证$\lambda(\Delta_c f(z)-R(z))=\rho(f)$. 令

$$F_2(z)=\frac{f(z+c)-R(z)}{f(z+c)-d}. \tag{3.11}$$

由(3.11)得

$$f(z)=\frac{R(z-c)-d}{1-F_2(z-c)}+d. \tag{3.12}$$

由(3.11)(3.12)与引理2.1得

$$\rho(F_2)=\rho(F_2(z-c))=\rho(f).$$

由引理2.1与$\lambda(f(z)-d)<\rho(f)$, 得

$$\lambda(\frac{1}{F_2})=\lambda(f(z+c)-d)=\lambda(f(z)-d)<\rho(f)=\rho(F_2),$$

$$\lambda(F_2)=\lambda(f(z+c)-R(z))<\rho(f)=\rho(F_2),$$

这意味着0和∞是F_2的borel例外值. 类似于(3.2)-(3.7)的过程可得

$$\lambda(\Delta_c f(z)-R(z))=\rho(f). \tag{3.13}$$

即

$$\max\{\lambda(\Delta_c f(z)-R(z)),\lambda(f(z+c)-R(z))\}=\rho(f).$$

证毕.

定理2的证明

假设$\lambda(f(z)-R(z))<\rho(f)$, 下证$\lambda(\Delta_c f(z)-R(z))=\lambda(f(z+c)-R(z))=\rho(f)$. 因为$\lambda(\frac{1}{f})<\rho(f)$, R是一个非常数有理函数, 由Hadamard因子分解定理, 有

$$f(z)-R(z)=\alpha(z)e^{p(z)}, \tag{3.14}$$

其中$\alpha(z)$是满足$\rho(\alpha)<\rho(f)$的亚纯函数, $p(z)$是满足$\deg p=\rho(f)$的非常数多项式. 因此

$$T(r,\alpha)=S(r,e^p), \quad T(r,f)=T(r,e^p)+S(r,f). \tag{3.15}$$

由(3.14)得

$$\begin{aligned}\Delta_c f(z)&=f(z+c)-f(z)\\&=R(z+c)+\alpha(z+c)e^{p(z+c)}-R(z)-\alpha(z)e^{p(z)}\\&=\left[\alpha(z+c)e^{p(z+c)-p(z)}-\alpha(z)\right]e^{p(z)}+R(z+c)-R(z)\\&=D_5(z)e^{p(z)}+R(z+c)-R(z),\end{aligned} \tag{3.16}$$

其中$D_5(z)=\alpha(z+c)e^{p(z+c)-p(z)}-\alpha(z)$. 由(3.15)得$T(r,D_5)=S(r,f)$. 因为$\Delta_c f(z)$是超越亚纯函数, 则$D_5(z)\not\equiv 0$. 由定理1的证明, $R(z+c)-R(z)\not\equiv 0$. 由(3.15)(3.16)与Nevanlinna 第二基本定理, 得

$$\begin{aligned}T(r,\Delta_c f)&=T(r,e^p)+S(r,f),\\N\left(r,\frac{1}{\Delta_c f-R}\right)&=T(r,e^p)+S(r,f).\end{aligned}$$

因此

$$\lambda(\Delta_c f(z)-R(z))=\rho(f). \tag{3.17}$$

即

$$\max\{\lambda(\Delta_c f(z)-R(z)),\lambda(f(z)-R(z))\}=\rho(f).$$

由(3.14)得

$$\begin{aligned}f(z+c)-R(z)&=R(z+c)+\alpha(z+c)e^{p(z+c)}-R(z)\\&=\left[\alpha(z+c)e^{p(z+c)-p(z)}\right]e^{p(z)}+R(z+c)-R(z)\\&=D_6(z)e^{p(z)}+R(z+c)-R(z),\end{aligned} \tag{3.18}$$

其中$D_6(z)=\alpha(z+c)e^{p(z+c)-p(z)}$. 类似于(3.16)(3.17)的过程可得

$$\lambda(f(z+c)-R(z))=\rho(f).$$

即

$$\max\{\lambda(f(z+c)-R(z)),\lambda(f(z)-R(z))\}=\rho(f).$$

假设$\lambda(f(z+c)-R(z))<\rho(f)$, 下证$\lambda(\Delta_c f(z)-R(z))=\rho(f)$. 类似于(3.14)-(3.17)的过程可得

$$\lambda(\Delta_c f(z)-R(z))=\rho(f).$$

即

$$\max\{\lambda(\Delta_c f(z)-R(z)),\lambda(f(z+c)-R(z))\}=\rho(f).$$

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